MATH 533 Lie Group
Final Essay

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1 Preface

This notes is the final essay of the MATH 533 Lie Group, which is taught by Professor Paul Baum. This notes will introduce what I have learned in this semester. We will start from the basic knowledge of the differential manifolds and end at Dynkin diagrams. The main reference is [1] and [2]. Before we come to mathematics topics, let's see some history of the Lie groups at first.

Lie groups, which is introduced by Sophus Lie in the winter of 1873-1874 and stated that all of the principal results by 1884. In 1884 a young German mathematician, Friedrich Engel, came to work with Lie to expose his theory of continuous groups. The result of their work is the three-volume Theorie der Transformationsgruppen, published in 1888, 1890, and 1893.

In 1888, the German mathematician Wilhelm Killing published the first paper in a series entitled Die Zusammensetzung der stetigen endlichen Transformationsgruppen (The composition of continuous finite transformation groups). Later, Elie Cartan and Hermann Wely refined Killing's works by ledding to classification of semisimple Lie algebras and description of representations of compact and semisimple Lie groups using highest weights, respectively.

"Weyl brought the early period of the development of the theory of Lie groups to fruition, for not only did he classify irreducible representations of semisimple Lie groups and connect the theory of groups with quantum mechanics, but he also put Lie's theory itself on firmer footing by clearly enunciating the distinction between Lie's infinitesimal groups (i.e., Lie algebras) and the Lie groups proper, and began investigations of topology of Lie groups."[3]

2 Differential Manifolds

In this section, we will review some basic definitions and properties of the differential manifolds, which will help us to understand the Lie group.

In order to give the definition of the differential manifold, we need the definition of the topological manifold,

Definition 1 (Topological Manifold)**.** *A topological space X is called locally Euclidean if there is a non-negative integer n such that every point in X has a neighborhood which is homeomorphic to the Euclidean space E^n (or, equivalently, to the real n-space R^n, or to some connected open subset of either of two). A topological manifold is a locally Euclidean Hausdorff space.*

With that definition, we can define what is differential manifold,

Definition 2 (Differential Manifold)**.** *A differentiable manifold is a topological manifold equipped with an equivalence class of atlases whose transition maps are all differentiable. In broader terms, a C^k-manifold is a topological manifold with an atlas whose transition maps are all k-times continuously differentiable.*

Then, for a manifold M and a point $p \in M$, we denoted by T_mM the tangent space to M at the point m, and TM the tangent bundle to M. For a morphism $f : X \to Y$ and a point $x \in X$, we denote the $f_* : T_xX \to T_{f(x)}Y$ the corresponding map of tangent space.

An immersed submanifold in a manifold is a subset $N \subset M$with a structure a manifold such that the inclusion map $i : N \to M$ is an immersion, which means that the $rank\ \ i_* = \dim T_xN$ for every $x \in N$.

An embedded submanifold $N \subset M$ is an immersed manifold such that the inclusion map $i : N \to M$ is a homeomorphism. Note that in this case the smooth structure on N is uniquely

determined by the smooth structure on M and this is not the case when we think about the immersed submanifold.

At last, we will introduce the Sard Theorem, which will play an important role in Lie group. But we need the definition of the regular value and critical value at first,

Definition 3 (Regular Value, Critical Value)**.** *Let M and N be differentiable manifolds and $f : M \to N$ be a differentiable map between them. The point $p \in M$ is called as a regular point if*

$$Df_p : T_pM \to T_{f(p)}N \tag{1}$$

is a surjective map. The point $p \in M$ which is not a regular point is a critical point. The point $q \in N$ is a regular value of f if all point p in its pre-image is regular point. The critical value i the image of the critical point.

Once we have these definitions, we state the Sard theorems,

Theorem 1 (Sard Theorem)**.** *The set of the critical values is null.*

3 Basic Definition and Theory of Lie Groups

3.1 Definitions of the Lie Groups

In this section, we will introduce some basic definitions, notations, theorems and some interesting examples of the Lie groups. At beginning, we give the definition of the Lie groups,

Definition 4 (Lie Group)**.** *A Lie group is a set G with two structures: G is a group and G is a manifold. These structures agree in the following sense: multiplication map $G \times G \to G$ and inversion map $G \to G$ are smooth structures.*

Right now, we can see some basic examples of the Lie groups,

Example 1. *Here are some simple examples,*

- *$\mathscr{R}^n$ with the group operation given by addition*
- *Any finite-dimensional vector space V over R is a Lie group, with product Mult given by addition*
- *$SL(n,\mathscr{R})$, $GL(n,\mathscr{R})$, $O(n,\mathscr{R})$, $U(n)$, $SO(n,\mathscr{R})$, $SU(n)$, $Sp(n,\mathscr{R})$(we will prove it later)*

Before introducing the closed Lie subgroup, we state the following two useful theorem, specific proof can be found in [2].

Theorem 2. *Let G be a Lie group. Denote by G^0 the connected component of identity. Then G^0 is a normal subgroup of G and is a Lie group itself. The quotient group G/G^0 is discrete.*

Theorem 3. *If G is a connected Lie group, then its universal cover $\tilde{G}$ has a canonical structure of a Lie group such that the covering map $p : \tilde{G} \to G$ is a morphism of Lie groups whose kernel is isomorphic to the fundamental group of G: $\ker p = \pi_1(G)$ as a group. Moreover, in this case $\ker p$ is a discrete central subgroup in $\tilde{G}$.*

Then, we can define the closed Lie subgroup,

Definition 5 (Closed Lie Subgroup)**.** *A closed Lie subgroup H of a Lie subgroup G is a subgroup which is also a submanifold.*

In order to describe the closed Lie subgroup, we have following theorem,

Theorem 4. *Any closed Lie subgroup is closed in G.*

Besides the closed Lie subgroup, we also need to introduce a more general definition for the subgroup,

Definition 6. *An Lie group in a Lie group $H \subset G$ is an immersed submanifold which is also a subgroup.*

At the end of this section, we mention a very interesting theorem which describe the structure of the connected Lie groups,

Theorem 5. *If G is a connected Lie group and U is a neighborhood of 1, then U generates G.*

3.2 Action and Representation of Lie groups

In this section, we will briefly introduce the action and representation of Lie group.

Definition 7 (Action of Lie Group)**.** *An action of a Lie group G on a manifold M is an assignment to each $g \in G$ a diffeomorphism $\rho(g) \in DiffM$ such that $\rho(1) = id$, $\rho(gh) = \rho(g)\rho(h)$ and such that the map,*

$$G \times M \to M : (g, m) \longmapsto \rho(g) \cdot m \tag{2}$$

is a smooth map.

Example 2.

- *The group $GL(n, \mathscr{R})$ acts on $\mathscr{R}^n$.*
- *The group $O(n, \mathscr{R})$ acts on the sphere $S^{n-1} \subset \mathscr{R}^n$.*
- *The group $U(n$ acts on the sphere $S^{2n-1} \subset \mathscr{C}^n$.*

Once we have the notion of the action, we can define the representation,

Definition 8. *A representation of a Lie group G is a vector space V together with a group morphism $\rho : G \to End(V)$. If V is finite-dimensional, we requires that ρ be smooth, so it is a morphism of Lie groups. A morphism between two representations V, W of the same group G is a linear map $f : V \to W$ which commutes with the group action: $f\rho_V(g) = \rho_W(g)f$.*

Example 3. *Representation of G on the space of functions $C^\infty(M)$ defined by,*

$$(\rho(g)f)(m) = f(g^{-1}m) \tag{3}$$

3.3 Left, right and adjoint action

At first, we define these three actions,

Definition 9 (Left, Right and Adjoint Action)**.** *Actions of G on itself,*

- *Left action: $L_g : G \to G$ is defined by $L_g(h) = gh$;*

- *Right action:* $R_g : G \to G$ *is defined by* $R_g(h) = hg^{-1}$*;*
- *Adjoint action:* $Ad_g : G \to G$ *is defined by* $Ad(g) = ghg^{-1}$*.*

Notice that adjoint action preserves the identity element $1 \in G$, thus we can define an action of G on the space T_1G, i.e.

$$Ad_g : T_1G \to T_1G. \tag{4}$$

Then, we can define the corresponding invariant vector fields with respect to these three actions,

Definition 10. *A vector field* $v \in Vect(G)$ *is left-invariant if* $(L_g)_*v = v$ *for every* $g \in G$ *and right-invariant if* $(R_g)_*v = v$ *for every* $g \in G$*. A vector field is called bi-invariant if it is both left- and right-invariant.*

Right now, we can identify the space that constitute by these invariant vector fields with tangent space at the unit.

Theorem 6. *The map* $v \longmapsto v(1)$ *defines an isomorphism of the vector space of left-invariant vector fields on* G *with the vector space* T_1G *and similarly for right-invariant vector spaces. As for the space of the bi-invariant vector fields on* G*, the map* $v \longmapsto v(1)$ *defines an isomorphism of that space and the vector space of invariants of adjoint action,*

$$(T_1G)^{Ag \quad G} = \{x \in T_1G | Ad \quad g(x) = x \quad for \quad all \quad g \in G\} \tag{5}$$

3.4 Exponential map for Matrix Group

In this section, we wil see some basic propety for exponential map for matrix group and prove some interesting facts.

More precisely, we define the exponenetial map and log map for the matrices as following,

$$\begin{aligned} \exp(x) &= \sum_{k=0}^{\infty} \frac{x^k}{k!} \\ \log(1+x) &= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}x^k}{k} \end{aligned} \tag{6}$$

It is no hard to see that exponential map is an analytic map for all $n \times n$ matrices and log map is an analytic map defined in a neighborhood of 1 in all $n \times n$ matrices.

With that definition, we can introduce some basic properties of the exponential map of the matrices group.

Proposition 1.
- $\log(\exp(x)) = x$, $\exp(\log(X)) = X$.
- $\exp(0) = 1$, $d\exp(0) = id$.
- *If* $xy = yx$ *then* $\exp(x+y) = \exp(x)\exp(y)$*. If* $XY = YX$*, then* $\log(XY) = \log(X) + \log(Y)$ *in some neighborhood of the identity.*
- *The map* $h : \mathscr{K} \to GL(n, \mathscr{K})$*,* $h(t) = \exp(tx)$ *for some fixed* x *in* $n \times n$ *matrices is a morphism of Lie groups.*
- $\exp(AxA^{-1}) = A\exp(x)A^{-1}$, $\exp(x^t) = (\exp(x))^t$.

By the third part of the previous property, we know that $\exp(x) \in GL(n, \mathscr{K})$.

Once we have the log map and exponential map, we can check the Lie group for some classical groups.

Definition 11 (Classical Group)**.** *$GL(n, \mathscr{K})$, $SL(n, \mathscr{K})$, $O(n, \mathscr{K})$, $SO(n, \mathscr{K})$, $Sp(n, \mathscr{K})$, $U(n)$, $SU(n)$, $Sp(n) = Sp(n, \mathscr{C}) \cap SU(2n)$ are called as the classical groups.*

Theorem 7. *For each classical group $G \subset GL(n, \mathscr{K})$, there exists a vector space $\mathfrak{g} \subset \mathfrak{gl}(n, \mathscr{K})$ ($\mathfrak{gl}(n, \mathscr{K})$ is the set of all $n \times n$ matrices) such that for some neighborhood U of 1 in $GL(n, \mathscr{K})$ and some neighborhood u of 0 in $\mathfrak{gl}(n, \mathscr{K})$ the following maps are mutually inverse,*

$$(U \cap G)^{\log} \Longleftrightarrow_{\exp} (u \cap \mathfrak{g}) \tag{7}$$

By this theorem, we can have the following corollary,

Corollary 1. *Each classical group is a Lie group with tangent space at identity $T_1 G = \mathfrak{g}$ and $\dim G = \dim \mathfrak{g}$.*

In fact, the proof of the theorem and corollary are not very hard. For the theorem, we just need to find the corresponding restriction for each case and then we done; for the corollary, we just need to move the neighborhood along the manifold and notice that $\mathfrak{g}$ is a vector space.

So, we have already proved that example 1 are Lie groups.

3.5 Lie Algebras

In this section, we will give the definition of the Lie algebra and some related topics.

At first, we define the general exponential map for Lie groups as we do not have multiplication in $\mathfrak{g}$.

Definition 12. *Let G be a real or complex Lie group, $\mathfrak{g} = T_1 G$. Then the exponential map $\exp : \mathfrak{g} \to G$ is defined by,*

$$\exp(x) = \gamma_x(1) \tag{8}$$

where $\gamma_x(t)$ is the one-parameter subgroup with tangent vector at 1 equal to x.

The existence and the uniqueness of the one-parameter group can be got from the basic theory of the ODE, we will not present that here.

Example 4. *Let $G = \mathscr{R}$, so that $\mathfrak{g} = \mathscr{R}$. Then for any $a \in \mathfrak{g}$, the corresponding one-parameter subgroups is $\gamma_a(t) = ta$, so the exponential map is given by $\exp(a) = a$.*

Then, for the general exponential map, we have following,

Proposition 2. *Let G be a real Lie group and $\mathfrak{g} = T_1 G$.*

- $\exp(0) = 1$ *and* $\exp_*(0) : \mathfrak{g} \to T_1 G$ *is the identity map.*
- *The exponential map is a diffeomorphism between some neighborhood of 0 in $\mathfrak{g}$ and a neighborhood of 1 in G.*
- $\exp((t+s)x) = \exp(tx)\exp(sx)$ *for every $s, t \in \mathscr{K}$.*
- *For any morphism of Lie groups $\varphi : G_1 \to G_2$ and any $x \in \mathfrak{g}_1$, we have $\exp(\varphi_*(x)) = \varphi(\exp(x))$.*

- *For any $X \in G$, $y \in \mathfrak{g}$, we have $X \exp(y) X^{-1} = \exp(Ad \quad X \cdot y)$.*

According to this property and Theorem 5, we can describe the Lie group morphism from connect Lie groups as following,

Theorem 8. *Let G_1 and G_2 be Lie groups. If G_1 is connected, then any Lie group morphism $\varphi : G_1 \to G_2$ is uniquely determined by the linear map $\varphi_* : T_1 G_1 \to T_1 G_2$.*

Right now, let's define what is a commutator, which is one of the most important conceptions in Lie groups.

Consider the sufficiently small $x, y \in \mathfrak{g}$, the product $\exp(x)\exp(y)$ will be close to $1 \in G$ and can be written as the form $\exp(x)\exp(y) = \exp(\mu(x, y)))$, where $\mu : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is a smooth map defined in a neighborhood of $(0, 0)$. By direct calculation, it is easy to see that $\mu(x, y) = x + y + \lambda(x, y) + \ldots$ where dots stand for terms of order large equal than 3 and $\lambda : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is a bilinear skew-symmetric map.

With all these preparation works, we define the $[x, y] = 2\lambda(x, y)$ and call this bilinear skew-symmetric map as the commutator.

According to the definition of the commutator, it is no hard to see that the induced map by the morphism of Lie group φ_* will preserves the commutator, more precisely, we have,

Proposition 3. *Let $\varphi : G_1 \to G_2$ be a morphism of Lie groups and $\varphi_* : \mathfrak{g}_1 \to \mathfrak{g}_2$, i.e. the corresponding map of tangent spaces at identity. Then φ_* preserves the commutator:*

$$\varphi_*[x, y] = [\varphi_* x, \varphi_* y]. \tag{9}$$

Remark 1. *By this proposition, we see that this commutator is almost same when it is compared with the group commutator in G, i.e. if G is a commutative Lie group, then $[x, y] = 0$ for all $x, y \in \mathfrak{g}$.*

So far, we have enough preparation to define the Lie Algebra. Before we give the formal definition, let's see one useful and critical theorem at first.

Theorem 9 (Jacobi Identity)**.** *Let G be a Lie group and $\mathfrak{g} = T_1 G$ and let the commutator $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ be defined as before. Then it satisfies the following identity, called Jacobi identity,*

$$[x, [y, z]] = [[x, y], z] + [y, [x, z]]. \tag{10}$$

Besides this, denote $ad = Ad_* : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$, i.e. the map of tangent spaces corresponding to the map $Ad : G \to GL(\mathfrak{g})$, then we have,

$$ad \quad x \cdot y = [x, y]. \tag{11}$$

Right now, we can define the Lie algebra,

Definition 13 (Lie Algebra)**.** *A Lie algebra over a field $\mathscr{K}$ is a vector space $\mathfrak{g}$ over $\mathscr{K}$ with a $\mathscr{K}$-bilinear map $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ which is skew-symmetric: $[x, y] = -[y, x]$ and satisfies the Jacobi identity. A morphism of Lie algebras is a $\mathscr{K}$-linear map $f : \mathfrak{g}_1 \to \mathfrak{g}_2$ which preserves the commutator.*

Once we have the definition for the Lie algebra, it is no hard to see the following theorem, which describe the Lie algebra of general Lie groups.

Theorem 10. *Let G be a Lie group. Then $\mathfrak{g} = T_1 G$ has a canonical structure of a Lie algebra over $\mathscr{K}$ with the commutator defined previous, we denote the Lie algebra by $Lie(G)$.*

As we have Lie subgroup, it is natural to define the Lie subalgebra,

Definition 14 (Lie subalgebra and Ideals)**.** *Let $\mathfrak{g}$ be a Lie algebra over $\mathscr{K}$. A subspace $\mathfrak{h} \subset \mathfrak{g}$ is called a Lie subalgebra is it is closed under the commutator. A subspace $\mathfrak{h} \subset \mathfrak{g}$ is called an ideal if for any $x \in \mathfrak{g}$, $y \in \mathfrak{h}$, we have $[x, y] \in \mathfrak{h}$.*

With these interesting definitions, it is easy to see that the tangent space of Lie subgroup at identity is a Lie subalgebra, which describe the structure relation between the Lie subgroup and Lie subalgebra.

3.6 Campbell-Hausdorff Formula and Fundamental Theorems of Lie Theory

In this section, we will give a very quick review about some very important topics in Lie groups, i.e. Campbell-Hausdorff formula and Fundamental theorems of Lie theory.

As we just show in previous sections, the commutator only used the lowest non-trivial terms of the group law in some special local coordinates. Thus, it is natural question to ask that whether the higher terms give much more complicate phenomenon on $\mathfrak{g}$? However, the answer is negative, we have the Campbell-Hausdorff formula,

Theorem 11. *For small enough $x, y \in \mathfrak{g}$, one has $\exp(x)\exp(y) = \exp(\mu(x, y))$ for some $\mathfrak{g}$-values function $\mu(x, y)$ which is given by the following series convergent in some neighborhood of $(0, 0)$:*

$$\mu(x, y) = x + y + \sum_{n \geq 2} \mu_n(x, y) \tag{12}$$

where $\mu_n(x, y)$ is a Lie polynomial in x, y of degree n. More precisely, an expression consisting of commutators of x, y, their commutators.... This expression is universal: it does not depend on the Lie algebra $\mathfrak{g}$ or on the choice of x, y.

By this theorem, we can recover some global property of the connected Lie groups by its local property,

Corollary 2. *The group operation in a connected Lie group G can be recovered from the commutator in $\mathfrak{g} = T_1 G$.*

Right now, let's go to the relation of the Lie groups and Lie algebras. It is natural to ask following questions about the Lie groups and Lie algebras,

- Given a morphism of Lie algebras $\mathfrak{g}_1 \to \mathfrak{g}_2$, where $\mathfrak{g}_1 = Lie(G_1)$ and $\mathfrak{g}_2 = Lie(G_2)$, can this morphism be always lifted to a morphism of the Lie groups?
- Given a Lie subalgebras $\mathfrak{h} \subset \mathfrak{g} = Lie(G)$, does there always exist a corresponding Lie subgroup $H \subset G$?
- Can every Lie algebra be obtained as a Lie algebra of a Lie group?

In fact, these questions looks like the questions in abstract algebra of the Sylow groups, and another similarly thing is that we also has Fundamental theorem of Lie theory here to answer these questions, just as what Sylow theorems do in abstract algebras.

Theorem 12 (Fundamental Theorems of Lie Theory)**.** *This theorem has three parts, each part answer one of the previous questions,*

- *For any Lie group G, there is a bijection between connected Lie subgroups $H \subset G$ and Lie subalgebras $\mathfrak{h} \subset \mathfrak{g}$ given by $H \to \mathfrak{h} = Lie(H) = T_1 H$.*
- *If G_1, G_2 are Lie groups and G_1 is connected and simply connected, then $Hom(G_1, G_2) = Hom(\mathfrak{g}_1, \mathfrak{g}_2)$, where $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are Lie algebras of G_1, G_2 respectively.*
- *Any finite-dimensional Lie algebra is isomorphic to a Lie algebra of a Lie group.*

Notice that in the second part of the Fundamental theorems of the Lie theory requires the simply connected condition, if the Lie groups are not simply connected, we have following interesting counterexamples.

Example 5. *Let $G_1 = S^1 = \mathscr{R}/\mathscr{Z}$ and $G_2 = \mathscr{R}$. Then the Lie algebras are $\mathfrak{g}_1 = \mathfrak{g}_2 = \mathscr{R}$, which commutator is just zero. Then, consider the identity map from $\mathfrak{g}_1$ to $\mathfrak{g}_2$, $a \mapsto a$. In this way, the corresponding morphism of Lie groups should be $\theta \mapsto \theta$, but also notice that this morphism should satisfies $f(\mathscr{Z}) = \{0\}$, thus, we know that the identity morphism of the Lie algebras cannot lift as a morphism of the Lie groups.*

4 Basic Representation Theory of Lie Groups

In this section, we will introduce the basic representation theory of Lie groups and Lie algebras. And almost all the Lie groups and Lie algebras in this section will be finite dimension and all representations are complex.

4.1 Basic Definitions and Properties

Definition 15 (Representation of Lie groups and Lie algebras)**.** *A representation of Lie group G is a vector space V together with a morphism $\rho : G \to GL(V)$. A representation of a Lie algebra $\mathfrak{g}$ is a vector space V together with a morphism $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$.*

Besides the representation of Lie groups and Lie algebras, we also need to define the morphism between the two representations.

Definition 16. *A morphism between two representations V, W of the same group G is a linear map $f : V \to W$ which commutes with the action of $G : f\rho_V(g) = \rho_W(g)f$. The morphism of a representation of a Lie algebra is same. The space of all G-morphisms between V and W will be denoted by $Hom_G(V, W)$, similarly, the space of all $\mathfrak{g}$-morphisms between V and W will be denoted by $Hom_{\mathfrak{g}}(V, W)$.*

With these interesting definitions, we can describe the relation between the representation of the Lie groups and lie algebras,

Theorem 13. *Let G be a Lie group with Lie algebra $\mathfrak{g}$.*

- *Every representation $\rho : G \to GL(V)$ defines a representation $\rho_* : \mathfrak{g} \to \mathfrak{gl}(V)$, and every morphism of representations of G is automatically a morphism of representation of $\mathfrak{g}$.*
- *If G is connected, simply connected, then $\rho \mapsto \rho_*$ gives an equivalence of categories of representation of G and categories of representation of $\mathfrak{g}$. In particular, every representation of $\mathfrak{g}$ can be uniquely lifted to a representation of G, and $Hom_G(V, W) = Hom_{\mathfrak{g}}(V, W)$.*

These theorems are extremely important in Representation theory. Due to these theorems connected the representation of Lie groups and Lie algebra and notice that Lie algebras are finite dimensional vector space, which will help us a lot.

Right now, we can go a little further and introduce some interesting notations and theorems. At first, let's see the definition for the subrepresentation.

Definition 17 (Subrepresentation)**.** *Let V be a representation of G. A subrepresentation is a vector subspace $W \subset V$ stable under the action: $\rho(g)W \subset W$ for all $g \in G$. We can define the subrepresentation almost same for the Lie algebras.*

Once we have this definition, by the $\mathscr{K}$-linear of the ρ_*, it is no hard to see that if G is a connected Lie group with Lie algebra $\mathfrak{g}$, then $W \subset V$ is a subrepresentation for G if and only if it is a subrepresentation for $\mathfrak{g}$.

Then, we can define the representation on direct sum and tensor product as following,

Theorem 14. *Let V, W be representations of G(respectively $\mathfrak{g}$). Then there is a canonical structure of a representation on $V^*, V \oplus W, V \otimes W$.*

Here we give a very interesting example to explain the previous theorem.

Example 6. *Let V be a representative of G(respectively $\mathfrak{g}$). Then the space $End(V) \simeq V \otimes V^*$ of linear operators on V is also a representation, with the action given by $g : A \mapsto \rho_V(g)A\rho_V(g^{-1})$ for $g \in G$(respectively, $x : A \mapsto \rho_V(x)A - A\rho_V(x)$ for $x \in \mathfrak{g}$).*

Then, we give the definition of the invariants of the representation theory.

Definition 18. *Let V be a representation of a Lie group G. A vector $v \in V$ is called invariant if $\rho(g)v = v$ for all $g \in G$. The subspace if invariant vectors in V is denoted by V^G. As for the Lie algebra, let V be a representation of a Lie algebra $\mathfrak{g}$. A vector $v \in V$ is called invariant if $\rho(x)v = 0$ for all $x \in \mathfrak{g}$. The subspace of invariant vectors in V is denoted by $V^{\mathfrak{g}}$.*

By noticing that the $\mathscr{K}$-linear of the ρ_* and the definition of the exponential map, it is easy to get that when G is a connected Lie group with the Lie algebra $\mathfrak{g}$, then for any representation $V \subset G$, we have $V^G = V^{\mathfrak{g}}$.

We use another interesting example to end this section.

Example 7. *Let V, W be representations and $Hom(V, W)$ be the space of linear maps $V \to W$, with the action of G defined by $g : A \mapsto \rho_W(g)A\rho_V(g^{-1})$ for $g \in G$. Then $(Hom(V, W))^G = Hom_G(V, W)$, which is the space of morphisms between representations. In particular, when $V = \mathscr{C}$, we have $V^G = (Hom(\mathscr{C}, V))^G = Hom_G(\mathscr{C}, V)$ when $\mathscr{C}$ considered as a trivial representation.*

4.2 Irreducible Representation

In order to understand the structure of the Lie groups, more precisely, the classification of all presentations of Lie group or a Lie algebra under some special assumption, we can begin our studying from simplest case, which called as the irreducible or simple,

Definition 19. *A non-zero representation V of G or $\mathfrak{g}$ is called irreducible or simple if it has no subrepresentation other than $0, V$. Otherwise V is called reducible.*

According to the definition, it is easy to see that 1-dimensional representations of any group are irreducible.

Once we have this definition, we can consider the decomposition of the representation in some sense.

Definition 20. *A representation is called completely reducible or semisimple if it is isomorphic to a direct sum of irreducible representations: $V \simeq \oplus V_i$, V_i irreducible.*

However, it is worth to notice that not every representation is completely reducible. For example, let $G = \mathscr{R}$, so $\mathfrak{g} = \mathscr{R}$. Notice that in this case, a representation of $\mathfrak{g}$ is the same as a vector space V with a linear map $\mathscr{R} \to End(V)$. More precisely, every such map is of the form $t \mapsto tA$ for some $A \in End(V)$ which can be arbitrary and the corresponding representation of the group G is just $t \mapsto \exp(tA)$. By some very simple calculation, it is no hard to see that writing a representation given by $t \mapsto \exp(tA)$ as a direct sum of irreducible part is equivalent to diagonalizing of A. But notice that not every matrix can be diagonalize, thus we see that not every representation is completely reducible.

At last of this section, let's mention one interesting theorem that will help us decompose the representation into direct sums.

Theorem 15. *Let V be representation of G and let $Z \in Z(G)$ be a central element of G such that $\rho(Z)$ is diagonlizable. For a representation of G, $V = \oplus V_\lambda$, where V_λ is the eigenspace for $\rho(Z)$ with eigenvalue λ. Similar results holds for central elements in $\mathfrak{g}$.*

In fact, this theorem just using the some basic properties in linear algebra about the eigenspace.

4.3 Representation of finite groups and Haar Measure on Compact Lie Groups

In this section, we will mainly concentrate on these two topics and finally show that a large class of representations is completely reducible. At first, we need to introduce what is unitary property for Lie group.

Definition 21. *A complex representation V of a real Lie group G is called unitary if there is a G-invariant inner product:*

$$(\rho(g)v, \rho(g)w) = (v, w) \quad \forall g \in G. \tag{13}$$

For Lie algebra, we can replace the previous equation by $(\rho(x)v, w) + (v, \rho(x)w) = 0$, $\forall x \in \mathfrak{g}$.

Once we have this definition, we will have some interesting description of the representation of the finite groups. More precisely, we have following two theorems,

Theorem 16. *Each unitary representation is completely reducible.*

And,

Theorem 17. *Any representation of a finite group is unitary.*

Combining these two theorems, we will easily have,

Theorem 18. *Every representation of a finite group is completely reducible.*

Before we go to the Haar measure topic, let's say some words about the proof of the Theorem 16 and Theorem 17. In fact, for Theorem 16, we can prove it by induction on dimension, once noticing that the complementary subspace of subrepresentation is also a subrepresentation, then you will easily prove it. For the Theorem 17, what you need to do is just average on the whole finite group and then you will easily get a unitary inner product.

Right now, we can go to the second topic of this section, i.e. Haar measure on compact groups.

Definition 22. *A right Haar measure on a real Lie group G is a Borel measure dg which is invariant under the right action of G on itself.*

More precisely, the right invariance implies $\int f(gh)dg = \int f(g)dg$ for $\forall h \in G$ and integrable function f.

As for the existence of the Haar measure on compact groups, as it is a little complicate, [2] contains a complete proof of it, and there is also a proof for a compact metrizable abelian group in the appendix.

So we just give the statement of the following theorem,

Theorem 19. *Let G be a compact real Lie group. Then is has a canonical Borel measure dg which is both left- and right-invariant and invariant under $g \mapsto g^{-1}$ and which satisfies $\int_G dg = 1$. This measure is called the Haar measure on G, which will be denoted as dg.*

Before we go to our last theorem in this section, let's see an interesting and important example at first.

Example 8. *Let $G = U(n)$ and let f be a smooth function on G such that $f(ghg^{-1}) = f(h)$, then,*

$$\int_{U(n)} f(g)dg = \frac{1}{n!}\int_T f\begin{pmatrix} t_1 & & & \\ & t_2 & & \\ & & \ddots & \\ & & & t_n \end{pmatrix} \Pi_{i<j}|t_i - t_j|^2 dt \tag{14}$$

where

$$T = \{\begin{pmatrix} t_1 & & & \\ & t_2 & & \\ & & \ddots & \\ & & & t_n \end{pmatrix}, t_k = e^{i\phi_k}\} \tag{15}$$

and $dt = \frac{1}{(2\pi)^n} d\phi_1 \dots d\phi_n$ is the Haar measure on T. And this formula called as Weyl Integration Formula. Due to its proof is very complicate thus we omit it here. The reason that this example is interesting is that we have explicitly formula for Haar measure in this special case.

We will use the following theorem to end this section,

Theorem 20 (Representation of Compact Lie Groups)**.** *Any finite-dimensional representation of a compact Lie group is unitary and thus completely reducible.*

The proof of this theorem is almost same with the Theorem **??**, the only thing we need to do is just replace the average procedure by integration of Haar measure and then we done.

5 Root System

5.1 Cartan Subalgebra and Root Decomposition

In this section, we will give a very short introduction of the Cartan subalgebra and root decomposition, which will help us understand the important of the root system. At first, let's see the some definitions about these topics.

Definition 23 (Semisimple Operator)**.** *An operator $A : V \to V$ is called semisimple if any A-invariant subspace has an A-invariant complement: if $W \subset V$. $AW \subset W$, then, there exists $W' \subset V$ such that $V = W' \oplus W$ and $Aw' \subset W'$.*

Definition 24 (Semisimple Element)**.** *An element $x \in \mathfrak{g}$ is called semisimple is adx is a semisimple operator $\mathfrak{g} \to \mathfrak{g}$.*

Definition 25 (Toral Subalgebra)**.** *A subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is called toral if it is commutative and consists of semisimple elements.*

Definition 26 (Cartan Subalgebra)**.** *Let $\mathfrak{g}$ be a complex semisimple Lie algebra. A Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is a toral subalgebra which concindes with its centralizer: $C(\mathfrak{h}) = \{x|[x, \mathfrak{h}] = 0\} = \mathfrak{h}$.*

Right now, we can use the following example to understand these definitions.

Example 9. *Let $\mathfrak{g} = \mathfrak{sl}(n, \mathscr{C})$ and $\mathfrak{h} = \{$diagonal matrices with trace zero$\}$. Then $\mathfrak{h}$ is a Cartan subalgebra. It is no hard to see the commutative and semisimple(by Jordan decomposition), so we know it is a toral subalgebra. By choosing $h \in \mathfrak{h}$ to be a diagonal matrix with distinct eigenvalues, it is easy to see that $[x, h] = 0$ implies x must be diagonal, which proves that $\mathfrak{h}$ is a Cartan subalgebra.*

Here we introduce an interesting result which connect the Cartan subalgebra and toral subalgebra in some sense.

Theorem 21. *Let $\mathfrak{h} \subset \mathfrak{g}$ be a maximal toral subalgebra, i.e. a toral subalgebra which is not properly contained in any other toral subalgebra. Then $\mathfrak{h}$ is a Cartan subalgebra.*

Notice that by this theorem, we also guarantee the existence of the Cartan subalgebra.

For the conjugacy of the Cartan subalgebra, due to the limit of the time, we will not introduce it here but just mention the result, i.e. Cartan subalgebras are all conjugate in $\mathfrak{g}$. By this result, we can easily know that the dimension of the Cartan subalgebra is a fixed number, which we will denote as the rank of the $\mathfrak{g}$. For example the rank of $\mathfrak{sl}(n, \mathscr{C})$ is $n - 1$.

Right now, we have already have enough preparation work for root system and root decomposition.

Theorem 22 (Root Decomposition)**.** *For a complex Lie algebra $\mathfrak{g}$ and a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$.*

- *We have the following decomposition for $\mathfrak{g}$, called the root decomposition,*

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha \tag{16}$$

  where $\mathfrak{g}_\alpha = \{x|[h, x] =< \alpha, h > x \text{for all } h \in \mathfrak{h}\}$ and $R = \{\alpha \in \mathfrak{g}^* - \{0\}|\mathfrak{g}_\alpha \neq 0\}$. The set R is called the root system of $\mathfrak{g}$ and subspaces $\mathfrak{g}_\alpha$ are called the root subspaces.

- $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$, *where we denote* $\mathfrak{g}_0 = \mathfrak{h}$.
- *If* $\alpha + \beta \neq 0$, *then* $\mathfrak{g}_\alpha$, $\mathfrak{g}_\beta$ *are orthogonal with respect to the Killing form* K, *where Killing form means that a bilinear form on* $\mathfrak{g}$ *defined by* $K(x, y) = tr(adx \quad ady)$.
- *For any* α, *the Killing form gives a non-degenerate pairing* $\mathfrak{g}_\alpha \otimes \mathfrak{g}_{-\alpha} \to \mathscr{C}$. *In particular, the restriction of* K *to* $\mathfrak{h}$ *is non-degenerate.*

By the previous theorem, we define what is root system. Right now, we can see some basic properties of the root system.

Theorem 23. *Let* $\mathfrak{g}_1, \ldots, \mathfrak{g}_n$ *are simple Lie algebras and let* $\mathfrak{g} = \oplus \mathfrak{g}_i$,

- *Let* $\mathfrak{h}_i \subset \mathfrak{g}_i$ *be Cartan subalgebras of* $\mathfrak{g}_i$ *and* $R_i \subset \mathfrak{h}_i^*$ *the corresponding root system of* $\mathfrak{g}_i$. *Then,* $\mathfrak{h} = \oplus \mathfrak{h}_i$ *is a Cartan subalgebra in* $\mathfrak{g}$ *and the corresponding root system is* $R = \sqcup R_i$.
- *Each Cartan subalgebra in* $\mathfrak{g}$ *must have the form* $\mathfrak{h} = \oplus \mathfrak{h}_i$ *where* $\mathfrak{h}_i \subset \mathfrak{g}_i$ *is a Cartan subalgebra in* $\mathfrak{g}_i$.

So far, we have enough ingredients to introduce the main theorem of the structure of semisimple Lie algebras. Due to the proof of this theorem requires many lemmas and properties, we omit the proof. The detailed proof is contained in [2].

Theorem 24 (Structure Theorem)**.** *Let* $\mathfrak{g}$ *be a complex semisimple Lie algebra with Cartan subalgebra* $\mathfrak{h}$ *and root decomposition* $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$. *Let* $(,)$ *a non-degenerate symmetric invariant bilinear form on* $\mathfrak{g}$.

- R *spans* $\mathfrak{h}^*$ *as a vector space, and elements* $h_\alpha, \alpha \in R$ *span* $\mathfrak{h}$ *as a vector space, where* h_α *defined as,*
$$h_\alpha = \frac{2H_\alpha}{(\alpha, \alpha)}, H_\alpha \text{ is the corresponding element of } \mathfrak{h} \text{ for } \alpha \in \mathfrak{h}^* \tag{17}$$
- *For each* $\alpha \in R$, *the root subspace* $\mathfrak{g}_\alpha$ *is one-dimensional.*
- *For any two roots* α, β, *the number*
$$< h_\alpha, \beta >= \frac{2(\alpha, \beta)}{\alpha, \alpha} \tag{18}$$
is integer.
- *For* $\alpha \in R$, *define the reflection operator* $s_\alpha : \mathfrak{h}^* \to \mathfrak{h}^*$ *by*
$$s_\alpha(\lambda) = \lambda - < h_\alpha, \lambda > \alpha = \lambda - \frac{2(\alpha, \lambda)}{(\alpha, \alpha)}\alpha. \tag{19}$$
Then for any roots $\alpha, \beta, s_\alpha(\beta)$ *is also a root. In particular, if* $\alpha \in R$, *then* $-\alpha = s_\alpha(\alpha) \in R$.
- *For any root* α, *the only multiples of* α *which are also roots are* $\pm\alpha$.
- *For roots* $\alpha, \beta \neq \pm\alpha$, *the subspace*
$$V = \bigoplus_{k \in \mathscr{Z}} \mathfrak{g}_{\beta + k\alpha} \tag{20}$$
is an irreducible representation of $\mathfrak{sl}(2, \mathscr{C})_\alpha$.
- *If* α, β *are roots such that* $\alpha + \beta$ *is also a root, then* $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$.

5.2 Abstract Root System

Right now, we can go to the topic about the abstract root system, which is a very useful tool.

Definition 27. *An abstract root system is a finite set of elements $R \subset E \setminus \{0\}$, where E is a Euclidean vector space, such that the following properties hold:*

- *R generates E as a vector space.*
- *For any two roots α, β, the number*

$$n_{\alpha,\beta} = \frac{2(\alpha,\beta)}{(\beta,\beta)} \tag{21}$$

  is integer.

- *Let $s_\alpha : E \to E$ be defined by*

$$s_\alpha(\lambda) = \lambda - \frac{2(\alpha,\lambda)}{(\alpha,\alpha)}\alpha. \tag{22}$$

  Then for any roots α, β, $s_\alpha(\beta) \in R$.

The number $r \dim E$ is called the rank of R.

If R satisfies the following property, i.e.

- *If $\alpha, c\alpha$ are both roots, then $c = \pm 1$.*

Then R is called a reduced root system.

Note that the reduced condition in the root system cannot be derived from previous three conditions, i.e. there exists non-reduced root systems.

Example 10 (Non-reduced Root Systems)**.** *Let $R \in \mathscr{R}^n$ be given by,*

$$R = \{\pm e_i, \pm 2e_i | 1 \le i \le n\} \cup \{\pm e_i \pm e_j | 1 \le i, j \le n, i \ne j\} \tag{23}$$

where e_i is the standard basis in $\mathscr{R}^n$. It is easy to see that R is a non-reduced system.

With the language of the abstract root system, we can describe the theorem we mention in the previous section as following,

Theorem 25. *Let $\mathfrak{g}$ be a simple complex Lie algebra, with root decomposition as $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$. Then the set of roots $R \subset \mathfrak{h}^*_{\mathscr{R}} \setminus \{0\}$ is a reduced root system.*

In order to simplify the notation, we denote the *coroot* $\alpha^\vee \in E^*$ as

$$< \alpha^\vee, \lambda >= \frac{2(\alpha,\lambda)}{(\alpha,\alpha)}. \tag{24}$$

By this notation, it is obviously that $s_\alpha(\lambda) = \lambda - < \lambda, \alpha^\vee > \alpha$ and $n_{\alpha\beta} =< \alpha, \beta^\vee >$.

5.3 Weyl Groups

Once we have the abstract root systems, we can consider the isomorphism between the root systems, i.e.

Definition 28 (Root Isomorphism)**.** *Let $R_1 \subset E_1$, $R_2 \subset E_2$ be two root systems. An isomorphism $\varphi : E_1 \to E_2$ such that $\varphi(R_2) = R_2$ and $n_{\varphi(\alpha)\varphi(\beta)} = n_{\alpha\beta}$ for any $\alpha, \beta \in R$.*

Note that according to the definition of the root isomorphism, if φ preserve the inner product, then $n_{\varphi(\alpha)\varphi(\beta)} = n_{\alpha\beta}$ will be obvious, but we still have root isomorphism that not keep inner product. For example, consider the root system R and $c \in \mathscr{R}^+$, then R and $cR = \{c\alpha | \alpha \in R\}$ are isomorphic by $\alpha \mapsto c\alpha$, which does not preserve the inner product.

Right now, we can define the Weyl group as following,

Definition 29. *The Weyl group W of a root system R is the subgroup of $GL(E)$ generated by reflections $s_\alpha \in R$.*

As for the Weyl groups, we have the following interesting properties,

Theorem 26.
- *The Weyl group W is a finite subgroup in the orthogonal group $O(E)$, and the root system R is invariant under the action of W.*
- *For any $w \in W$, $\alpha \in R$, we have $S_{\omega(\alpha)} = \omega s_\alpha \omega^{-1}$.*

In fact, these two properties are very important. The first part describe the general structure of the Weyl group and the second part describes the combined action of the Weyl group and reflections.

We use a very simple example and a useful remark to end this section.

Example 11. *Suppose that e_i is the standard basis of the $\mathscr{R}^n$ with usual inner product. Then, consider the space E and R define as: $E = \{\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathscr{R}^n | \sum_{i=1}^n \lambda_i = 0\}$, $R = \{e_i - e_j | 1 \le i, j \le n, i \ne j\} \subset E$. With these definitions, it is easy to see that R is a reduced root system. Then, for the $\alpha \in R$, $\alpha = e_i - e_j$ for some i, j. As for the s_α, it is just the exchange of the ith coordinate and jth coordinate, which can be denoted as s_{ij},*

$$s_\alpha(\ldots, \lambda_i \ldots, \lambda_j, \ldots) = (\ldots, \lambda_j, \ldots, \lambda_i, \ldots) \tag{25}$$

As a result, the Weyl group of R is generated by s_{ij}.

According to the action of s_{ij}, it is easy to see that the Weyl group of the R is same as the symmetric group S_n, which means that the Weyl group of R is just S_n.

Remark 2. *Not all automorphisms of a root system are given by elements in Weyl group. See for $n > 3$ in the previous example and automorphism $\alpha \mapsto -\alpha$.*

5.4 Rank Two Systems, Positive Roots and Simple Roots

In this section, we will introduce many different tools that will help us to do some classification of the reduced systems. The first step is to do some classification work about rank two systems.

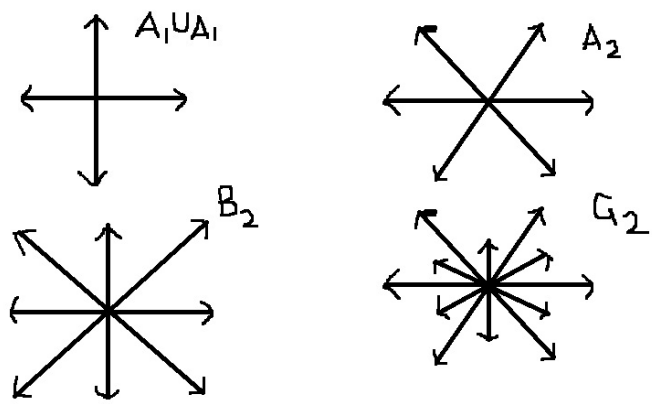
Suppose R is a reduced root system, then with the second and third part of the definition of the root system, we have the following theorem,

Theorem 27. *Let $\alpha, \beta \in R$ be roots which are not multiple of one another, with $|\alpha| \ge |\beta|$, and let φ be the angle between them. Then we must have one of the following possibilities:*

- $\varphi = \frac{\pi}{2}$, $n_{\alpha\beta} = n_{\beta\alpha} = 0$;
- $\varphi = \frac{2\pi}{3}$, $|\alpha| = |\beta|$, $n_{\alpha\beta} = n_{\beta\alpha} = -1$;
- $\varphi = \frac{\pi}{3}$, $|\alpha| = |\beta|$, $n_{\alpha\beta} = n_{\beta\alpha} = 1$;
- $\varphi = \frac{3\pi}{4}$, $|\alpha| = \sqrt{2}|\beta|$, $n_{\alpha\beta} = -2$, $n_{\beta\alpha} = -1$;
- $\varphi = \frac{\pi}{4}$, $|\alpha| = \sqrt{2}|\beta|$, $n_{\alpha\beta} = 2$, $n_{\beta\alpha} = 1$;
- $\varphi = \frac{5\pi}{6}$, $|\alpha| = \sqrt{3}|\beta|$, $n_{\alpha\beta} = -3$, $n_{\beta\alpha} = -1$;
- $\varphi = \frac{\pi}{6}$, $|\alpha| = \sqrt{3}|\beta|$, $n_{\alpha\beta} = 3$, $n_{\beta\alpha} = 1$.

In fact, each of the possibilities listed in previous theorem can be realized as following.

Theorem 28. *Let $A_1 \times A_1, A_2, B_2, G_2$ be the sets of vectors in $\mathscr{R}^2$ as following picture. Then, each of them is a rank two system. What is more, any rank two reduced system is isomorphic to one of previous root systems.*



For $A_1 \cup A_1$, all angles are $\frac{\pi}{2}$, lengths are equal. For A_2, all angles are $\frac{\pi}{3}$, lengths are equal. For B_2, all angels are $\frac{\pi}{4}$, lengths are 1 and $\sqrt{2}$. For G_2, all angles are $\frac{\pi}{6}$, lengths are 1 and $\sqrt{3}$.

Then we can introduce the positive roots and simple roots, which will help us to do the classification of root system from the view of the "generating set".

Definition 30 (Positive Roots and Negative Roots)**.** *Let $t \in E$ be such that for any root $\alpha \in R$, $(t, \alpha) \neq 0$. Then we can split the root system as following,*

$$R = R_+ \bigsqcup R_- \tag{26}$$

where $R_+ = \{\alpha \in R | (\alpha, t) > 0\}$, $R_- = \{\alpha \in R | (\alpha, t) < 0\}$. We will call such decomposition as a polarization of R and the roots of R_+ will be called positive, the roots $\alpha \in R_-$ will be called negative.

Definition 31 (Simple Roots)**.** *A root $\alpha \in R_+$ is called simple if it can not be written as a sum of two positive roots. We will denote the set of simple roots by $\Pi \subset R_+$.*

By the definition of simple roots, it is easy to see that every positive root can be written as a sum of simple roots by induction. And what is more, by contradiction method, we can prove that if $\alpha, \beta \in R_+$ are simple, then $(\alpha, \beta) \leq 0$. Combining these results, we will have following theorem,

Theorem 29. *Let $R = R_+ \bigsqcup R_- \subset E$ be a root system. Then the simple roots form a basis of the vector space E.*

We will use a very straightforward example to end this section.

Example 12. *Let R be the root system of the previous example, then choose the polarization as following,*

$$R_+ = \{e_i - e_j | i < j\} \tag{27}$$

Then, it is easy to see the simple roots are,

$$\alpha_1 = e_1 - e_2, \quad \alpha_2 = e_2 - e_3, \ \ldots \quad \alpha_{n-1} = e_{n-1} - e_n. \tag{28}$$

By the construction as previous, any positive root can be written as a sum of simple roots with non-negative coefficients, for example, $e_1 - e_3 = (e_1 - e_2) + (e_2 - e_3) = \alpha_1 + \alpha_2$.

5.5 Weyl Chambers and Simple Reflections

In the section, we will introduce the Weyl chambers, which will help us understand the structure of the set of simple roots. More precisely, right now, we have already done the following reduced procedure,

$$R \to R_+ \to \Pi = \{\alpha_1, \ldots, \alpha_k\} \tag{29}$$

but the first reduced procedure needs the choosing of the polarization, which depends on the choose of the $t \in E$. Our goal in this section is to understand the influence to simple root system Π of choosing different polarization $t \in E$.

At first, we need to introduce the concepts of the Weyl chambers.

Definition 32. *A Weyl chamber is a connected component of the complement to the hyperplanes,*

$$C = \textit{connected component of } (E \setminus \bigcup_{\alpha \in R} L_\alpha) \tag{30}$$

where $L_\alpha = \{\lambda \in E | (\alpha, \lambda) = 0\}$.

Notice that the polarization is unchanged if we change t as long as it still in Weyl chamber.

In fact, there is another way to specify the Weyl chambers, which is defined as a system of inequalities as following,

$$\pm(\alpha, \lambda) > 0 \tag{31}$$

The solution of any such system is either an empty set or a Weyl chamber follow.

Here we mention a very interesting relation between the Weyl chambers and polarizations as following,

Theorem 30. *There is a bijection between the set of all polarizations of R and the set of Weyl chambers. More precisely, any Weyl chamber C define a polarization by,*

$$R_+ = \{\alpha \in R | (\alpha, t) > 0\}, \quad t \in C. \tag{32}$$

Conversely, given a polarization $R = R_+ \bigsqcup R_-$, we define the corresponding positive Weyl chamber as following,

$$C_+ = \{\lambda \in E | (\lambda, \alpha) > 0 \textit{ for all } \alpha \in R_+\} = \{\lambda \in E | (\lambda, \alpha_i) > 0 \textit{ for all } \alpha_i \in \Pi\}. \tag{33}$$

Right now, we can consider the action of the Weyl group on Weyl chambers,

Theorem 31 (Transitively Action)**.** *The Weyl group acts transitively on the set of Weyl chambers.*

As for the proof of this theorem, we just need to use the relation between reflection and Weyl chambers and then we will done.

So far, we have enough tools to go back to the original goal, to understand the influence to simple root system Π of choosing different polarization $t \in E$. By the previous theorem and the corresponding relation between the Weyl chambers and polarizations, we have,

Corollary 3. *Let $R = R_+ \bigsqcup R_- = R'_+ \bigsqcup R'_-$ be two polarizations of the same root system and Π, Π' the corresponding sets of simple roots. Then there exists an element $w \in W$ such that $\Pi = w(\Pi')$.*

Notice that Weyl group is generated by the reflections. Thus this corollary tells us that sets of simple roots obtained form different polarizations can be related by an orthogonal transformation of E.

Recall that at the beginning of this section, we mention the following relation,

$$R \to R_+ \to \Pi = \{\alpha_1, \ldots, \alpha_k\}. \tag{34}$$

It is natural to think about form a similarly relation on the converse direction, i.e. to recover the root system from its simple roots. Fortunately, we have the following theorem,

Theorem 32. *Let R be a reduced root system, with fixed polarization $R = R_+ \bigsqcup R_-$. Let $\Pi = \{\alpha_1, \ldots, \alpha_r\}$ be the set of simple roots. Consider the reflection corresponding to simple roots $s_i = s_{\alpha_i}$, which called as simple reflection, then*

- *The simple reflections s_i generate W;*
- *$W(\Pi) = R$, i.e. every $\alpha \in R$ can be written in the form $w(\alpha_i)$ for some $w \in W$ and $\alpha_i \in \Pi$.*

By this theorem, we can have the result we want,

Corollary 4. *The root system R can be recovered from the set of simple roots Π.*

We finish all material in this section.

5.6 Dynkin Diagrams and Classification of Root Systems

So far, we have already show that given a reduced root system R, we can choose a polarization by separating the R into R_+ and R_-, and then define the set of simple roots; on the other hand, we have also shown that R can be recovered form simple roots set Π and simple roots that depend on different choices of polarizations can be related by the action of the Weyl group. By all these results, the classifying of the root system is equivalent to classify the possible simple roots Π.

At first, we need to notice that we can construct the large root system from smaller ones, i.e. if $R_1 \subset E_1$, $R_2 \subset E_2$, then define $R = R_1 \bigsqcup R_2$ and $E = E_1 \oplus E_2$, with inner product on E such that $E_1 \perp E_2$. By all these constructions, R is a root system.

By this procedure, it is necessary to define the "basic brick" in simple roots.

Definition 33 (Reducible and Irreducible)**.** *A root system R is called reducible if it can be written in the form $R = R \bigsqcup R_2$, with $R_1 \perp R_2$. Otherwise R is called irreducible.*

Once we have this definition, we can generalize the construct procedure we mentioned before.

Theorem 33. *Let R be a reduced root system with given polarization, and let Π be the set of simple roots.*

- *If R is reducible: $R = R_1 \bigsqcup R_2$, $R_1 \perp R_2$, then $\Pi = \Pi_1 \bigsqcup \Pi_2$, where $\Pi_i = \Pi \cap R_i$ is the set of simple roots for R_i.*
- *Conversely, if $\Pi = \Pi_1 \bigsqcup \Pi_2$ with $\Pi_1 \perp \Pi_2$, then $R = R_1 \bigsqcup R_2$, where R_i is the root system generated by Π_i.*

Noticed that it can be shown that every reducible root system can be uniquely written in the form $R = R_1 \bigsqcup \ldots \bigsqcup R_r$, where R_i are mutually orthogonal irreducible root systems. Thus, once we classify all irreducible root systems, we will classify all root systems.

In order to classify the root system, it is necessary to introduce the Cartan matrix, which will be invariant under the isomorphisms of the root systems.

Definition 34 (Cartan Matrix)**.** *The Cartan matrix A of a set of a simple roots $\Pi \subset R$ is the $r \times r$ matrix with entries,*

$$a_{ij} = n_{\alpha_j \alpha_i} \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}. \tag{35}$$

As for the Cartan matrix, it has following straightforward properties,

Proposition 4.

- *For any i, $a_{ii} = 2$.*
- *For any $i \neq j$, a_{ij} is a non-positive integer: $a_{ij} \in \mathscr{Z}$, $a_{ij} \leq 0$.*
- *For any $i \neq j$, $a_{ij} a_{ji} = 4 \cos^2 \varphi$, where φ is the angle between α_i, α_j.*

Here we give a quick example about the Cartan matrix,

Example 13. *For root system R that describe in the previous example with dimension n, then the Cartan matrix is,*

$$\begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & & \vdots & & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{pmatrix}. \tag{36}$$

Right now, we can define a Dynkin diagram as following, which can be interpret as the graphical way to represent the Cartan matrix.

Definition 35 (Dynkin Diagram)**.** *Let Π be a set of simple roots of a root system R. The Dynkin diagram of Π is the graph constructed as following,*

- *For each simple root α_i, we construct a vertex v_i of the Dynkin diagram;*

- *For each pair of simple roots $\alpha_i \neq \alpha_j$, we connect corresponding vertices by n edges, where n depends on the angel φ between α_i, α_j:*
  For $\varphi = \frac{\pi}{2}$, $n = 0$;
  For $\varphi = \frac{2\pi}{3}$, $n = 1$;
  For $\varphi = \frac{3\pi}{4}$, $n = 2$;
  For $\varphi = \frac{5\pi}{6}$, $n = 3$;

- *Finally, for every pair of distinct simple roots $\alpha_i \neq \alpha_j$, if $|\alpha_i| \neq |\alpha_j|$ and they are not orthogonal, we orient the corresponding edges by putting on it an arrow pointing towards the shorter root.*

Let's see some very simple examples about the Dynkin diagram.

Example 14. *The Dynkin diagram in the previous picture about the rank two system has following Dynkin diagram,*

$$\begin{aligned} &A_1 \cup A_1 : \bigcirc \quad \bigcirc \\ &A_2 : \bigcirc - \bigcirc \\ &B_2 : \bigcirc \Longrightarrow \bigcirc \\ &G_2 : \bigcirc \Rrightarrow \bigcirc \end{aligned} \tag{37}$$

Once we have the definition for the Dynkin diagram, we investigate some properties of the Dynkin diagram as following.

Proposition 5. *Let Π be a set of simple roots of reduced root system R.*

- *The Dynkin diagram of Π is connected if and only if R is irreducible;*

- *The Dynkin diagram determines the Cartan matrix A;*

- *R is determined by the Dynkin diagram uniquely up to an isomorphism: if R, R' are two reduced root systems with the same Dynkin diagram, then they are isomorphic.*

In fact, the proof of this proposition only depends on the definition of the Dynkin diagram and root system, combining all these definitions, we can easily get the proof.

Finally, we can introduce the main theorem of this section, the classification of the Dynkin diagram,

Theorem 34. *Let R be a reduced irreducible root system. Then its Dynkin diagram is isomorphic to one of the diagrams below(Due to the limit of the graph skill, I use the "=" to replace the "-" in the vertical direction of the Dynkin diagram):*

- $A_n (n \geq 1) : \bigcirc - \bigcirc - \cdots - \bigcirc$

- $B_n (n \geq 2) : \bigcirc - \bigcirc - \cdots \Longrightarrow \bigcirc$

- $C_n (n \geq 2) : \bigcirc - \bigcirc - \cdots \Longleftarrow \bigcirc$

- $D_n (n \geq 4) :$

$$\begin{array}{ccccccc} & & & \bigcirc & & & \\ & & & \| & & & \\ \bigcirc & - & \bigcirc - \cdots - & \bigcirc & - & \bigcirc & \end{array}$$

- E_6 :
$$\begin{array}{c} \bigcirc \\ \| \\ \bigcirc-\bigcirc-\bigcirc-\bigcirc-\bigcirc \end{array}$$
- E_7 :
$$\begin{array}{c} \bigcirc \\ \| \\ \bigcirc-\bigcirc-\bigcirc-\bigcirc-\bigcirc-\bigcirc \end{array}$$
- E_8 :
$$\begin{array}{c} \bigcirc \\ \| \\ \bigcirc-\bigcirc-\bigcirc-\bigcirc-\bigcirc-\bigcirc-\bigcirc \end{array}$$
- F_4 : $\bigcirc-\bigcirc \Longrightarrow \bigcirc-\bigcirc$
- G_2 : $\bigcirc \Rrightarrow \bigcirc$

Conversely, each of these diagrams does appear as the Dynkin diagram of a reduced irreducible root system.

Remark 3. *As for the previous theorem, we use some notations that we did not explain, we will explain these notations here.*

A_n	$\mathfrak{sl}(n+1, \mathscr{C})$
B_n	$\mathfrak{so}(2n+1, \mathscr{C})$
C_n	$\mathfrak{sp}(n, \mathscr{C})$
D_n	$\mathfrak{so}(2n, \mathscr{C})$

And E_6, E_7, E_8, F_4, G_2 are some exceptional cases that we will not describe it here.

By this theorem, we accomplish our goal.

6 Acknowledgement

The author would like to thank to Professor Paul Baum from whom he learned Lie groups and Lie theory.

7 Appendix

In this appendix, I will show one proof about the existence of Haar measure on compact metrizable abelian group.

Proof. **7.1 Birkhoff-Kakutani Theorem**

In fact, compact group is the topological group for which the topology is compact. Thus, we can use the Birkhoff-Kakutani theorem, which statement is the following,

Theorem 35 (Birkhoff-Kakutani)**.** *A topological group G is metrizable if and only if it is separated and the neutral element e has a countable fundamental system of neighborhoods. A metrizable topological group admits a left-invariant(or a right-invariant) compatible metric.*

Proof. Before we begin the proof, we construct several lemmas which we will prove after the theorem.

Lemma 1. *If G is a topological group, then the class $\mathscr{V}$ of all neighborhoods of the neutral element e has the following properties:*
(1)$e \in V$ for all $V \in \mathscr{V}$;
(2)if $V, W \in \mathscr{V}$ then $V \cap W \in \mathscr{V}$;
(3)if $V \in \mathscr{V}$ then there exists $W \in \mathscr{V}$ such that $WW \subset V$;
(4)if $V \in \mathscr{V}$ then $V^{-1} \in \mathscr{V}$;
(5)if $V \in \mathscr{V}$ and $a \in G$ then $aVa^{-1} \in \mathscr{V}$;
(6)if $V \in \mathscr{V}$ and $W \supset V$, then $W \in \mathscr{V}$.

Lemma 2. *If G is a topological group and $\mathscr{B}$ is any fundamental system of neighborhoods of the neutral element e, then the following conditions are equivalent:*
(a)G is separated;
(b)$\{e\}$ is a closed subset of G;
(c)$\bigcap_{B \in \mathscr{B}} B = \{e\}$.

Lemma 3. *Let X be a set and suppose $f : X \times X \to R$ is a function satisfying the following conditions:*
(1)$f(x, y) \geq 0$ for all x, y;
(2)$f(x, x) = 0$ for all x;
(3)if $\epsilon > 0$, the relations $f(w, x) \leq \epsilon$, $f(x, y) \leq \epsilon$, $f(y, z) \leq \epsilon$ imply $f(w, z) \leq 2\epsilon$.
Define a function $d : X \times X \to R$ as follows. If $(x, y) \in X \times X$ and $\rho = \{x = x_0, x_1, \ldots, x_n = y\}$ is any finite system of points in X that begins at x and ends at y, write

$$|\rho| = \sum_{k=1}^{n} f(x_{k-1}, x_k) \tag{38}$$

and define

$$d(x, y) = \inf |\rho|, \tag{39}$$

where ρ varies over all such finite systems. Then, d has the following properties:
(4)$\frac{1}{2} f(x, y) \leq d(x, y) \leq f(x, y)$;
(5)$d(x, z) \leq d(x, y) + d(y, z)$;
(6)if $f(x, y) = f(y, x)$ for all, then $d(x, y) = d(y, x)$ for all x, y.
If $f(x, y) = f(y, x)$ for all x, y, and if $f(x, y) > 0$ whenever $x \neq y$, then d is a metric on X.

Then, let's begin the proof.

Only if :

Any metrizable space is separated and first countable.

If :

Let G be a separated topological group possessing a fundamental sequence of neighborhoods $U_n (n = 1, 2, 3, \ldots)$ of e.

The first step is to construct an 'improved' fundament sequence of neighborhoods V_n. Replacing U_n by $U_n \cap U_n^{-1}$, one can assume that the U_n are symmetric($U_n = U_n^{-1}$). Let $V_1 = U_1$. By lemma 1, there exists U_k such that $U_k^3 \subset U_2 \cap V_1$; let V_2 be the first such U_k. Inductively, let V_n be the first U_k such that $U_k^3 \subset U_n \cap V_{n-1}$. Since $V_n \subset U_n$ for all n, the sequence of neighborhoods V_n is also fundamental; also

$$\bigcap_{1}^{\infty} V_k = \{e\} \tag{40}$$

because G is separated and lemma 2.

And by construction

$$V_{k+1}^3 \subset V_k \quad (k = 1, 2, 3, \ldots). \tag{41}$$

Set $V_0 = G$. From the equation (41), we have

$$G = V_0 \supset V_1 \supset V_2 \supset V_3 \supset \ldots; \tag{42}$$

thus, every $x \in G$ belongs to some V_k, and it follows from equation (40) and equation (42) that if $x \neq e$ then V_k excludes x from some k onward, that is, x belongs to only finitely many V_k.(If G admits a finite fundamental system of neighborhoods of e, then G is obviously discrete; in this case the discrete metric, $d(x, x) = 0$ for all x and $d(x, y) = 1$ when $x \neq y$, is a left-invariant compatible metric . Having disposed of this case, let us assume G nondiscrete.)

Each V_k represents a degree of 'nearness' to the 'origin' e; alternately, $x^{-1}y \in V_k$ is a measure of the nearness of x to y. The problem is to express such qualitative statements in terms of real numbers. Left-invariance will then follow from the fact that the germinal relation $x^{-1}y \in V_k$ is itself left-invariant, i.e., $(ax)^{-1}(ay) = x^{-1}y$.

Suppose $x \neq y$. A qualitative assertion is that $x^{-1}y \in V_k$ for some k. A quantitative assertion is that there exists a largest such k; this permits the definition

$$f(x, y) = \min\{(\frac{1}{2})^k : x^{-1}y \in V_k\}. \tag{43}$$

On the other hand, if $x = y$ then $x^{-1}y = e \in V_k$ for all k; setting $f(x, x) = 0$, one has

$$f(x, y) = \inf\{(\frac{1}{2})^k : x^{-1}y \in V_k\} \tag{44}$$

for all $x, y \in G$. The desired metric d will be derived from f via the lemma 3; we now show hat the hypotheses of the lemma are fulfilled.

Obviously $f(x, y) \geq 0$, and $f(x, y) = 0$ iff $x = y$. Also, $f(x, y) = f(y, x)$ since the V_n are symmetric. To apply the lemma 3, we need only verify the condition (3) in lemma 3; the left-invariance of d will follow at once from the evident property $f(ax, ay) = f(x, y)$.

For condition (3) in lemma 3, assuming $\epsilon > 0$, $f(w, x) \leq \epsilon$, $f(x, y) \leq \epsilon$, $f(y, z) \leq \epsilon$, it is to be shown that $f(w, z) \leq 2\epsilon$. This is trivial if $\epsilon \geq \frac{1}{2}$(Because $f \leq 1$); suppose $0 < \epsilon < \frac{1}{2}$. According to equation (44) there exist positive integers i, j, k such that

$$w^{-1}x \in V_i \;\; and \;\; (\frac{1}{2})^i \leq \epsilon, \tag{45}$$

$$x^{-1}y \in V_j \;\; and \;\; (\frac{1}{2})^j \leq \epsilon, \tag{46}$$

$$y^{-1}z \in V_k \;\; and \;\; (\frac{1}{2})^k \leq \epsilon, \tag{47}$$

If $r = \min\{i, j, k\}$, then $(\frac{1}{2})^r \leq \epsilon$ and it follows from equation (41) that

$$w^{-1}z = (w^{-1}x)(x^{-1}y)(y^{-1}z) \in V_iV_jV_k \subset V_r^3 \subset V_{r-1}, \tag{48}$$

therefore $f(w, z) \leq (\frac{1}{2})^{r-1} = 2(\frac{1}{2})^r \leq 2\epsilon$.

The lemma 3 is now applicable, and yields a left-invariant metric d; it remains to show that d generates the given topology.

For any $\epsilon > 0$ and any $a \in G$, define

$$U_\epsilon(a) = \{x : f(a,x) < \epsilon\}. \tag{49}$$

We assert that the sets $U_\epsilon(a) > 0$, form a fundamental system of neighborhoods of a for the given topology. First, every $U_\epsilon(a)$ is a neighborhood of a; for, if k is a positive integer such that $(\frac{1}{2})^k < \epsilon$, then as

$$x \in aV_k \Rightarrow a^{-1}x \in V_k \Rightarrow f(a,x) \leq (\frac{1}{2})^k < \epsilon \tag{50}$$

we have $aV_k \subset U_\epsilon(a)$.

On the other hand, if A is any neighborhood of a, let us show that $U_\epsilon(a) \subset A$ for some $\epsilon > 0$. Let k be a positive integer such that $aV_k \subset A$($a^{-1}A$ is a neighborhood of e, and the V_k are basic), and set $\epsilon = (\frac{1}{2})^k$. If $x \notin aV_k$ then $a^{-1}x$ can belong to V_j only for $j < k$, therefore $f(a,x) > (\frac{1}{2})^k = \epsilon$ and so $x \notin U_\epsilon(a)$; thus $U_\epsilon(a) \subset aV_k \subset A$.

By (4) of the lemma 3, $\frac{1}{2}f(x,y) \leq d(x,y) \leq f(x,y)$; it follows that, for $\epsilon > 0$,

$$f(x,y) < \epsilon \Rightarrow d(x,y) < \epsilon \Rightarrow \frac{1}{2}f(x,y) < \epsilon \tag{51}$$

thus

$$U_\epsilon(x) \subset \{y : d(x,y) < \epsilon\} \subset U_{2\epsilon}(x); \tag{52}$$

since the $U_\epsilon(x)$ are a fundamental system of neighborhoods of x for the given topology, and the open balls $\{y : d(x,y) < \epsilon\}$ are a fundamental system of neighborhoods of x for the topology derived from the metric d, it is immediate from equation (52) that the two topologies coincide.

Finally, if G is metrizable then G is separated and first countable by the 'only if' part of the proof, and therefore G possesses a left-invariant compatible metric by the 'if' part of the proof. □

Then, we can use Birkhoff-Kakutani theorem to prove the problem.

As our condition is a compact metrizable abelian group, then we have a metrizable abelian topological group, by Birkhoff-Kakutani theorem, this group admits a left-invariant compatible metric. By abelian, this left-invariant metric is also right-invariant metric. Thus we find a bi-invariant metric on the given group, which finishes our proof of the problem. □

7.2 Proof of lemma 1

Proof. $(1), (2), (6)$ are general properties of the filter of neighborhoods of a point of a topological space.

(3) The mapping $f(x,y) = xy$ is continuous at (e,e), and $f(e,e) = e$. Given any neighborhood V of e, choose a neighborhood A of (e,e) in $G \times G$ such that $f(A) \subset V$. One can suppose that $A = W \times W$ for some $W \in \mathscr{W}$. Thus, $WW = f(W \times W) \subset V$.

$(4), (5)$ $x \mapsto x^{-1}$ and $x \mapsto axa^{-1}$ are homeomorphisms of G mapping e onto e. □

7.3 Proof of lemma 2

Proof. (a) implies (b):

In a separated space every singleton is closed.

(b) implies (c):

Assuming $x \neq e$ it is to be shown that some B in $\mathscr{B}$ excludes x. Since $\{x\}$ is also closed and $e \notin \{x\}$, there exists a neighborhood V of e such that $V \cap \{x\} = \emptyset$, i.e., $x \notin V$. Choose $B \in \mathscr{B}$ with $B \subset V$.

(c) implies (a):

Assuming $x \neq y$, we seek a neighborhood V of e such that $(xV) \cap (yV) = \emptyset$.

Since $y^{-1}x \neq e$, by (c) there exists $B \in \mathscr{B}$ with $y^{-1}x \notin B$. Choose $C \in \mathscr{B}$ with $CC \subset B$(lemma 1(3)). Then $V = C \cap C^{-1}$ is a neighborhood of e, and if $(xV) \cap (yV) \neq \emptyset$ would imply $y^{-1}x \in VV^{-1} = VV \subset CC \subset B$, a contradiction. □

7.4 Proof of lemma 3

Proof. It is obviously that (6) holds and that the final assertion is immediate from (4), (5), (6); it remains only to verify (4) and (5).

First, we note that for each $\epsilon > 0$, f satisfies the following 'weak triangle inequality': if $f(x, y) \leq \epsilon$ and $f(y, z) \leq \epsilon$ then $f(x, z) \leq 2\epsilon$(by (3)). In particular, if $f(x, y) = 0$ and $f(y, z) = 0$, we can get that $f(x, z) \leq 2\epsilon$ for any $\epsilon > 0$, which means that $f(x, z) = 0$. It follows by induction that if $f(x_0, x_1) = f(x_1, x_2) = f(x_2, x_3) = \ldots = f(x_{n-1}, x_n) = 0$, then $f(x_0, x_n) = 0$; in other words, if $\rho = \{x = x_0, x_1, \ldots, x_n = y\}$ is a system such that $|\rho| = 0$, then $f(x, y) = 0$.

(4)

For the system $\varrho = \{x = x_0, x_1 = y\}$, we have,

$$d(x, y) \leq |\varrho| = f(x, y) \tag{53}$$

which completes one side of the inequality.

To prove that $\frac{1}{2}f(x, y) \leq d(x, y)$, we need to show that $\frac{1}{2}f(x, y) \leq |\rho|$ for every finite system $|\rho| = \{x = x_0, x_1, \ldots, x_n = y\}$. We will prove by induction.

If $n = 1$ then $\rho = \{x = x_0, x_1 = y\}$ and so $|\rho| = f(x, y) \geq \frac{1}{2}f(x, y)$ by condition (1) in the lemma 3. Suppose that $n \geq 2$ and assume that the assertion true for system of length $< n$. We will have the following three cases:

Case1:

$f(x_0, x_1) \geq \frac{1}{2}|\rho|$. Then, with the help of the induction, we have the following inequality,

$$\frac{1}{2}|\rho| = |\rho| - \frac{1}{2}|\rho| \geq |\rho| - f(x_0, x_1) = \sum_{k=2}^{n} f(x_{k-1}, x_k) \geq \frac{1}{2}f(x_1, x_n) \tag{54}$$

In fact, this inequality is $|\rho| \geq f(x_1, x_n)$. It is obviously that $f(x_0, x_1) \leq |\rho|$, therefore $f(x_0, x_n) \leq 2|\rho|$ as f has weak triangle inequality. Thus we prove that $\frac{1}{2}f(x, y) \leq |\rho|$.

Case2:

$f(x_{n-1}, x_n) \geq \frac{1}{2}|\rho|$. It is the same procedure as previous case.

Case3:

$f(x_0, x_1) < \frac{1}{2}|\rho|$ and $f(x_{n-1}, x_n) < \frac{1}{2}|\rho|$. In particular, $|\rho| > 0$ and $n \geq 3$. Let r be the largest integer such that

$$\sum_{k=1}^{r} f(x_{k-1}, x_k) \leq \frac{1}{2}|\rho| \tag{55}$$

Here $r \geq 1$ as $f(x_0, x_1) < \frac{1}{2}|\rho|$. Since $f(x_{n-1}, x_n) < \frac{1}{2}|\rho|$, we have,

$$\sum_{k=1}^{n-1} f(x_{k-1}, x_k) = |\rho| - f(x_{n-1}, x_n) > \frac{1}{2}|\rho| \tag{56}$$

Thus we know that $r < n-1$. As a result $1 \leq r \leq n-2$. By the maximality of the r, $\sum_{k=1}^{r+1} f(x_{k-1}, x_k) > \frac{1}{2}|\rho|$, therefore,

$$\sum_{k=r+2}^{n} f(x_{k-1}, x_k) < \frac{1}{2}|\rho| \tag{57}$$

By induction and inequality (55), we have,

$$\frac{1}{2} f(x_0, x_r) \leq \sum_{k=1}^{r} f(x_{k-1}, x_k) \leq \frac{1}{2}|\rho| \tag{58}$$

Thus,

$$f(x_0, x_r) \leq |\rho| \tag{59}$$

And it is obviously that

$$f(x_r, x_{r+1}) \leq |\rho| \tag{60}$$

Then by induction and inequality (57), we have,

$$\frac{1}{2} f(x_{r+1}, x_n) \leq \sum_{k=r+2}^{n} f(x_{k-1}, x_k) < \frac{1}{2}|\rho| \tag{61}$$

Thus, we have,

$$f(x_{r+1}, x_n) < |\rho| \tag{62}$$

According to condition (3) in lemma 3 and inequalities (59), (60), (62), we know that $f(x_0, x_n) \leq 2|\rho|$, i.e. $\frac{1}{2} f(x, y) \leq |\rho|$.

(5)

Fix $x, y, z \in X$, Given any pair of systems,

$$\rho = \{x = x_0, x_1, \ldots, x_n = y\} \quad \varrho = \{y = y_0, y_1, \ldots, y_m = z\} \tag{63}$$

let ς be the concatenation of the two systems,

$$\varsigma = \{x = x_0, x_1, \ldots, x_n = y = y_0, y_1, \ldots, y_m = z\} \tag{64}$$

Obviously, $d(x, z) \leq |\varsigma| = |\rho| + |\varrho|$, varying ρ and ϱ independently, we have,

$$d(x, z) \leq d(x, y) + d(y, z) \tag{65}$$

□

8 Reference

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