

Iterated elimination of dominated strategies in countable-strategy games [working title]

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Abstract

For an arbitrary countable ordinal α , we construct a game in which weak and strong domination coincide, and in which α rounds of elimination of dominated strategies are required to solve a game.

1 Introduction

An agent making a decision in an environment decides what to do by *reasoning* about the structure of the situation. Typically this reasoning encompasses *higher-order reasoning* about other agents in the environment, with whose decisions the agent's own decision will interact.

Game theory is the body of theory regarding how to model such multi-person, interactive-decision environments. Yet, to date, the benchmark approach to game-theoretic modeling has been to abstract from reasoning by assuming that agents possess “savant” competence to identify their respective strategically optimal decisions.

Apart from “bounded rationality” theories of agents who reason in unsound ways or face extrinsic constraints (such as finite memory) on reasoning in sound ways, the notable exceptions to this generalization has been the

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study of various procedures fictitious play and of iterated dominated-strategy elimination. The research reported here has to do with iterated-elimination procedures. These procedures are the multi-person extensions of procedures for single-person decision problems that are justified by classic results of statistical decision theory. and their employment in finite-strategy games leads to selection of precisely the set of strategies, from among which agents whose rationality is common knowledge would choose. (See especially Brandenburger and Dekel [1987], Tan and da Costa Werlang [1988], and Keisler and Lee [2011].¹)

In the finite-strategy-games context, an iterated-elimination process must terminate in finitely many steps. Specifically, the number of rounds of elimination cannot exceed the sum of the cardinalities of all players' strategy sets. When strategy spaces are compact and payoff functions are jointly continuous in all players' strategies, iterated elimination of strictly dominated strategies is still well behaved, in the following sense. Iterated elimination is *confluent*: the class of sets of strategies that can be obtained by iterated elimination is closed under intersection, the intersection of all of them is non-empty, and every sequence of eliminations reaches, or can be extended to reach, that intersection set. There is a particular elimination sequence, the *greedy sequence* in which *all* of the strategies dominated at a particular stage are removed immediately at that stage, the stages of which are indexed by the natural numbers, that converges to the intersection set.²

However, without both a compact strategy space and continuous payoffs, a natural-number-indexed sequence of elimination steps cannot be guaranteed to terminate even in this weak sense. Lipman [1994] constructs an example of a symmetric, 2-player game with a compact strategy set, in which the intersection of such a sequence of strategy sets obtained by iterated greedy elimination contains a pair of elements, one of which is strictly dominated by the other with respect to opponent's strategies in the set. Chen et al. [2007] invoke the axiom of choice to provide an example of a game in which there is a sequence having uncountable cardinality of greedy elimination steps, with

¹Caveat: These papers deal with dominance with respect to opponents' mixed strategies, while the analysis to be presented here has to do with dominance with respect to opponents' pure strategies. I conjecture that the distinction is immaterial for the examples that are studied.

²This result is not surprising, in view of closely related results of Bernheim [1984] regarding such games. It is proved explicitly by Dufwenberg and Stegeman [2002].

a strategy being deleted at each step.³

The games presented in these examples are not very nice games to play. The game studied by Chen et al., which can only be proved to exist by invoking a nonconstructive axiom will not have any regularity property that would enable a player to “get a handle on it” by a process of constructive reasoning. Even in Lipman’s game, which is explicitly defined, the payoff function fails to be upper semicontinuous.⁴ Joint upper semicontinuity of the payoff function in the players’ strategies, combined with compactness of the strategy space is the condition that ensures that payoffs do not precipitously become worse at the limit, when all players simultaneously make convergent sequences of payoff-increasing adjustments of their respective strategies. If this condition is not fulfilled, then perhaps it is not very surprising that iterated elimination of strategies is not “well behaved.”

This background information explains the first contribution of this paper: to produce an example of a symmetric, two-player game with compact strategy set and jointly upper semicontinuous payoffs, for which the shortest sequence of strict-dominance-elimination steps that converges is transfinite.⁵ Thus, short of restricting attention to games with continuous payoffs, the phenomenon of transfinite-length elimination sequences seems impossible to avoid.

The second contribution is to gauge how serious a problem this phenomenon is. When one opens the door to transfinite iteration of the elimination procedure by retaining the assumption of a compact strategy space, but weakening joint continuity of payoffs slightly to joint upper semicontinuity, how wide an opening is created? To put this question into perspective, con-

³Chen, Long, and Luo claim (footnote 9, p. 303) that they do not require the axiom of choice to formulate their example. However, their proposed alternative is to use the *well ordering principle*, and that principle is equivalent to the axiom of choice. (Cf. Jech [2002].)

⁴Lipman notes (footnote 11, p. 126) that the strategy space should be topologized in such a way that $a_1 = \lim_{k \rightarrow \infty} a_k$. Payoffs (specified on pp. 123–4) satisfy $\lim_{k \rightarrow \infty} u(a_{k+1}, a_k) = 1$ but $u(a_1, a_1) = 0$, violating upper semicontinuity. Alternatively, if a_1 is not regarded as a limit point, then the strategy space is a countable set of discrete points, which is not compact.

⁵*Transfinite* means a procedure with a *too long* a sequence of stages to be indexed by the natural numbers. *Uncountable* means that there are *more* stages than there are natural numbers. It is possible for a sequence to be transfinite, but not to be uncountable. Cf. Jech [2002].

sider the terms in which logicians regard infinite-stage cognitive procedures.

One of the issues in [Hilbert’s] program concerned the elimination of ideal objects. Actual statements (in Hilbert’s setting) are statements which are directly verifiable . . . Hilbert’s “dream” of obtaining a *justification of the infinite by finite reasoning* failed. Eliminating ideal objects in a proof of an actual statement needs in general infinitely many steps. . . . [H]owever, Hilbert’s idea can be generalized. . . . Instead of regarding only natural numbers as actual objects, we may also count infinite ordinals among the actual objects, [if they are sufficiently short transfinite orderings].⁶

That is, we do not regard players in a game as *actually thinking through* transfinitely many elimination stages, but rather as *reasoning soundly about* a transfinite sequence. In particular, a player should know—that is, should be able to prove to himself—that the ordering that indexes the stages is a well-founded ordering, so that transfinite induction is a sound procedure. Consider that the supremum of the ordinals for which well foundedness can be proved in Peano arithmetic is

$$\varepsilon_0 = \omega^{\omega^{\omega^{\dots}}},^7$$

where ω is the order type of the set of natural numbers. In contrast, the supremum of the ordinals that would be required to index transfinite elimination procedures in Lipman’s examples is “only”

$$\omega + \omega + \omega + \dots.^8$$

Thus, if Lipman’s example could be amended to make the payoffs jointly upper semicontinuous without “breaking” that bound, and if game theorists were to consider players to be capable of the same degree of sophistication in reasoning that proving results in Peano arithmetic requires, then those game theorists should not regard the limits of procedural rationality—players’ reasoning capacities—as being a binding constraint on substantively rational play.

⁶Pohlers [2009], p. 355.

⁷Pohlers [2009], p. 119

⁸Lipman [1994], p. 126.

However, it will be shown here that every countable ordinal corresponds to the minimum-length iteration of dominated-strategy elimination required by some (compact and jointly upper semicontinuous) game. The ordinal ε_0 is countable, as are the ordinals that characterize the strength of more powerful mathematical theories, and there is no longest countable ordinal. One view of the fact that an elimination sequence may need to be indexed by an arbitrarily long ordinal, is that there are “well behaved” games, the solution of which by iterated dominated-strategy elimination would require too long a sequence of steps for players plausibly to conceptualize soundly. A different view—particularly applicable to the game forms that are invented as allocation mechanisms—is that a game is a human cognitive construct, and that the limitations of people’s abilities to construct exceedingly complicated games may imply that the solution of those games by iterated elimination of dominated strategies is also bounded by some specific, countable ordinal. In particular, merely to require compactness and joint upper semicontinuity is not an appropriately tight characterization of what it means for a game to be well behaved, and cognitive sophistication may turn out not be a binding constraint on iterated strategy elimination (or on player’s ability to identify substantively rational strategies, in general) when an appropriately tight characterization is studied.

2 Countable ordinals

A set that can be mapped 1–1 into the natural numbers ($\mathbb{N} = \{0, 1, \dots\}$) is *countable*. The *countable ordinals* are a structure (Ω, \preceq) that satisfies the following two conditions.⁹ In this paper, lower-case Greek variables and constants denote countable ordinals and lower-case Latin variables and constants denote natural numbers, strategies, and functions. The asymmetric part of \preceq is denoted by \prec .

Axiom 1. Ω is not countable.

Axiom 2. The relation \preceq is a complete, transitive, antisymmetric, well founded relation on Ω , for which each element of Ω has a countable set of

⁹The theory of ordinals that is outlined here, is treated in various textbooks of set theory, such as Jech [2002]. The results cited here are theorems of ZFC set theory, which is the implicit setting for virtually all economic theory.

predecessors.¹⁰

A structure (Ω, \preceq) that satisfies these two axioms, also satisfies the *principle of transfinite induction*: If $A \subseteq \Omega$, then

$$[\forall \beta [\forall \alpha \prec \beta [\alpha \in A]] \implies \beta \in A] \implies [A = \Omega]. \quad (1)$$

It can be proved by transfinite induction that any two structures, each of which satisfies the two axioms, are isomorphic.

Following Zermelo, define a canonical structure (Ω, \preceq) by identifying each countable ordinal θ with its set of predecessors.

$$\theta = \{\zeta \mid \zeta \prec \theta\}. \quad (2)$$

That is, Ω can be specified to be the set of all such sets, and \prec can be specified to be the membership relation. The equivalence (2) between $\zeta \in \theta$ and $\zeta \prec \theta$ will be assumed for the remainder of the paper.

The natural numbers \mathbb{N} , represented in this way, are a subset of Ω —the finite ordinals—and are an initial segment of \preceq .¹¹ Let ω denote the first infinite ordinal.¹² In the canonical structure, $\omega = \mathbb{N}$.

Defining $A = (B \cap \gamma) \cup (\Omega \setminus \gamma)$ in (1), yields the following form of transfinite induction for a countable ordinal γ :

$$[\forall \beta \prec \gamma [\beta \subseteq B \implies \beta \in B]] \implies \gamma \subseteq B. \quad (3)$$

The following result shows that every countable ordinal can be embedded in an order-preserving way into the open unit interval. This embedding will be used to define payoffs in the games that will be studied below.

Proposition 1. *For every countable ordinal α , there is an order-preserving, 1–1 function $f : \alpha + 1 \rightarrow (0, 1)$.*

¹⁰Being well founded means that there is no mapping $f : \mathbb{N} \rightarrow \Omega$ such that, for all n , $f(n+1) \prec f(n)$.

¹¹Zermelo's representation is defined recursively by $0 = \emptyset$, $1 = \{\emptyset\}$, $2 = \{\emptyset, \{\emptyset\}\}$, and so forth. Ordinary recursion of this construction yields a set-theoretic representation of the natural numbers, and transfinite recursion extends the representation to all ordinals.

¹²Since \prec is well founded and Ω is uncountable (so not all countable ordinals can be natural numbers), there exists a first infinite ordinal.

Proof. Let $g : \alpha \rightarrow \mathbb{N}$ be 1-1. For each $\beta \preceq \alpha$, define

$$f(\beta) = \frac{1}{4} + \sum_{\gamma \in \beta} 2^{-(2+g(\gamma))}. \quad (4)$$

It is routine to verify that f satisfies the proposition. \square

Addition of ordinals is defined as concatenation of ordered sets. A successor ordinal β is an ordinal, the set of predecessors of which has a greatest element γ (so that $\beta = \gamma + 1$). An ordinal that is not 0 or a successor ordinal is a limit ordinal. Let $\Lambda \subseteq \Omega$ denote the set of countable limit ordinals. Zermelo's canonical representation of ordinals has the feature that

$$\beta \in \Lambda \implies \beta = \bigcup_{\gamma \prec \beta} \gamma. \quad (5)$$

This fact, and also the following analogue of the division algorithm for integers, will be used in the subsequent proof of the main theorem.

Proposition 2. *For every countable ordinal β , there are unique ordinals $\lambda \in \{0\} \cup \Lambda$ and $n \in \mathbb{N}$ such that $\beta = \lambda + n$.*

3 Games

For purposes of this paper, a game is a symmetric, two-person game (A, u) with a countable strategy set A and a payoff function $u : A \times A \rightarrow (-1, 1)$.

For $B \subset A$, define domination relative to B to be strict domination with respect to the opponent's pure strategies. This concept is represented by the following binary relation on strategies.

$$a \ll_B b \iff \forall c \in B \ u(a, c) < u(b, c). \quad (6)$$

For $B \subseteq A$, define the dominated subset of B to be

$$D(B) = \{a \mid a \in B \text{ and } \exists b \in B \ a \ll_B b\}. \quad (7)$$

Note that B plays two roles in this definition: as a subset of a player's strategies and as a subset of that player's opponent's strategies. This double duty reflects the symmetry of the game.

Define an α -*elimination sequence* to be a function $S : \alpha + 1 \rightarrow 2^A$ that satisfies:

$$S(0) = A; \tag{8a}$$

$$S(\beta) \setminus D(S(\beta)) \subseteq S(\beta + 1) \subset S(\beta) \tag{8b}$$

$$S(\beta) = \bigcap_{\gamma \prec \beta} S(\gamma) \text{ if } \beta \in \Lambda. \tag{8c}$$

An α -elimination sequence *terminates* if

$$D(S(\alpha)) = \emptyset. \tag{8d}$$

Then $S(\alpha)$ is called the *terminal point* of S .

The *greedy sequence* of a game is the function $G : \Omega \rightarrow 2^A$ such that, for all β ,

$$S(\beta + 1) = S(\beta) \setminus D(S(\beta)).^{13} \tag{9}$$

A game is *confluent* if

- Every elimination sequence can be extended to an elimination sequence that terminates; and
- all sequences that terminate, share the same terminal point.¹⁴

A game *requires α steps for elimination to terminate* if the greedy sequence terminates and $\alpha + 1$ is its domain. This definition is motivated by the following lemma.

Lemma 1. *If elimination sequence $S : \alpha + 1 \rightarrow 2^A$ is the restriction to $\alpha + 1$ of the greedy sequence of a confluent game and T is a β -elimination sequence that terminates, then $\alpha \preceq \beta$.*

¹³Transfinite recursion is the generalization to ordinals of definition by (ordinary) recursion on the natural numbers. The existence and uniqueness of the defined object can be proved by transfinite induction. Note that conditions (8a), (9), and (8c) define a unique sequence $G : \Omega \rightarrow 2^A$ by transfinite recursion. If this sequence is eventually constant, then its initial part is a greedy elimination sequence.

¹⁴The concept of confluence defined here is a consequence of a more primitive concept, originating in mathematical logic, which has been applied to dominance elimination by Apt [2004].

Proof. Suppose, to obtain a contradiction, that $\beta \prec \alpha$. Transfinite induction shows that, for all $\gamma \preceq \beta$, $S(\gamma) \subseteq T(\gamma)$. Thus $S(\alpha) \subseteq S(\beta + 1) \subset T(\beta)$, contradicting the game being confluent. \square

A *one-by-one elimination sequence* is an α -elimination sequence such that, for every $\beta < \alpha$, $S(\beta) \setminus S(\beta + 1)$ is a singleton.

A game is α -*solvable*, if there is an α -elimination sequence such that $S(\alpha)$ is a singleton.

Note that, if the greedy elimination sequence is a one-by-one sequence, then it is the unique sequence that terminates, so the game is confluent.

4 Games requiring long iteration sequences to solve

Theorem 1. *For every countable ordinal α , there is a game that is α -solvable but is not β -solvable for any $\beta \prec \alpha$. This game requires α steps for elimination to terminate.*

Proof. The proof is by construction. Let f be as defined in (4), and let h be the function that subtracts 1 from an ordinal, if the natural-number “remainder” in the representation provided in proposition 2 is at least 1.

$$h(\beta) = \begin{cases} \lambda & \text{if } \beta = \lambda \in \{0\} \cup \Lambda; \\ \lambda + (n - 1) & \text{if } \beta = \lambda + n, \lambda \in \{0\} \cup \Lambda, 1 \leq n \in \mathbb{N}. \end{cases} \quad (10)$$

Now let $A = \alpha + 1$ and define $u : A \times A \rightarrow (-1, 1)$ as follows.

$$u(\beta, \theta) = \begin{cases} -f(\beta) & \text{if } \theta \prec h(\beta); \\ f(\beta) & \text{if } h(\beta) \preceq \theta. \end{cases} \quad (11)$$

We prove the theorem by applying, to this game, the transfinite-induction schema (3) for $\gamma = \alpha + 1$ and

$$B = \{\beta \mid S(\beta) = (\alpha + 1) \setminus \beta\}. \quad (12)$$

If $\alpha + 1 \subseteq B$, then $S(\alpha) = (\alpha + 1 \setminus \alpha) = \{\alpha\}$ and, for $\beta \prec \alpha$, $\{\beta, \alpha\} \subseteq S(\beta)$, so α is the first ordinal β for which the game is β -solvable. That is, the theorem holds for α .

By schema (3), to prove $\alpha + 1 \subseteq B$, it is sufficient to prove for every $\beta \preceq \alpha$ that

$$\beta \subseteq B \implies \beta \in B. \quad (13)$$

In turn, to prove (13), it is sufficient to assume that

$$\beta \subseteq B \quad (14)$$

and to prove, under that hypothesis, that

$$\beta \in B. \quad (15)$$

There are three cases to consider: that β is 0, a successor ordinal, or a limit ordinal.

If $\beta = 0$, then (13) holds by definition, because $S(0) = \alpha + 1$ and $0 = \emptyset$.

If $\beta = \delta + 1$, then $S(\delta) = (\alpha + 1) \setminus \delta$ by assumption (14), and $S(\beta) = (S(\delta)) \setminus D(S(\delta))$. In order to prove (15) from (14), it is sufficient to prove that $D(S(\delta)) = \{\delta\}$. That is, it is sufficient to prove that δ is dominated by some other strategy in $S(\delta)$, against an opponent's strategies in $S(\delta)$, and no other strategy in $S(\delta)$ is so dominated.¹⁵

The strategy that dominates δ in $S(\delta)$ is $\delta + 1$, since the only strategies $\theta \in A$ such that $u(\delta + 1, \theta) \leq u(\delta, \theta)$ are those that precede $h(\delta + 1)$. Those strategies have already been removed from $S(\delta)$, since $S(\delta) = (\alpha + 1) \setminus \delta = (\alpha + 1) \setminus h(\delta + 1)$ by assumptions (2) and (15).

To show that no other strategy in $S(\delta)$ except δ is dominated at stage $\beta = \delta + 1$, let γ be any ordinal satisfying $\delta + 1 \preceq \gamma \preceq \alpha$. It must be shown that no other strategy $S(\delta)$ dominates γ against all strategies in $S(\delta)$. Note that, for every ζ and θ , $-f(\zeta) \leq u(\zeta, \theta) \leq f(\zeta)$; and that if $\zeta \preceq \theta$, then $u(\zeta, \theta) = f(\zeta)$ because $h(\zeta) \preceq \zeta \preceq \theta$.

If $\gamma \prec \eta$, then η does not dominate γ . The reason is that $\delta < h(\eta)$, so $u(\eta, \delta) = -f(\eta) < -f(\gamma) \leq u(\gamma, \delta)$. If $\eta \prec \gamma$ then η does not dominate γ , because $u(\eta, \gamma) = f(\eta) < f(\gamma) = u(\gamma, \gamma)$.

¹⁵For the remainder of this argument regarding the case that $\beta = \delta + 1$, the phrase “against opponent's strategies in $S(\delta)$ ” will always be implicit.

Thus the theorem holds for successor ordinals.

If β is a limit ordinal, then $S(\beta) = \bigcap_{\gamma \prec \beta} S(\gamma)$. Assumption (15) implies that, for every $\gamma \prec \beta$, $S(\gamma) = (\alpha + 1) \setminus \gamma$. Thus, by (5), $S(\beta) = \bigcap_{\gamma \prec \beta} ((\alpha + 1) \setminus \gamma) = (\alpha + 1) \setminus \bigcup_{\gamma \prec \beta} \gamma = (\alpha + 1) \setminus \beta$. S is the greedy elimination sequence, and it is a one-by-one sequence, so it is the only sequence that terminates. Thus the game is confluent. By lemma 1, it requires α stages for elimination to terminate. \square

5 How contrived are the examples?

One possible view of the family of examples constructed to prove theorem 1, is that these are contrived examples that are far removed from games that might be of economic interest. It is true that the examples are constructed for the purpose of proving the theorem, rather than “found” in applied economics. However, these games resemble mathematically some games that arise in applied economics. In particular, each countable successor ordinal can be topologized as a compact space. When the set of strategy profiles in the proof of the theorem is given the product topology of the two players’ strategy spaces under this topology, the payoff function is upper semicontinuous. These are the conditions under which Dufwenberg and Stegeman [2002] show that iterated removal of dominated strategies is a well behaved solution process.¹⁶ It should be kept in mind that the payoffs of games arising in applied economics are not always continuous, or even upper semicontinuous.

A *base* of a topology is a family of sets, such that any non-empty set is open if and only if it is a union basic sets. The *interval topology* is defined on an ordinal α by taking sets $(\alpha, \beta) = \{\gamma \mid \beta \prec \gamma \prec \alpha\}$ for $\beta \prec \alpha$, together with $\{0\}$, to be a base. A successor ordinal is compact in its interval topology.

¹⁶Dufwenberg and Stegeman [2002] show that, if strategy spaces are compact and each player’s payoff is u.s.c. in his own strategy, and if one sequence of eliminations terminates (that is, with no further elimination being possible) in a finite number of stages with a non-empty set of surviving strategies, then all such terminating elimination sequences (not necessarily proceeding in the same order) have the same set of surviving strategies. They show that, if strategy spaces are compact and each player’s payoffs are jointly continuous in all players’ strategies, then iteration must terminate in finitely many stages, and with a non-empty set of surviving strategies. I conjecture that joint uppersemicontinuity of payoffs is sufficient for this non-emptiness result, except that termination may occur at an infinite ordinal as in the current examples.

With respect to the interval topology on $A = \alpha + 1$, the function $f : A \rightarrow [0, 1]$ defined by (1) is continuous, and the payoff function $u : A \times A \rightarrow [-1, 1]$ specified by equations (4), (10), and (11) is jointly upper semicontinuous in the two strategies.

It may be thought that, regardless of these considerations, a game having an ordinal as its strategy space is unintuitive. This game easily extends to a game in which each player's strategy space is the unit interval, however. Define a $a : [0, 1] \rightarrow \alpha + 1$ by

$$a(x) = \min\{\gamma \mid \gamma = \alpha \text{ or } x \leq f(\gamma)\}, \quad (16)$$

where \min selects from each non-empty set the minimum element with respect to \preceq . Thus $a(x)$ is defined for every set, because the right side of (16) is nonempty and \preceq well orders $\alpha + 1$.

Now define a symmetric game in which each player has strategy set $[0, 1]$, and in which the payoff function $v : [0, 1] \times [0, 1] \rightarrow (-1, 1)$ is defined by

$$v(x, y) = \begin{cases} -f(f(a(x))) & \text{if } y \prec h(a(x)); \\ f(a(x)) & \text{if } h(a(x)) \preceq y. \end{cases} \quad (17)$$

Restricted to $f(\alpha + 1) \times f(\alpha + 1)$, v is just a “translation” of u in which $f(\beta)$ “names” β . This embedding of u is extended to the full unit square by taking $f(a(x))$ and $f(a(y))$ to approximate x and y respectively, for each point (x, y) of the square. The resulting payoff function v is upper semicontinuous. For any strategy of the opponent, a player's payoff is quasi-concave in his own strategy, so the set of best responses to a given strategy of the opponent is a compact interval. In short, this game bears close qualitative resemblance to a number of models that are studied in applied economics, and it even possesses some features that make it tractable (that is, compactness, upper semicontinuity, and quasi-concavity) but that various applied-economics models lack.

Nevertheless, in this game, $S(\beta) = [f(\beta), 1]$, so transfinitely iterated elimination of dominated strategies works isomorphically to iterated elimination in the ordinal game. In particular, $[f(\alpha), 1]$ is the set of iteratively undominated strategies, and convergence to this set via iterated elimination of dominated strategies requires α stages—a transfinite ordinal—to be completed.

To re-state what has been proved here:

Theorem 2. *For every countable ordinal α , there is a symmetric, 2-player game*

- *of which $[0, 1]$ is the strategy space;*
- *of which the payoff function is jointly upper semicontinuous in the two players' strategies; and*
- *that is α -solvable but is not β -solvable for any $\beta \prec \alpha$.*

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