ably the most useful of all instruments in the study of educational problems. Strange to say, altho this method was invented and developed by skillful mathematicians, they have not applied it to the solution of their own teaching problems. We should be first rather than last to use our weapons in the pedagogy of our subject. We should try to find out by this method what mathematics a student can learn, and how and when he could learn it to the best advantage. That would constitute a scientific basis for both text books and teaching. We would then know how to develop the laboratory spirit of proceeding from the known to the unknown. This is the spirit which tries to find and point out the practical, usable features of a course, that knows when to turn aside from the beaten path long enough to revel in the beauties of the wild flowers by the wayside, kindling the enthusiasm which shall spur one on thru the dull monotonous stretches of the journey.

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THE HISTORY OF THE SOLUTION OF THE CUBIC EQUATION

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In order to trace the history of the solution of the cubic equation, it is necessary that we go back to the history of ancient peoples. Naturally, the first solutions were geometric, for ancient Egyptians and Greeks knew nothing of algebra. Both of these nations were familiar with the simple cubic equation which resulted from problems in land measuring and especially from the duplication of the cube.

The Egyptians considered the solution impossible, but the Greeks came nearer to a solution. Hippocrates of Chios, about 430 B. C., was the first to show that the duplication of a cube could be reduced to finding two mean proportionals between a given line and one twice its length, for instance, $a:x=x:y=y:2a$; since $x^2=ay$ and $y^2=2ax$, $x^4=a^2y^2$ and $x^4=2xa^3$, then $x^3=2a^3$;

but he did not find the mean proportionals. The Greeks could not solve this equation which arose, as we have said, in the problem of the duplication of a cube, and also in the trisection of an angle, by ruler and compasses, but only by mechanical curves. The foundation of the solution of cubic equations by intersecting conics was laid by the Greeks, for, it was Menaechmus, born 429 B. C., who first invented conic sections. Diophantus of Alexandria, about 300 A. D., who is preeminently the Greek writer on algebra, solved only one cubic equation, viz; the equation $x^2+2x+3=x^3+3x-3x^2-1$, which arose in the solution of a problem to find a right angled triangle such that the sum of its area and hypotenuse is a square, and its perimeter equals a cube. He did not give negative or irrational numbers as solutions, but he did give rational fractions for roots.

The Arabs improved the methods of Hippocrates of Chios and Menaechmus and at the same time developed a method originating with Diophantus and improved by the Hindoos for finding approximate roots of numerical equations by algebraic process. Al Mahani of Bagdad was the first to state the problem of Archimedes demanding the section of a sphere by a plane so that the two segments shall be a prescribed ratio in the form of a cubic equation. Abu Jafar Al Hazin was the first to solve the equation by conic sections. Abul Gud solved the equation $x^3-x^2-2x+1=0$.

The solution of cubic equations by intersecting conics was the greatest achievement of the Arabs in algebra. The foundation had been laid by the Greeks whose aim had been not to find the number corresponding to x, but simply, to determine the side x of a cube double another cube of side a. The Arabs, on the other hand, had another object in view, namely to find the roots of given numerical equations. Omar Al Hay of Chorassan, about 1079 A. D. did most to elevate to a method the solution of the algebraic equations by intersecting conics. He believed that cubics could not be solved by calculation.

The Arabs did practically no original work. They developed the work of the Greeks and the Hindoos, and by allowing themselves to be influenced more by the Greeks than by the Hindoos, they barred the road of progress for themselves.

The Hindoos did no actual work on the cubic equation, but they developed the form and spirit of our modern algebra and arithmetic.

In Europe in 1202 Leonardo of Pisa published the Liber Abaci. This book contained all the knowledge the Arabs possessed in algebra and arithmetic and treated the subject in a free and independent way.

Leonardo was presented to Emperor Frederick II of Hohenstaufen, who was a great patron of learning. On that occasion several problems were proposed to Leonardo which he solved promptly. His methods were partly borrowed from the Arabs and partly original. One problem was the solving of $x^3+2x^2+10x=20$. As yet cubic equations had not been solved algebraically. Leonardo, changing his method of inquiry showed by clear and rigorous demonstration that the roots could not be represented by Euclidean irrational quantities; that is, constructed with ruler and compassess. He obtained close approximations to the required root.

We now pass on to Lucas Pacioli, who states in his book published in 1497 that the solution of equations such as $x^3+nx=n$ and $x^3+n=nx$ is impossible in the present state of science.

The first step in the algebraic solution of cubics was taken by Scipo Ferro (died 1526), professor of mathematics at Bologna, who solved the equation $x^3+mx=n$. Nothing more is known of his discovery than that he imparted it to his pupil Floridas in 1505. It was the practice, then, to keep discoveries secret in order to secure by that means an advantage over rivals by proposing problems beyond their reach.

A second solution is given by Nicolo of Brescia 1506 (?)—1557. When a boy he was so badly cut by a French soldier that he never regained the use of his tongue, so he was called Tartaglia, "the Stammerer". He was too poor to go to school so he learned to read and picked up, by himself, a knowledge of Latin and Greek, and mathematics. Possessing a mind of extraordinary power, he became a teacher of mathematics at one of the universities.

Tartaglia found an imperfect method in 1530 for solving $x^3+px^2=q$ but kept it secret. He spoke of his discovery in public and when Ferro's pupil, Floridas, heard of it he proclaimed his own knowledge of the form $x^3+mx=n$. Tartaglia believing him to be a braggart challenged him to a contest, a sort of mental duel popular at that time. Then Tartaglia put forth all his zeal, industry and skill to find a rule and succeeded ten days before the contest. The difficult step was passing from the quadratic irrationals to the cubic. Placing $x=\sqrt[3]{t}-\sqrt[3]{u}$, Tartaglia perceived that irrationals disappeared from the equation making $n=t-u$. But this, together with $(m/3)^3=tu$ gives at once $t=\sqrt{(n/2)^3+(m/2)^3}+n/2$, $u=\sqrt{(n/2)^3+(m/2)^3}-n/2$. Tartaglia solved thirty problems while Floridas solved none.

In 1541 Tartaglia discovered the general solution for $x^3+px^2=+q$ by transforming it to the form $x^3+mx=+n$. This knowledge was obtained by Cardan under promise of secrecy, but he published it in his Ars Magna. This destroyed Tartaglia's hope of giving the world an immortal work, for the crown intended for his work had been snatched away. He challenged Cardan and his pupil Ferrari to a contest. Cardan did not appear. Tartaglia solved most of the questions in seven days, the other two did not give their results before five months and most of the solutions were incorrect.

Tartaglia then started to publish his work but he died before he reached the consideration of the cubic equation. Although he sustained his claim for priority, posterity has not conceded to him the honor of his discovery The solution is called today "Cardan's solution".

Cardan, 1501-1576, was an Italian physican and mathematician who received all advantages offered by the universities of his day. He went one step beyond Tartaglia in recognizing negative roots of an equation, calling them "fictitious".

Vieta pointed out how the construction of the roots of a cubic equation depended upon the celebrated problems of the duplication of the cube and the trisection of an angle. He reached the interesting conclusion that the former problem includes solution of all cubics in which the radical in Tartaglia's formula is real but the latter problem includes only those leading to irreducible cases.

We have seen how the cubic equation discussed by Egyptians, Greeks, and Hindoos was finally solved algebraically by the Italians of the sixteenth century. One unit of our modern algebra was discussed for practically two thousand years and from this discussion "Cardan's solution" resulted.

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THE ACTUAL AND THE ARTIFICIAL IN ANALYTIC GEOMETRY

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Beginners in the study of analytic geometry are not apt to be discriminating in their work. Text books contain many mechanical exercises that are intended for exercise only. That in itself is no harm. But some exercises have to do with the actual properties of curves and some do not. It would no doubt add to the effectiveness of the study of analytic geometry in schools if more attention were given to this phase.

Ordinarily we use rectangular coordinates. Is the slope of the tangent at a point fixed on the curve the same for all rectangular systems of coordinates? Obviously a rotation of axes will alter the value of the slope at a point fixed on the curve. The ordinary use of the slope does not express a characteristic of the curve any more than it does of the coordinate system in which the equation is written. On the other hand if two lines or two curves intersect at a point in one system of coordinates they will intersect at the same point relative to the curves in another system. Moreover the angle of intersection is unchanged by a change of coordinates.

When an equation expressed in one system of coordinates is transformed by a rotation of axes into another system points that were at first maximum points of the curve may no longer be maximum points in the new system. The maximum ordinate of a curve then is not independent of the coordinates. Once the attention is called to this fact it is recognized immediately.

A familiar exercise is—Show that the tangent to the para-