Introduction to Gromov Hyperbolic Space and Hyperbolic Groups

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MATH 530 Final Presentation

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History of the Gromov Hyperbolic Space

The theory of Gromov hyperbolic space, introduced by M. Gromov in the eighties, has been considered in many books and papers.

1. D. Burago, Y. Burago and S. Ivanov, A Course in metric geometry, AMS 2001
2. H. Short, Notes on word hyperbolic groups, 1991
3. B. H. Bowditch, Notes on Gromov's hyperbolicity criterion for path-metric spaces, World Scientific, 1991
4. J. Roe, Lectures on coarse geometry, University Lecture Series 31, AMS 2003
5. .....

Basic Assumptions

- In this presentation, by a space, we mean a metric space.
- The distance between points x and y usually written as $|x - y|$.
- An arc in a space X is a subset homeomorphic to a real interval. Unless otherwise stated, this interval is closed.
- For an arc α, we will write it as $x \curvearrowright y$ if α is an arc with endpoints x and y.
- Occasionally, we consider a singleton $\{x\}$ as an arc $\alpha : x \curvearrowright x$.
- $\bar{B}(a, r) = \{x : |x - a| \leq r\}$, $\bar{B}(A, r) = \{x \in X : d(x, A) \leq r\}$.

Basic Definitions and Properties

Intrinsic Space

A space X is intrinsic if $|x-y| = \inf\{l(\alpha)|\alpha : x \curvearrowright y\}$. We can also call intrinsic spaces as length spaces or path-metric spaces.

h-short

Let $h > 0$, we say that an arc $\alpha : x \curvearrowright y$ is h-short if $l(\alpha) \leq |x-y| + h$. Every subarc of an h-short are is h-short.

Hausdorff Distance

The Hausdorff distance between two nonempty sets A and A' is defined as following,

$$d_H(A, A') = \inf\{r : A' \subset \bar{B}(A, r), A \subset \bar{B}(A', r)\}. \tag{1}$$

Basic Definitions and Properties

Gromov Product

For $x, y, p \in X$, we define the Gromov product $(x|y)_p$ by,

$$(x|y)_p = \frac{1}{2}(|x-p| + |y-p| - |x-y|). \tag{2}$$

Gromov Product Properties

- $(x|y)_p = (y|x)_p$, $(x|y)_y = (x|y)_x = 0$.
- $|x-y| = (x|z)_y + (y|z)_x$.
- $0 \leq (x|y)_p \leq |x-p| \wedge |y-p|$.
- $|(x|y)_p - (x|y)_q| \leq |p-q|$, $|(x|y)_p - (x|z)_p| \leq |y-z|$.
- If $\alpha : p \curvearrowright y$ is h-short and $x \in \alpha$, then, $|x-p| - \frac{h}{2} \leq (x|y)_p \leq |x-p|$.

First Definition of Gromov Hyperbolic Space

δ-Hyperbolic Space and Gromov Hyperbolic Space

Let $\delta > 0$. A space X is Gromov δ-hyperbolic if

$$(x|z)_p \geq (x|y)_p \wedge (y|z)_p - \delta \tag{3}$$

for all $x, y, z, p \in X$. A space is Gromov hyperbolic if it is δ-hyperbolic for some $\delta \leq 0$.

- Every bounded geodesic metric space is hyperbolic. If $d(x, y) \leq B$ for all x, y. In this case, we can just take $\delta = B$, then everything is fine.
- Every tree is a hyperbolic metric space. It is clearly geodesic, since any two points are connected by a shortest path.

Second Definition of Gromov Hyperbolic Space

Slim Triangles

Let $\delta \leq 0$, A triangle $\triangle$ in a space X is δ-slim if each side τ of $\triangle$ is contained in $\bar{B}(|\triangle|\backslash\tau, \delta)$.

Let $\mathscr{A}$ be a family of arcs in X such that,

- If $\alpha \in \mathscr{A}$, then every subarc of α is in $\mathscr{A}$.
- For each $x, y \in X$, $x \neq y$, and $h > 0$, there is an h-short member $\alpha : x \curvearrowright y$ of $\mathscr{A}$.

Rips condition

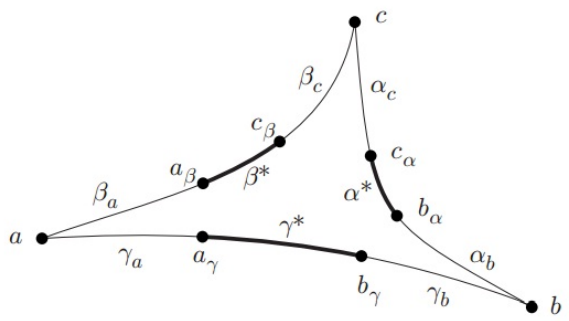
We say that X is a $(\delta, h, \mathscr{A})$-Rips space if every h-short triangle in X with sides in $\mathscr{A}$ is δ-slim. In the case where $\mathscr{A}$ is the family of all arcs in X, we simply say that X is (δ, h)-Rips.

Second Definition of Gromov Hyperbolic Space

In fact, for intrinsic spaces, the $(\delta, h, \mathscr{A})$-Rips condition is quantitatively equivalent to δ-hyperbolicity. More precisely, we have following theorem,

Equivalence Theorem

If X is $(\delta, h, \mathscr{A})$-Rips, then X is δ'-hyperbolic with $\delta' = 3\delta + \frac{3h}{2}$.
If X is δ-hyperbolic, then X is $(\delta', h, \mathscr{A})$-Rips with $\delta' = 3\delta + \frac{3h}{2}$ for each $h > 0$ and for each $\mathscr{A}$.



Examples

Right now, we can use this new definition of the hyperbolic space to see much more examples.

- For previous first example, automatically any side of a triangle is contained in the B-neighborhood of the union of the other two sides.
- For previous second example, any side of a triangle is contained in the union of the other two sides.
- The hyperbolic plane H^2 is a hyperbolic metric space, by the thin triangles and Gauss-Bonnet formula.
- More generally, hyperbolic n-space H^n is a hyperbolic metric space by the thin triangles property for H^2 (since any geodesic triangle is contained in a plane).
- Euclidean space E^n is not a hyperbolic metric space for $n \geq 2$, since E^2 does not satisfy the thin triangles property.

Terminology

(λ, μ)-Quasi-Isometry

Let $\lambda \geq 1$, $\mu \geq 0$. We say a map $f : X \to Y$ between metric spaces is a (λ, μ)-quasi-isometry if,

$$\lambda^{-1}|x - y| - \mu \leq |fx - fy| \leq \lambda|x - y| + \mu \tag{4}$$

for all $x, y \in X$. The map f need not be continuous, In the case where $f : I \to Y$ is a map of a real interval I, we say that such a map is (λ, μ)-quasi-isometry path.

λ-Bilipschitz Condition

For $\mu = 0$, the previous case reduces to the λ-bilipschitz condition,

$$\lambda^{-1}|x - y| \leq |fx - fy| \leq \lambda|x - y|. \tag{5}$$

Examples of Quasi-Isometric Spaces

- If S and T are finite generating sets for a group G, then (G, d_S) and (G, d_T) are quasi-isometric. Indeed, let λ be the maximum length of any element of S expressed as a word in T or vice versa. Then the identity map $G \to G$ is a $(\lambda, 0)$-quasi-isometry from (G, d_S) to (G, d_T) and vice versa. Hence, we can omit mention of the particular generating set, and make statements like 'G is quasi-isometric to H' without ambiguity.
- $(\mathscr{Z}, d)$ and $(\mathscr{R}, d)$ are quasi-isometric, where d is the usual metric: $d(x; y) = |x - y|$. The natural embedding $\mathscr{Z} \to \mathscr{R}$ is an isometry, so a $(1, 0)$-quasi-isometry. It is not surjective, but each point of $\mathscr{R}$ is at most $\frac{1}{2}$ away from $\mathscr{Z}$. We can define a $(1, \frac{1}{2})$-quasi-isometry $f : \mathscr{R} \to \mathscr{Z}$ by $f(x) = x$ 'rounded to the nearest integer'.

Examples of Quasi-Isometric Spaces

- We can generalize the above example. Let G be a group with a finite generating set S, and let $\Gamma = \Gamma(G, S)$ be the corresponding Cayley graph. We can regard Γ as a topological space in the usual way, and indeed we can make it into a metric space by identifying each edge with a unit interval $[0, 1] \subset R$ and defining $d(x, y)$ to be the length of the shortest path joining x to y. This coincides with the path-length metric d_S when x and y are vertices. Since every point of Γ is in the $\frac{1}{2}$-neighborhood of some vertex,we see that (G, d_S) and $(\Gamma(G; S), d)$ are quasi-isometric for this choice of d.
- This example is extremely important in hyperbolic group.

Stability Theorem and Inverse Theorem

Stability Theorem

Suppose that X is an intrinsic δ-hyperbolic space and that $\phi : [a, b] \to X$ and $\phi' : [a, b] \to X$ are (λ, μ)-quasi-isometric paths with $\phi(a) = \phi'(a)$ and $\phi(b) = \phi'(b)$. Then, $d_H(im\phi, im\phi') \leq M(\delta, \lambda, \mu)$.(Quasi-Geodesically Property)

Inverse Theorem

Let $h > 0$, $\delta > 0$, Suppose that X is an intrinsic space such that $\tau \subset \bar{B}(\alpha, \delta)$ whenever τ and α are arcs in X with common endpoints such that α is h-short and $l(\alpha[u, v]) \leq 3|u - v| + 4h$ for all $u, v \in \tau$. Then, X is (δ, h)-Rips.

Remark: By this inverse theorem, we have the third version definition for Gromov hyperbolic space.

Third Definition and Preserving Hyperbolicity Map

Third Definition

A length space (X, d) is said to be Gromov hyperbolic if it is quasi-geodesically stable.

μ-Roughly Surjective

A map $f : X \to Y$ is μ-roughly surjective if for each $y \in Y$, there is $x \in X$ with $|fx - y| \leq \mu$.

Preserving Hyperbolicity Map

Suppose that X and Y are intrinsic metric spaces and that $f : X \to Y$ is a μ-roughly surjective (λ, μ)-quasi-isometry. If X is δ-hyperbolic, then Y is δ'-hyperbolic with $\delta' = \delta'(\delta, \lambda, \mu)$.

First Definition

Hyperbolic Groups

A group Γ is said to be hyperbolic (with respect to some finite generating set S) if the Cayley graph $C(\Gamma)$ is a Gromov hyperbolic metric space.

- With this definition and previous examples, we can easily see that this property independent from the generating set.
- A trivial example hyperbolic group is $\mathscr{Z}$, which Cayley graph is an infinite path.

Examples of Hyperbolic Group

- Every finite group is hyperbolic, because its Cayley graphs are all bounded.
- Every free group is hyperbolic, because it has Cayley graphs that are trees. Moreover, if G has a free subgroup of finite index, then G is quasi-isometric to a free group, and hence hyperbolic.
- $Z \times Z$ is quasi-isometric to E^2, and hence is not hyperbolic.

Fundamental Group

Let M be a compact Riemannian manifold with sectional curvature bounded above by a negative number. Then the fundamental group $\pi_1(M)$ is hyperbolic.

Tree Version Definition

Subcone

Let $K = \{p_1, p_2, \ldots, p_n\}$ be a finite metric space. K is said to be a finite subcone at infinity for X if there is a sequence $N_i \to \infty$ and a sequence of n-tuples of points $q_1^i, q_2^i, \ldots, q_n^i \in X$ with,

$$N_i \frac{d(p_l^i, p_m^i)}{d(q_l^i, q_m^i)} \to 1, \quad l, m \in \{1, 2, \ldots, n\}, \quad i \to \infty \tag{6}$$

A metric space Y is said to be a subcone at infinity for X if every finite subset of Y is a finite subcone at infinity for X.

Second Definition For Hyperbolic Group

A finitely-generated group Γ is said to be hyperbolic if every subcone at infinity for $C(\Gamma)$ is a topological tree.

Linear Isoperimetric Inequality

Simple Modifications

Suppose that Γ be a group with a symmetrical generating set G and relations R_i, $i = 1, 2, \ldots, l$. By a simple modification of a word ω, we mean

- Inserting one of the words R_i anywhere in ω;
- Crossing out a subword of ω identical to one of the R_i;
- Crossing out a generator and its inverse if they appear next to each other in ω.

Linear Isoperimetric Inequality

Γ is said to satisfy a linear isoperimetric inequality if there is a constant C such that ever word $\omega = \omega_1\omega_2 \ldots \omega_n$, whose value is the identity can be transformed into an empty word in at most Cn simple modifications.

Third Definition of Hyperbolic Group

Third Definition

A finitely-presented group is hyperbolic if it satisfies a linear isoperimetric inequality.

In his 1987 monograph "Hyperbolic groups", Gromov proved that a finitely presented group is word-hyperbolic if and only if it satisfies a linear isoperimetric inequality.

Remark

Recall that contracting a closed curve to a point sweeps a topological disc. Now if γ is closed (that is, the value of ω is the identity), the number of simple modifications required to transform ω into an empty word is the same as the combinatorial area (the number of 2-cells) in a topological disc bounded by γ.

Reference

- Burago, Dmitri, Burago, Yuri, and Sergei Ivanov. A course in metric geometry. Vol. 33. Providence: American Mathematical Society, 2001.
- Vaisala, Jussi. "Gromov hyperbolic spaces." Expositiones Mathematicae 23.3 (2005): 187-231.
- Howie, James. "Hyperbolic groups lecture notes." Available electronically at http://citeseerx. ist. psu. edu/viewdoc/summary (1999).