CAN WE MAKE MATHEMATICS INTELLIGIBLE?

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Why is it that we mathematicians have such a hard time making ourselves understood? Many people have negative feelings about mathematics, which they blame, rightly or wrongly, on their teachers [**1**]. Students complain that they cannot understand their textbooks; they have been doing this ever since I was a student, and presumably for much longer than that. Professionals in other disciplines feel compelled to write their own accounts of the mathematics they had trouble with. However, it was not until after I became editor of this MONTHLY that I quite realized how hard it is for mathematicians to write so as to be understood even by other mathematicians (outside of fellow specialists). The number of manuscripts rejected, not for mathematical deficiencies but for general lack of intelligibility, has been shocking. One of my predecessors had much the same experience 35 years earlier [**2**].

To put it another way, why do we speak and write about mathematics in ways that interfere so dramatically with what we ostensibly want to accomplish? I wish I knew. However, I can at least point out some principles that are frequently violated by teachers and authors. Perhaps they are violated because they contradict what many of my contemporaries seem to consider to be self-evident truths. (They also have little in common with the MAA report on how to teach mathematics [**3**].)

Abstract Definitions. Suppose you want to teach the "cat" concept to a very young child. Do you explain that a cat is a relatively small, primarily carnivorous mammal with retractile claws, a distinctive sonic output, etc.? I'll bet not. You probably show the kid a lot of different cats, saying "kitty" each time, until it gets the idea. To put it more generally, generalizations are best made by abstraction from experience. They should come one at a time; too many at once overload the circuits.

There is a test for identifying some of the future professional mathematicians at an early age. These are students who instantly comprehend a sentence beginning "Let X be an ordered quintuple $(a, T, \pi, \sigma, \mathfrak{B})$, where . . ." They are even more promising if they add, "I never really understood it before." Not all professional mathematicians are like this, of course; but you can hardly succeed in becoming a professional unless you can at least understand this kind of writing.

However, unless you are extraordinarily lucky, most of your audience will not be professional mathematicians, will have no intention of becoming professional mathematicians, and will never become professional mathematicians. To begin with, they won't understand anything that starts off with an abstract definition (let alone with a dozen at once), because they don't yet have anything to generalize from. Please don't immediately write me angry letters explaining how important abstraction and generalization are for the development of mathematics: I *know* that. I also am sure that when Banach wrote down the axioms for a Banach space he had a lot of specific spaces in mind as models. Besides, I am discussing only the communication of mathematics, not its creation.

For example, if you are going to explain to an average class how to find the distance from a point to a plane, you should first find the distance from $(2, -3, 1)$ to $x - 2y - 4z + 7 = 0$. After that, the general procedure will be almost obvious. Textbooks used to be written that way. It is a good general principle that, if you have made your presentation twice as concrete as you think you should, you have made it at most half as concrete as you ought to.

Remember that *you* have been associating with mathematicians for years and years. By this

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time you probably not only think like a mathematician but imagine that everybody thinks like a mathematician. Any nonmathematician can tell you differently.

Analogy. Sometimes your audience will understand a new concept better if you explain that it is similar to a more familiar concept. Sometimes this device is a flop. It depends on how well the audience understands the analogous thing. An integral is a limit of a sum; therefore, since sums are simpler (no limiting processes!), students will understand how integrals behave by analogy with how sums behave. Won't they? In practice, they don't seem to. Integrals are simpler than sums for many people, and there may be some deep reason for this [**4**].

Vocabulary. Never introduce terminology unnecessarily [**5**]. If you are going to have to mention a countable intersection of open sets—just once!—there is no justification for defining G_δ's and F_σ's.

I have been assured that nobody can really understand systems of linear equations without all the special terminology of modern linear algebra. If you believe this you must have forgotten that people understood systems of linear equations quite well for many years before the modern terminology had been invented. The terminology allows concise statements; but concision is not the alpha and omega of clear exposition. Modern terminology also lets one say more than could be said in old-fashioned presentations. Nevertheless, at the beginning of the subject a lot of the students' effort has to go into memorizing *words* when it could more advantageously go into learning mathematics. Paying more attention to vocabulary than to content obscures the content. This is what leads some students to think that the real difference between Riemann and Lebesgue integration is that in one case you divide up the x-axis and in the other you divide up the y-axis.

If you think you can invent better words than those that are currently in use, you are undoubtedly right. However, you are rather unlikely to get many people except your own students to accept your terminology; and it is unkind to make it hard for your students to understand anyone else's writing. One Bourbaki per century produces about all the neologisms that the mathematical community can absorb.

In any case, if you *must* create new words, you can at least take the trouble to verify that they are not already in use with different meanings. It has not helped communication that "distribution" now means different things in probability and in functional analysis. On the other hand, if you need to use old but unfashionable words it is a good idea to explain what they mean. A friend of mine was rebuked by a naïve referee for "inventing" bizarre words that had actually been invented by Kepler.

It is especially dangerous to assume either that the audience understands your vocabulary already or that the words mean the same to everybody else that they do to you. I know someone who thinks that everybody from high school on up knows all about Fourier transforms, in spite of considerable evidence to the contrary. Other people think that everybody knows what they mean by Abel's theorem, and therefore never say which of Abel's many theorems they are appealing to.

An even more serious problem comes from what (if it didn't violate my principles) I would call geratologisms: that is, words and phrases that, if not actually obsolete in ordinary discourse, are becoming so. Contemporary prose style is simpler and more direct than the style of the nineteenth century—except in textbooks of mathematics. While I was writing this article I was teaching from a calculus book that begins a problem with, "The strength of a beam varies directly as . . ." I do not know whether the jargon of variation is still used in high schools, but in any case it isn't learned: only one student in a class of 45 had any idea what the book meant (and he was a foreigner). Blame the students if you will, blame the high schools; for my own part I blame the authors of the textbook for not realizing that contemporary students speak a different language. Another current calculus book says, "Particulate matter concentrations in parts per million theoretically decrease by an inverse square law." You couldn't get away with that in *Newsweek* or even in *The New Yorker*, but in a textbook . . .

Authors of textbooks (lecturers, too) need to remember that they are supposed to be addressing the students, not the teachers. What is a function? The textbook wants you to say something like, "a rule which associates to each real number a uniquely specified real number," which certainly defines a function—but hardly in a way that students will comprehend. The point that "a definition is satisfactory only if the students understand it" was already made by Poincaré [6] in 1909, but teachers of mathematics seem not to have paid much attention to it.

The difficulties of a vocabulary are not peculiar to mathematics; similar difficulties are what makes it so frustrating to try to talk to physicians or lawyers. They too insist on a rich technical language because "it is so much more precise that way." So it is, but the refined terminology is clearer only when rigorous distinctions are absolutely necessary. There is no use in emphasizing refined distinctions until the audience knows enough to see that they are needed.

Symbolism is a special kind of terminology. Mathematics can't get along without it. A good deal of progress has depended on the invention of appropriate symbolism. But let's not become so fascinated by the symbols that we forget what they stand for. Our audience (whether it is listening or reading) is going to be less familiar with the symbolism than we are. Hence it is not a good idea (to take a simple example) to say "Let f belong to L^2" instead of "Let f be a measurable function whose square is integrable," unless you are sure that the audience already understands the symbolism. Moreover, if you are not actually going to use L^2 as a Hilbert space, but want only the properties of its elements as functions, the structure of the space is irrelevant and calling attention to it is a form of showing off—mild, but it *is* showing off. If the audience doesn't know the symbolism, it is mystified; if it does know, it will be wondering when you are going to get to the point.

My advice about new terminology applies with even greater force to new symbolism. Do not create new symbolism, or change the old, unnecessarily; and admit (if necessary) that usage varies and explain the existing equivalences. If your $\Phi(x)$ also appears in the literature as $P(x)$ or $P(x)+\frac{1}{2}$ or $F(x)$, *say so*. Irresponsible improvements in notation have already caused enough trouble. I don't know who first thought of using θ in spherical coordinates to mean azimuth instead of colatitude, as it almost universally did and still does in physics and in advanced mathematics. It's superficially a reasonable convention because it makes θ the same as in plane polar coordinates; however, since r is different anyway, that isn't much help. The result is that students who go beyond calculus have to learn all the formulas over again. Such complications don't bother the true-blue pure mathematicians, those who would just as soon see Newton's second law of motion stated as $\mathbf{v}=(d/d\sigma)(\mathfrak{R}\mathbf{q})$, but they do bother many students, besides irritating physical scientists.

Proofs. Only professional mathematicians learn anything from proofs. Other people learn from explanations. I'm not sure that even mathematicians learn much from proofs in fields with which they are not familiar. A great deal can be accomplished with arguments that fall short of being formal proofs. I have known a professor (I hesitate to say "teacher") to spend an entire semester on a proof of Cauchy's integral theorem under very general hypotheses. A collection of special cases and examples would have carried more conviction and left time for more varied and interesting material, besides leaving the audience better equipped to understand, apply, generalize, and teach Cauchy's theorem.

I cannot remember who first remarked that a sweater is what a child puts on when its parent feels cold; but a proof is what students have to listen to when the teacher feels shaky about a theorem. It has been claimed [7] that "some of the most important results . . . are so surprising at first sight that nothing short of a proof can make them credible." There are fewer of these than you think.

Experienced parents realize that when a child says "Why?" it doesn't necessarily want to hear a reason; it just wants more conversation. The same principle applies when a class asks for a proof.

Rigor. This is often confused with generality or completeness. In spite of what reviewers are likely to say, there is nothing unrigorous in stating a special case of a theorem instead of the most general case you know, or a simple sufficient condition rather than a complicated one. For example, I prefer to give beginners Dirichlet's test for the convergence of a Fourier series: "piecewise monotonic and bounded" is more comprehensible than "bounded variation"; and, in fact, equally useful after one more theorem (learned later).

The compulsion to tell everything you know is one of the worst enemies of effective communication. We mathematicians would get along better with the Physics Department if, for example, we could bring ourselves to admit that, although their students need some Fourier analysis for quantum mechanics, they don't need a whole semester's worth—two weeks is nearer the mark.

Being more thorough than necessary is closely allied to **pedantry**, which (my dictionary says) is "excessive emphasis of trivial details."

Here's an example. Suppose students are looking for a local minimum of a differentiable function f, and they find critical points at $x = 2$, $x = 5$, and nowhere else. Suppose also that they do not want to use (or are told not to use) the second derivative. Some textbooks will tell them to check $f(2 + h)$ and $f(2 - h)$ for all small h. Students naturally prefer to check $f(3)$ and $f(1)$. The pedantic teacher says, "No"; the honest teacher admits that any point up to the next critical point will do.

Enthusiasm. Teachers are often urged to show enthusiasm for their subjects. Did you ever have to listen to a really enthusiastic specialist holding forth on something that you did not know and did not want to know anything about, say the bronze coinage of Poldavia in the twelfth century or "the doctrine of the enclitic *De*" [**8**]? Well, then.

Skills. A great deal of the mathematics that many mathematicians support themselves by teaching consists of subjects like elementary algebra or calculus or numerical analysis—skills, in short. It is not always easy to tell whether a student has acquired a skill or, as we like to put it, "really" learned a subject. The difficulty is much like that of deciding whether apes can use language in a linguistically interesting way or whether they have just become very clever at pushing buttons and waving their hands [**9**]. Mathematical skills are like any other kind. If you are learning to play the piano, you usually start by practicing under supervision; you don't begin with theoretical lectures on acoustical vibrations and the internal structure of the instrument. Similarly for mathematical skills. We often read or hear arguments about the relative merits of lectures and discussions, as if these were the only two ways to conduct a class. Having students practice under supervision is another and very effective way. Unfortunately it is both untraditional and expensive.

Even research in mathematics is, to a considerable extent, a teachable skill. A student of G. H. Hardy's once described to me how it was done. If you were a student of Hardy's, he gave you a problem that he was sure you could solve. You solved it. Then he asked you to generalize it in a specific way. You did that. Then he suggested another generalization, and so on. After a certain number of iterations, you were finding (and solving) your own problems. You didn't necessarily learn to be a second Gauss that way, but you could learn to do useful work.

Lectures. These are great for arousing the emotions. As a means of instruction, they ought to have become obsolete when the printing press was invented. We had a second chance when the Xerox machine was invented, but we seem to have muffed it. If you *have* to lecture, you can at least hand out copies of what you said (or wish you had said). I know mathematicians who contend that only through their lectures can they communicate their personal attitudes toward their subjects. This may be true at an advanced level, for pre-professional students. Otherwise I wonder whether these mathematicians' personalities are really worth learning about, and (if so) whether the students couldn't learn them better some other way (over coffee in the cafeteria, for example.)

One of the great mysteries is: How can people manage to extract useful information from incomprehensible nonsense? In fact, we can and do. Read, for example, in Morris Kline's book [**10**] about the history of the teaching of calculus. Perhaps this talent that we have can explain the popularity of lectures. One incomprehensible lecture is not enough, but a whole course may be effective in a way that one incomprehensible book never can. I still contend that a comprehensible book is even better.

Conclusion. I used to advise neophyte teachers: "Think of what your teachers did that you particularly disliked—and don't do it." This was good advice as far as it went, but it didn't go far enough. My tentative answer to the question in my title is, "Yes; but don't be guided by introspection." You cannot expect to communicate effectively (whether in the classroom or in writing) unless and until you understand your audience. This is not an easy lesson to learn.

References

1. See, for example, Sydney J. Harris, column for February 9, 1980, Chicago Sun-Times and elsewhere.

2. L. R. Ford, Retrospect, this MONTHLY, 53 (1946) 582–585.

3. College Mathematics: Suggestions on How to Teach It, Mathematical Association of America, 1972.

4. D. R. Stoutemyer, Symbolic computation comes of age, SIAM News, 12, no. 6 (December 1979) 1, 2, 9.

5. The same point has been made by P. R. Halmos in How to Write Mathematics, L'Enseignement mathématique, (2) 16 (1970) 123–152.

6. H. Poincaré, Science et méthode, 1909, Book II, Chapter 2.

7. H. and B. S. Jeffreys, Methods of Mathematical Physics, 2nd ed., Cambridge University Press, 1950, p. v.

8. Robert Browning, "A Grammarian's Funeral," in The Complete Poetic and Dramatic Works of Robert Browning, Houghton Mifflin, Boston and New York, 1895, pp. 279–280.

9. For example, E. S. Savage-Rumbaugh, D. M. Rumbaugh, and S. Boysen, Do apes use language? Amer. Scientist, 68 (1980) 49–61.

10. Morris Kline, Mathematics: The Loss of Certainty, Oxford University Press, New York, 1980.

A PERMANENT INEQUALITY

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A famous problem that has resisted the attacks of many of the world's greatest mathematicians was resolved in 1980 by G. P. Egorychev [**5**], who proved a conjecture that B. L. van der Waerden had made in 1926 [**15**]. The conjecture, which is now a theorem, states that the permanent of an $n \times n$ doubly stochastic matrix is never less than $n!/n^n$; the latter value, which is obtained when all entries of the matrix are equal to $1/n$, is therefore the minimum. The purpose of this note is to give an essentially self-contained exposition of Egorychev's proof and the auxiliary results that preceded it, using only elementary concepts of mathematics (except at one point).*

1. Introduction to Quadratic Forms. Our discussion will be based mostly on facts about matrices and quadratic forms that we will prove "from scratch." A *quadratic form* $f(x_1,\ldots,x_n)$ of n variables is an expression

$$f(x_1,\ldots,x_n) = \sum_{i,j} f_{ij} x_i x_j \tag{1.1}$$

defined by some $n \times n$ matrix of real coefficients f_{ij}. The sum is over $1 \leq i, j \leq n$, which we abbreviate to "i, j". Since the coefficient of $x_i x_j$ in $f(x_1,\ldots,x_n)$ is $f_{ij} + f_{ji}$ when $i \neq j$, we can

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The name conventionally transliterated Egorychev is pronounced (approximately) YeGORycheff.—*Editors*

*__Added in proof:__ See the additional information on p. 798, this issue.