Sir ISAAC NEWTON's
TREATISE
OF THE
Quadrature of CURVES.

INTRODUCTION to the Quadrature of Curves.

1. Consider mathematical Quantities in this Place not as consisting of very small Parts; but as describ'd by a continued Motion. Lines are describ'd, and thereby generated not by the Apposition of Parts, but by the continued Motion of Points; Superficies's by the Motion of Lines; Solids by the Motion of Superficies's; Angles by the Rotation of the Sides; Portions of Time by a continual Flux; and so in other Quantities. These Geneses really take Place in the Nature of Things, and are daily seen in the Motion of Bodies. And after this Manner the Ancients, by drawing moveable right Lines along immovable right Lines, taught the Genesis of Rectangles.

2. Therefore considering that Quantities, which increase in equal Times, and by increasing are generated, become greater or less according to the greater or less Velocity with which they increase and are generated; I sought a Method of determining Quantities from the Velocities of the Motions or Increments, with which they are generated; and calling these Velocities of the Motions or Increments Fluxions, and the generated Quantities Fluents, I fell by degrees upon the Method of Fluxions, which I have made use of here in the Quadrature of Curves, in the Years 1665 and 1666.
3. Fluxions are very nearly as the Augments of the Fluents generated in equal but very small Particles of Time, and, to speak accurately, they are in the first Ratio of the nascent Augments; but they may be expounded by any Lines which are proportional to them.

4. Thus if the Area's ABC, ABDG be described by the Ordinates BC, BD moving along the Base AB with an uniform Motion, the Fluxions of these Area's shall be to one another as the describing Ordinates BC and BD, and may be expounded by these Ordinates, because that these Ordinates are as the nascent Augments of the Area's.

5. Let the Ordinate BC advance from it's Place into any new Place bc. Complete the Parallelogram BCEb, and draw the right Line VTH touching the Curve in C, and meeting the two Lines bc and BA produc'd in T and V: and Bb, Ec and Cc will be the Augments now generated of the Abscis AB, the Ordinate BC and the Curve Line ACc; and the Sides of the Triangle CET are in the first Ratio of these Augments considered as nascent, therefore the Fluxions of AB, BC and AC are as the Sides CE, ET and CT of that Triangle CET, and may be expounded by these same Sides, or, which is the same thing, by the Sides of the Triangle VBC, which is similar to the Triangle CET.

6. It comes to the same Purpose to take the Fluxions in the ultimate Ratio of the evanescent Parts. Draw the right Line Cc, and produce it to K. Let the Ordinate bc return into it's former Place BC, and when the Points C and c coalesce, the right Line CK will coincide with the Tangent CH, and the evanescent Triangle CEC in it's ultimate Form will become similar to the Triangle CET, and it's evanescent Sides CE, Ec and Cc will be ultimately among themselves as the Sides CE, ET and CT of the other Triangle CET, are, and therefore the Fluxions of the Lines AB, BC and AC are in this same Ratio. If the Points C and c are distant from one another by any small Distance, the right Line CK will likewise be distant from the Tangent CH by a small Distance. That the right Line CK may coincide with the Tangent CH, and the ultimate Ratios of the Lines CE, Ec and Cc may be found, the Points C and c ought to coalesce and exactly coincide. The very smallest Errors in mathematical Matters are not to be neglected.

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Quadrature of Curves.

7. By the like way of reasoning, if a Circle describ'd with the Center B and Radius BC be drawn at right Angles along the Abfciis AB, with an uniform Motion, the Fluxion of the generated Solid ABC will be as that generating Circle, and the Fluxion of it's Superficies will be as the Perimeter of that Circle and the Fluxion of the Curve Line AC jointly. For in whatever Time the Solid ABC is generated by drawing that Circle along the Length of the Abfciis, in the same Time it's Superficies is generated by drawing the Perimeter of that Circle along the Length of the Curve AC. You may likewise take the following Examples of this Method.

8. Let the right Line PB, revolving about the given Pole P, cut another right Line AB given in Position: it is required to find the Proportion of the Fluxions of these right Lines AB and PB.

Let the Line PB move forward from it's Place PB into the new Place Pb. In Pb take PC equal to PB, and draw PD to AB in such manner that the Angle bPD may be equal to the Angle bBC; and because the Triangles bBC, bPD are similar, the Augment Bb will be to the Augment CB as Pb to Db. Now let Pb return into it's former Place PB, that these Augments may evanish, then the ultimate Ratio of these evanefcent Augments, that is the ultimate Ratio of Pb to Db, shall be the same with that of PB to DB, PDB being then a right Angle, and therefore the Fluxion of AB is to the Fluxion of PB in that same Ratio.

9. Let the right Line PB, revolving about the given Pole P, cut other two right Lines given in Position, viz. AB and AE in B and E: the Proportion of the Fluxions of these right Lines AB and AE is sought.

Let the revolving right Line PB move forward from it's Place PB into the new Place Pb, so as to cut the Lines AB, AE in the Points b and e: and draw BC parallel to AE meeting Pb in C, and it will be Bb : BC :: AB : Ae, and BC : Ee :: PB : PE, and by joining the Ratios, Bb : Ee :: AB x PB : Ae x PE. Now let Pb return into it's former Place PB, and the evanefcent Augment Bb will be to the evanefcent Augment Ee as AB x PB to AE x PE; and therefore the Fluxion of the right Line AB is to the Fluxion of the right Line AE in the same Ratio.
10. Hence if the revolving right Line PB cut any curve Lines given in Position in the Points B and E, and the right Lines AB, AE now becoming moveable, touch these Curves in the Points of Section B and E: the Fluxion of the Curve, which the right Line AB touches, shall be to the Fluxion of the Curve, which the right Line AE touches, as \( AB \times PB \) to \( AE \times PE \). The same thing would happen if the right Line PB perpetually touch'd any Curve given in Position in the moveable Point P.

11. Let the Quantity \( x \) flow uniformly, and let it be proposed to find the Fluxion of \( x^n \).

In the same Time that the Quantity \( x \), by flowing, becomes \( x + o \), the Quantity \( x^n \) will become \( x^n + nox^{n-1} + \frac{n^2}{2}nox^{n-2} + &c. \) And the Augments \( o \) and \( nox^{n-1} + \frac{n^2}{2}nox^{n-2} + &c. \) are to one another as \( 1 \) and \( nx^{n-1} + \frac{n^2}{2}nox^{n-2} + &c. \).

Now let these Augments vanish, and their ultimate Ratio will be \( 1 \) to \( nx^{n-1} \).

12. By like ways of reasoning, the Fluxions of Lines, whether right or curve in all Cases, as likewise the Fluxions of Superficies's, Angles and other Quantities, may be collected by the Method of prime and ultimate Ratios. Now to institute an Analysis after this manner in finite Quantities and investigate the prime or ultimate Ratios of these finite Quantities when in their nascent or evanescent State, is consonant to the Geometry of the Ancients: and I was willing to show that, in the Method of Fluxions, there is no necessity of introducing Figures infinitely small into Geometry. Yet the Analysis may be performed in any kind of Figures, whether finite or infinitely small, which are imagin'd similar to the evanescent Figures; as likewise in these Figures, which, by the Method of Indivisibles, use to be reckoned as infinitely small, provided you proceed with due Caution.

From the Fluxions to find the Fluents, is a much more difficult Problem, and the first Step of the Solution is equivalent to the Quadrature of Curves; concerning which I wrote what follows some considerable Time ago.