Euclid's Geometry

If our universe doesn't go on forever, and if it has no walls, doesn't that mean that the universe must be bent or curved somehow? To rephrase in terms of the language of the last chapter, shouldn't any compact three-dimensional manifold without boundary curve back on itself? How else can we explain going off in one direction and coming back to where we started? It is one thing to say that the surface of our world is two-dimensional in the sense that we can map regions of it on flat pieces of paper, but doesn't it need a third dimension in which to curve? And, shouldn't the same be true of our three-dimensional universe? If it is curved, whatever that means, doesn't it need a direction in which to curve? But if our universe includes everything, how can there be something else to curve into? Come to think of it, what do we mean by "curving"? Do these questions mean anything, or are they misguided wordplay with poorly defined terms?

These questions are meaningful, and turn out to be critical to the Poincaré conjecture and its proof. They also illustrate the reason that mathematicians insist on absolute rigor. Anytime we communicate with another person, we invoke years of shared experience. We know that a glass will not fall through a table, that buildings have insides accessible by doors, that one can be right-handed or left-handed. We know what it is to be in love or to feel pain, and we don't need precise definitions to communicate. The objects of mathematics lie outside common experience, however. If one doesn't define these objects carefully, one cannot manipulate them meaningfully or talk to others about them.
Artists and humanists embrace complexity and ambiguity. Mathematicians, in contrast, work by obsessively defining terms and stripping off extraneous meaning. The almost neurotic insistence that every term be rigorously defined, and every statement proved, ultimately frees one to imagine and talk about the unimaginable. Most people, traumatized by school experiences of mathematics, know all too well that mathematics is the most meticulous and demanding of disciplines, but few get to see that it is also the most liberating and imaginative of all human activities. Absolute precision buys the freedom to dream meaningfully.

But absolute precision comes at a price. Terms need very careful definitions, and every statement, even the seemingly obvious, must be proved. What seems obvious can be frighteningly difficult to prove—sometimes, it even turns out to be wrong. Seemingly tiny exceptions matter, details can overwhelm, and progress can be unbearably slow. Mathematics is the only field of human endeavor where it is possible to know something with absolute certainty, but the hard work of slogging through morasses of possible definitions and formulations too often foreclose the dreamy vistas it affords to all but a driven few. Nothing illustrates the tension between precision and dreaminess better than Euclid's *Elements*, the famous treatise on geometry that is essential to our story, and to which we now turn.

**THE ELEMENTS**

Euclid's *Elements* dates to the reign of Ptolemy Soter (the first Ptolemy) around 300 BCE in Alexandria. From the beginning, it was a sensation. The *Elements* codified the mathematics developed from the times of Thales and Pythagoras through Plato and Archimedes. It reinterpreted millennia-old Babylonian and Egyptian mathematics within a distinctively Greek framework.

Sadly, we know almost nothing about Euclid (c. 325–c. 265 BCE).32 We know even less about him than we do about Pythagoras, and what little we do know has been hotly contested by scholars. Euclid wrote at least ten books, only half of which have survived. A number of mutually consistent indications suggest that he lived after Aristotle and before Archimedes. He was one of the first mathematicians at the great library of Alexandria and there had gathered a group of talented mathematicians about him. Legends about him abound, many as (possibly apocryphal) insertions in other mathematicians'
works. One tells that Ptolemy asked Euclid for a quick way to master geometry and received the reply, “There is no royal road to geometry.” Another tells of a student who, after encountering the first proposition in the *Elements*, asked Euclid what practical use studying geometry could have. The mathematician allegedly turned to his slave and replied dismissively, “Slave, give this boy a threepence, since he must make gain of what he learns.”

The *Elements* contains thirteen books (chapters). Books 1 to 6 deal with plane geometry, 11 to 13 with solid geometry, and 7 to 10 with number theory. Everything is worked out from first principles. Book 1 starts with twenty-three definitions, five common notions, and five postulates. The definitions name the basic objects and concepts that Euclid will consider. The *common notions* are commonly accepted rules about reasoning and relationships that he makes explicit. The *postulates*, or axioms, are assertions about the objects under consideration that are assumed to be true without proof. Nowadays, we would treat the common notions as axioms as well. The definitions, common notions, and postulates are taken as the starting points from which further assertions, called *propositions*, are proved by strict logical rules. An especially significant proposition is called a *theorem*, a proposition whose main purpose is to prove a theorem is called a *lemma*, and a proposition which follows especially easily from a theorem is called a *corollary*. A *proof* of a proposition is an ordered, precise deductive argument in which each assertion is an axiom or previously proved proposition, or else follows from such by formal rules of logic. It begins with axioms and known propositions, and ends with the statement that is to be proved.

Here, for instance, are some of the definitions in book 1.33

1. A *point* is that which has no part.

2. A *line* is a breadthless length....

8. A *plane angle* is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line.

9. And when the lines containing the angle are straight, the angle is called *rectilinear*.

10. When a straight line set upon a straight line makes the adjacent angles equal to one another, each of the equal angles is called *right*, and the straight line standing on the other is called *perpendicular* to that on which it stands....
23. Parallel straight lines are straight lines which being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction.

Here are the five common notions.

1. Things which are equal to the same thing are equal to one another.
2. If equals be added to equals, the wholes are equal.
3. If equals be subtracted from equals, the remainders are equal.
4. Things which coincide with one another are equal to one another.
5. The whole is greater than the part.

And, here are the five postulates.

1. There is a straight line from any point to any point.
2. A finite straight line can be produced in any straight line.
3. There is a circle with any center and any radius.
4. All right angles are equal to one another.
5. If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which the angles are less than the two right angles.

Euclid then goes on to deduce all of the common truths of plane geometry, each using only the definitions, common notions, postulates and previously established propositions. So, for example, proposition 1 states that given any line segment, we can create an equilateral triangle one of whose sides is that line segment. Proposition 4 states that if two sides of one triangle and the angle they enclose are equal to two sides and the enclosed angle of another triangle, then the two triangles are equal. Most high school texts refer to this as the SAS (side-angle-side) property. Proposition 5 tells us that the angles at the base of an isosceles triangle are equal (the definition of isosceles requires only that two sides be equal; one must then prove that the angles these sides make
with the third side are equal). Proposition 32 states that the angles of a triangle add up to a rectilineal angle (that is, the sum of the angles of any triangle is 180 degrees) and proposition 37 gives the formula for the area of a triangle. Proposition 47 is the Pythagorean theorem: The sum of the areas of squares erected on two sides of a triangle that meet at a right angle (that is, at 90 degrees) equals the area of the square on the side opposite the right angle; and proposition 48 its converse (if the area of the square erected on the long side of a triangle equals the sum of the areas of the squares on the other sides, the angle opposite the long side is a right angle). Two thousand years of geometry in a few short, compelling pages!

Earlier texts had appeared before Euclid's with the same emphasis on deduction. But Euclid got it right. His choice of axioms, his arrangement of propositions, and the sheer coverage were brilliant and rendered other texts obsolete. Plato, Aristotle, and other Greek philosophers were enormously interested in mathematics, and the two centuries from the times of Thales and Pythagoras to the establishment of Alexandria saw much discussion of geometry, along with arguments about what followed from what, and learned exchanges over appropriate first principles. Euclid brought order to a large collection of scattered proofs and discussions. The findings of over two hundred years of Greek geometry and number theory, and fifteen hundred years before that of Babylonian mathematics, were rigorously established from first principles. Starting with a handful of simple statements, and proceeding inexorably one small step at time, Euclid obtains result after result of genuine depth. Achievements wrung over the ages with great difficulty were made to seem inevitable.

The appearance of the Elements in a culture that valued geometry, that puzzled over the different results, and that had worked out the rules of formal reasoning contributed to the explosively creative early years of Alexandria. Extraordinarily talented mathematicians, the most brilliant of whom were Archimedes (287–212 BCE) and Apollonius (262–190 BCE), created far-reaching mathematical results and theories that built on the foundations laid by the Elements. In the next two hundred years, mathematics and science would advance much further. Theodosius (160–90 BCE) and Menelaus (c. 70–140 CE) investigated geometry on the sphere. Hipparchus (190–120 BCE) and Erastostenes (276–194 BCE) brought mathematical geography and astronomy to a higher level. Much of this later work has been lost, or has come to us in a form that makes it difficult to determine what the author was saying.
RIGOR AND THE *Elements*

Any work as self-consciously rigorous as the *Elements* invites the question of just how rigorous it is. Contrary to the insistence of generations of well-meaning teachers, Euclid does not argue only from axioms and definitions. He tacitly uses other properties.

Consider, for example, proposition 1. It states that given any line segment, we can construct an equilateral triangle that has that segment as a side. The argument goes as follows. Let A and B denote the endpoints of the given segment. By postulate 3, we can draw the circle centered at point A having the segment AB as radius. By postulate 3 again, draw the circle at point B with the same radius.

These circles (see figure 23) meet at two points. Choose one. Call it C. By postulate 1, we can draw the lines AC and BC (see figure 24). Then the triangle with sides AB, BC, AC has to be equilateral because the lengths of AB, BC, and AC are equal (because all are radii of circles with equal radius).

Nice. But what postulate or property says that the circles centered at points A and B must meet?

This does not follow from the postulates or the definitions, and is an obvious gap, noticed almost from the very beginning and mentioned in many commentaries. If the game is to be completely explicit about all assumptions, and to take nothing for granted, what allows Euclid to implicitly assume that two lines or circles that cross each other must have a common point? We need some sort of betweenness principle stating that if a line or circle contains points on different sides of another line or circle, then the two must have at least one point in common. There are other gaps as well. Many proofs, not just the one of the first proposition, leave much to be desired. Moreover, some
of the postulates are unclear. Does postulate 2 mean that we can extend any line segment forever? Does it mean that we can cut up any segment? And if it means the first, who is to say that the resulting line is unique? And how seriously should we take the definitions? Are they just meant to provide guidance about a word that is essentially undefined (today's, and probably Euclid's, interpretation) or are they supposed to completely specify the object named? In the latter case, just what does the phrase "a breadthless length" mean?

Mathematicians and scholars know that there are gaps in Euclid, and there has been a great deal of discussion over the ages about alternate axioms, or possible additional ones. That has not stopped generations of worshipful schoolmasters, besotted with the majestic order, the accessibility and the patent usefulness of the Elements from rushing in and trumpeting it as the finest in human thought. However, to a thoughtful student, the Elements can seem less rational than capricious. The insistence that the Elements is flawless, and the apex of rigorous thought, turns some students away from mathematics. One wonders how much fear of mathematics stems from the disjuncture between the assertion that Euclid is perfect and some students' intuitive, but difficult to articulate, sense that some things in it are not quite right. Unless you are unusually rebellious, it is easy blame yourself and conclude that mathematics is beyond you.

It is worth bearing in mind that mathematical results, for all they are represented as eternal and outside specific human cultures, are in fact transmitted and understood within definite social and cultural contexts. Some argue, for example, that the Greeks invented proof in order to make sense of the statements of mathematical results of Babylon and Egypt without access to the context in which such results were used and discovered.35 In order to make use of the results, the Greeks needed to sort out different, seemingly
contradictory, computations and re-create them themselves in their own terms. This is certainly plausible. Even within the same civilization, each generation of mathematicians reinterprets and reframes the mathematics of previous generations. To learn mathematics is to reinvent it.

But the ambiguities run deeper still. Alexandria of twenty-three hundred years ago was a culture very foreign to our own. Although it was very advanced mathematically and technologically in its first centuries, large chunks of this learning were lost, and we know almost nothing of the context in which the *Elements* was created. In a provocative book that has just received a marvelous translation from Italian to English, Lucio Russo argues that science, in the modern sense of the term, flourished in Alexandria from 300–150 BCE and was subsequently lost. According to Russo, the geometric parts of the *Elements* were a theory of computation. In his account, the Greeks did computations by first translating them into geometry, then drawing the relevant geometric construction using a ruler and compass, and measuring. Like the slide rule centuries later, the ruler and compass were analogue means of computation, and the *Elements* was a manual of sorts, showing how and for what one could use them.

Russo's argument is quite a way out of the mainstream, and will surely draw some heated criticism. But the obviousness of the gap in the proof of proposition 1 supports Russo's contention that Euclid was creating a mathematical model of what a person could do with a ruler and compass on a piece of parchment, and not aiming for absolute purity. The first axiom states that we can draw a line between any two points, and the second that we can extend any line indefinitely. Taken together, these amount to saying that we have a ruler, but that we are going to ignore any complications that arise from its being too short. The third axiom, positing that we can construct a circle centered at any point of any desired size, says that we have an "ideal compass"—we can assume that it is as big, or small, as necessary. One can imagine Euclid saying that we are going to explore what we can do with a ruler and compass yet without worrying about their physical limitations. It would never occur to him to posit that lines and circles drawn with rulers and compasses intersect in a point if they cross one another. This would seem completely obvious if he had ideal physical objects in mind.

The plausibility of Russo's argument should teach us humility. We don't know the purpose or the intended readership of the *Elements*. But the widespread notion that it was a textbook for schoolchildren may not be accurate. Even the most casual reading of the *Elements* suggests that it was written for
grown-ups, not children. The assumption that it was a textbook for pupils associated with the Alexandria Library is just that: an assumption.

**THE LONGEVITY OF THE ELEMENTS**

Its flaws notwithstanding, one cannot read the *Elements* without coming away from it with a genuine admiration for the artfulness with which it is laid out, and the cleverness of the proofs. The relentless advance from the simplest notions to subtle, deep, and beautiful propositions testifies to the efficacy of human reason. Looking back from the vantage of today, it is easy to take the longevity of *Elements* for granted and to concoct after-the-fact rationalizations that explain its survival. Anything this good must endure, one would think. We tell ourselves that the really good books from ancient times are the ones that survived. According to this chestnut, the more authoritative a manuscript was, the more likely it was that it would be copied and recopied. The more likely, too, that it would have been translated into Arabic and survive that way even if the Greek manuscript were lost. Such wishful thinking is reassuring, but far too many books were lost to indulge in it. Euclid was the most famous geometer of ancient times, yet half of his books have disappeared. 37

The blithe assumption that anything as good as the *Elements* must endure detracts from the miracle of its survival. The *Elements* endured even as the creative energies and quality of Alexandrine scholarship slowly leached away (and they were largely past by the time Julius Caesar torched Alexandria harbor in 47 BCE). It survived the end of Alexandria as a place of learning, an extinction marked by the brutal murder of the Neoplatonist mathematician Hypatia (370–415) in March 415. Threatened by her charisma and the power of her lectures, a frenzied Christian mob stripped her naked and flayed her to death seeking to erase her authority, her beauty, and her learning. They did not succeed. The edition of the *Elements* on which she had worked with her father, Theon (335–c. 405), became the standard after her death. It was the basis of what Caliph al-Mansur (712–775, reigned 754–774) obtained from the Byzantine emperor and which the great Arab translator al-Hajjaj (c. 786–833) would subsequently translate not once, but twice. 38 Arab scholars were as fascinated by the *Elements* as had been the Greeks and the books would be retranslated, recopied, and reedited hundreds of times in the great centers of Arab learning. Countless commentaries, summaries, expositions, and translations were made.
Several hundred years later, the *Elements* and a number of works of Aristotle were among the very first Greek works to be translated back from Arabic into Latin. Gherard of Cremona (1114–1187) seems to have made the first such translation. Johannes Campanus, a chaplain to Urban IV who was pope 1261–1281, retranslated it. The onslaught of translations of classical texts by Gherard and others coincided with the near-spontaneous emergence of our oldest universities in Bologna, Paris, Oxford, Cambridge, Salamanca, and elsewhere in Europe. In a very real sense, Euclid is at the heart of our universities. 39

The *Elements* was the first scientific book to be published after printing presses emerged in the mid-fifteenth century. 40 Different editions of it were best-sellers in Renaissance Europe. It is widely conceded to be the second-most-read book in human history. 41 However, when you consider that it was translated into Chinese in 1607 and had penetrated the Indian subcontinent by the tenth century, it may well in fact be the most-read book of all times. This is all the more striking when one realizes that its only rivals are the Bible and the Quran.

The popularity of the *Elements* over time suggests that it responds to a deep human need. Many have hailed the salutary effect that studying it has on the development of one’s reasoning ability. While in Congress, Abraham Lincoln took it to bed with him every night. Thomas Jefferson advised an inquiring young man that he would find the most useful results in the later books. Others theorized that Euclid was especially useful for young women: As early as 1838, Mount Holyoke College, the oldest women’s college in the United States, required its students to own and study either Simson’s or Playfair’s Euclid. 42

Despite the survival and the popularity of the *Elements*, we really have no idea what the original looked like. This would be the case for any book that age: Before the invention of printing, documents were copied from one hand to another, with all the errors that are introduced in transcribing texts compounding with each transcription. It is especially so for a book as widely copied as the *Elements*. The complexity in the transmittal of the *Elements* to present day far exceeds that of any other ancient text. 43 Thousands of scholars and teachers have studied it, annotating it carefully and recopying it in ways that strike them as clearer. Different editions, annotations, and translations abound. The original Arabic translations of the Greek have been lost, as have the Greek manuscripts from which they were translated.

For a long time, it had been thought that the standard Arabic edition was actually older than extant Greek editions. However, in 1808, François Peyrard argued that a Greek manuscript copy of the *Elements* in the Vatican Library
that Napoleon had pilfered and taken to Paris was actually older than the Arab­ic edition. The vital clue was a comment that Theon made in his commen­tary to Ptolemy’s Almagest, indicating that he had added material to the final proposition of book 6 of the Elements. The Vatican manuscript did not contain this addendum. Peyrard went on to correct the then-authoritative Greek edition [that had been prepared by Simon Grynaeus (Basel, 1533)].

In 1883–84, the Danish scholar J. L. Heiberg published a very learned re­construction, starting from scratch with the original Greek text based on the Vatican manuscript and other manuscripts. There is no question that Heiberg’s result is “purer” than the version attributed to Theon and Hypatia, and his is the version most scholars start with today. It is the basis of the standard English translation, prepared by Heath in 1908. In spite of Heiberg’s massive scholarship, we do not know how much older the Vatican manu­script is than the version of the Elements of Theon and Hypatia. The latter appeared seven hundred years after Euclid’s Elements. The Vatican version could have appeared several hundred years earlier than the Theonine version and still have absorbed several hundred years’ worth of changes. For all we know, Heiberg’s reconstruction may differ greatly from Euclid’s original.

What is most inspiring about the Elements is not its majesty, but its contin­gency. The text itself is not what matters and indeed, appearances to the con­trary, is not what has endured. What matters is the curiosity that things geometric engender, the willingness to question received wisdom, and the way in which human knowledge builds on work of others. Virtually every word and every line of the Elements has received extended attention from thousands of commentators. Alternate phrasings have been proposed, different proofs mooted. Theorems are known in different places of the world by different pet names. Proposition 5 is known in England as the pons arsinorum, “Asses Bridge.” Proposition 47, known nowadays as the Pythagorean theorem, was in times past referred to as the “Wedding Theorem” or the “Theorem of the Bride.”

The most complete commentary to have survived from ancient times is Proclus’s (410–485) from Rome. He, in turn, seems to have had at hand no fewer than four earlier major commentaries, all lost excepting fragments, underscoring again the fragility of individual works. The Elements is a leitmotif for the human enterprise, for our dependence on one another, and for the unreliability of individual achievement. With regard to the Poincaré conjecture, what matters most about the Elements is the fifth postu­late and how it has given way to our current understanding of space.