Some Fine Combinatorics

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October 25, 2009

2009 Fall Eastern Section Meeting of the AMS
Penn State University

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Introduction

In *Basic Hypergeometric Series and Applications*, Fine studied the series

$$\begin{aligned} F(a,b;t:q) &= \sum_{n=0}^{\infty} \frac{(aq)_n t^n}{(bq)_n} \\ &= 1 + \frac{1-aq}{1-bq}t + \frac{(1-aq)(1-aq^2)}{(1-bq)(1-bq^2)}t^2 + \cdots \end{aligned}$$

where $(z)_n = (1-z)(1-zq)\cdots(1-zq^{n-1})$.

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where $(z)_n = (1-z)(1-zq)\cdots(1-zq^{n-1})$.

Some results from Chapter 1:

- Functional equations satisfied by $F(a,b;t:q)$
- Rogers-Fine Identity
- Symmetry result for $(1-t)F(a,b;t:q)$
- Specializations

Rogers-Fine Identity

Theorem

$$\sum_{n=0}^{\infty} \frac{(aq)_n t^n}{(bq)_n} = \sum_{n=0}^{\infty} \frac{(aq)_n (atq/b)_n b^n t^n q^{n^2} (1 - atq^{2n+1})}{(t)_{n+1} (bq)_n}$$

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We begin by making the substitutions

$$a \to -\frac{b}{aq} \qquad b \to c \qquad t \to a$$

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We begin by making the substitutions

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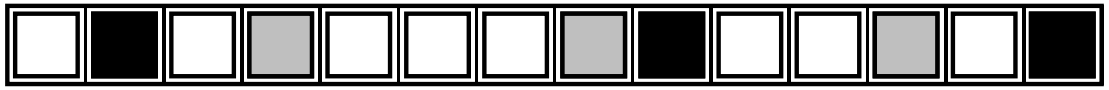
resulting in

$$\sum_{n=0}^{\infty} \frac{(-b/a)_n a^n}{(cq)_n} = \sum_{n=0}^{\infty} \frac{(-b/a)_n (-b/c)_n a^n c^n q^{n^2} (1 + bq^{2n})}{(a)_{n+1} (cq)_n}$$

Weighted Tilings

A tiling is a (finite) sequence of squares:



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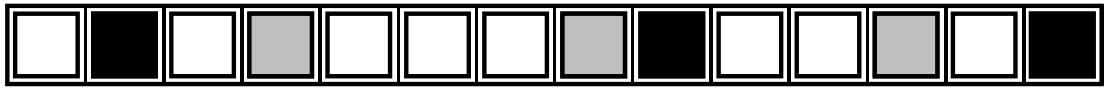

The weight of each tile is given by

$$w(t) = \begin{cases} aq^i & \text{if } t \text{ is a } \blacksquare \text{ with } i \text{ } \blacksquare \text{ or } \square \text{ to its left} \\ bq^i & \text{if } t \text{ is a } \blacksquare \text{ with } i \text{ } \blacksquare \text{ or } \square \text{ to its left} \\ c & \text{if } t \text{ is a } \square. \end{cases}$$

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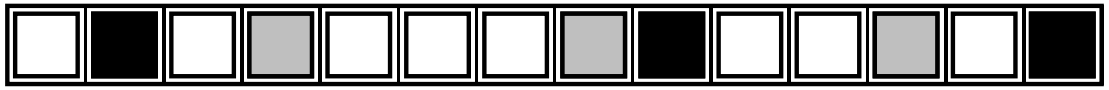

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The weight of a tiling T is given by

$$w_T(a, b, c; q) = \prod_{t \in T} w(t).$$

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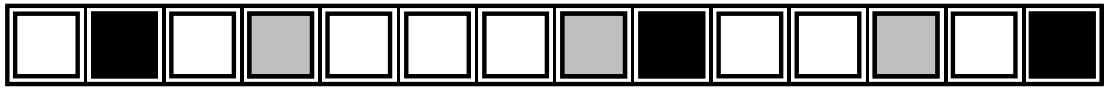

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$$\begin{aligned} w_T(a,b,c;q) &= c \cdot aq \cdot c \cdot bq^2 \cdot c \cdot c \cdot c \cdot bq^6 \cdot aq^7 \cdot c \cdot c \cdot bq^9 \cdot c \cdot aq^{11} \\ &= a^3 b^3 c^8 q^{36}. \end{aligned}$$

Proof of Rogers-Fine, Preliminaries

- Only considering tilings that **do not end** with $\square$.
- Tilings can consist of any finite number of tiles
- $G(a,b,c;q) = \sum_T w_T(a,b,c;q)$

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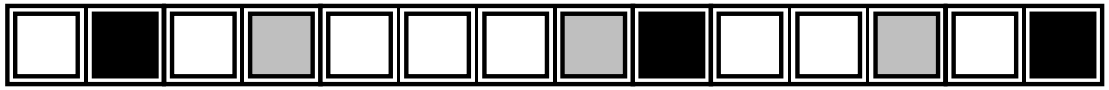
We will show that

$$\sum_{n=0}^{\infty} \frac{(-b/a)_n a^n}{(cq)_n} = G(a,b,c;q) = \sum_{n=0}^{\infty} \frac{(-b/a)_n(-b/c)_n a^n c^n q^{n^2}(1+bq^{2n})}{(a)_{n+1}(cq)_n}$$

Proof Of Rogers-Fine, Part I

$$\sum_{n=0}^{\infty} \frac{(-b/a)_n a^n}{(cq)_n} = G(a, b, c; q)$$

Consider all tilings with exactly n black or gray squares.



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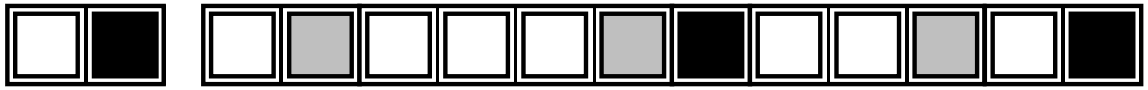

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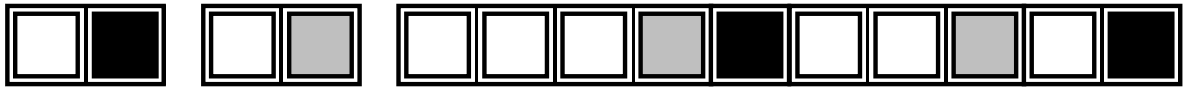

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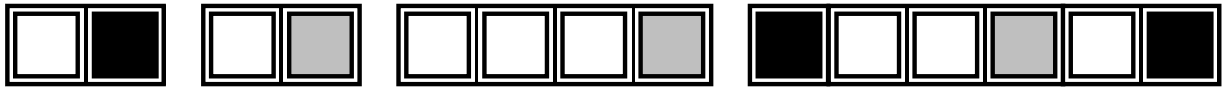

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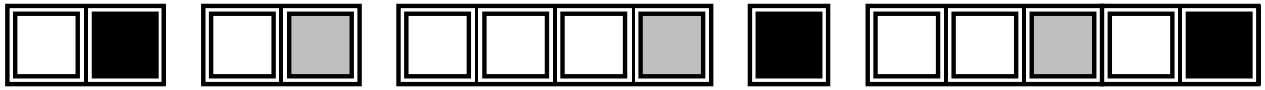

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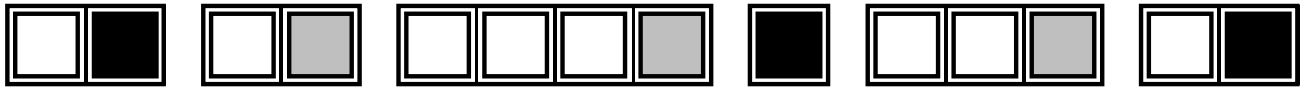

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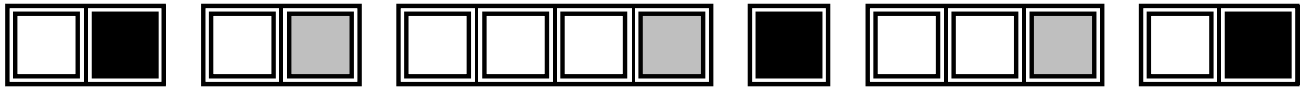

Weight of the ith segment of tiles:

$$(a + bq^{n-i}) \sum_{j=0}^{\infty} c^j q^{j(n+1-i)} = \frac{a + bq^{n-i}}{1 - cq^{n+1-i}}.$$

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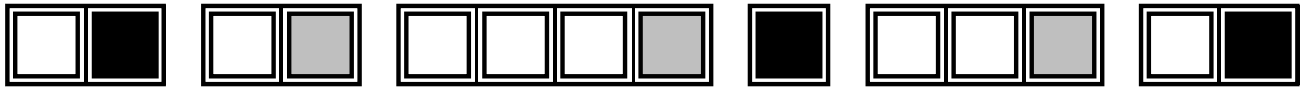

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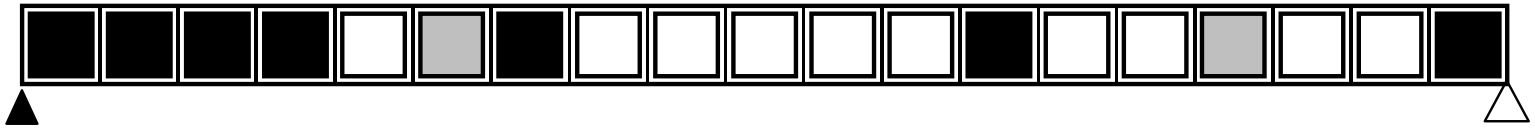
Generating function for tilings with exactly n black or gray squares is

$$\prod_{i=1}^{n} \frac{a + bq^{n-i}}{1 - cq^{n+1-i}} = \frac{(-b/a)_n a^n}{(cq)_n}.$$

Weighted Center

The *weighted center* of a tiling is the place on the board where the number of **gray or white squares strictly to its left** is the same as the number of **black or gray squares strictly to its right**.



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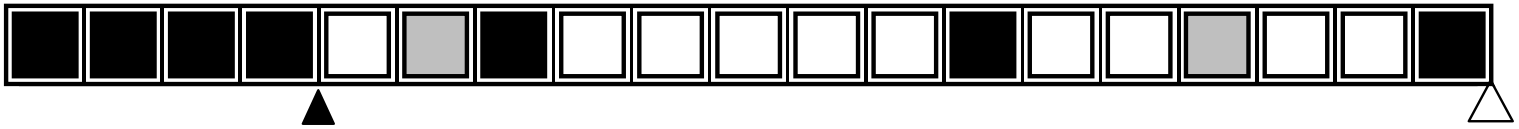

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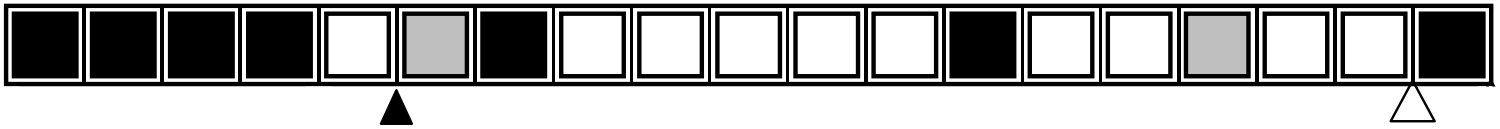

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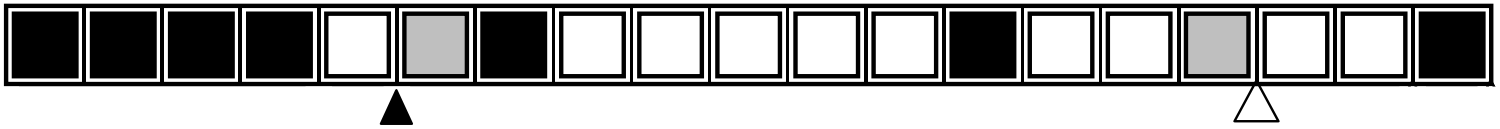

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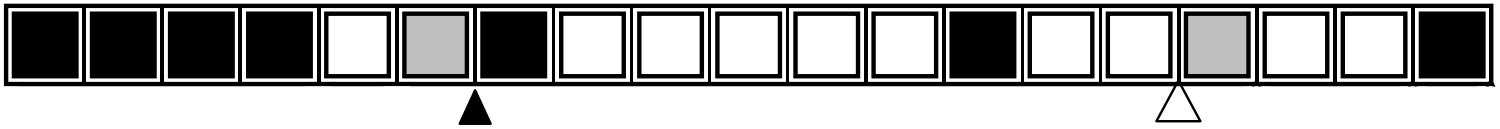

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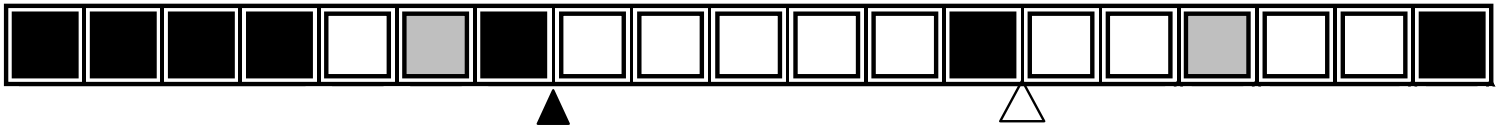

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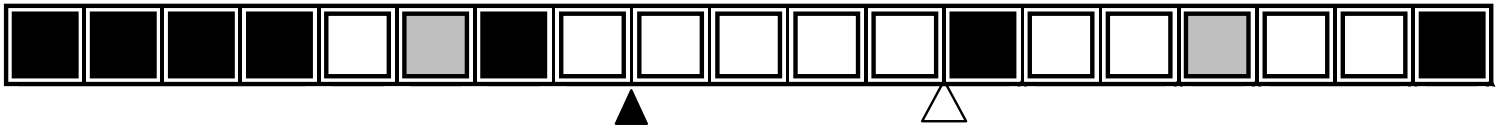

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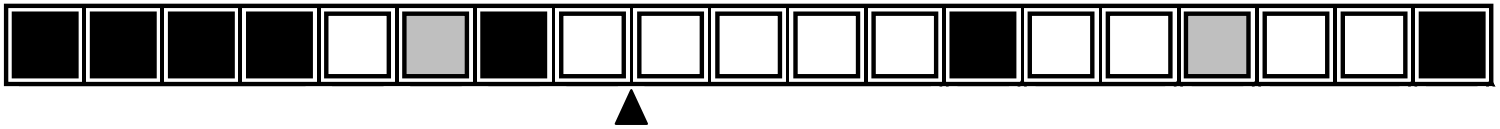

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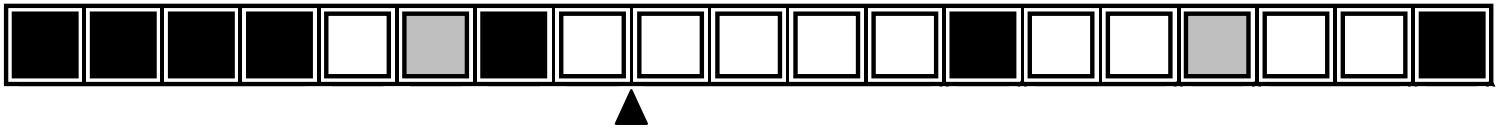

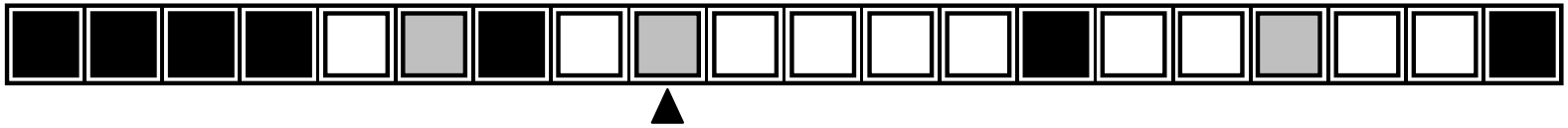

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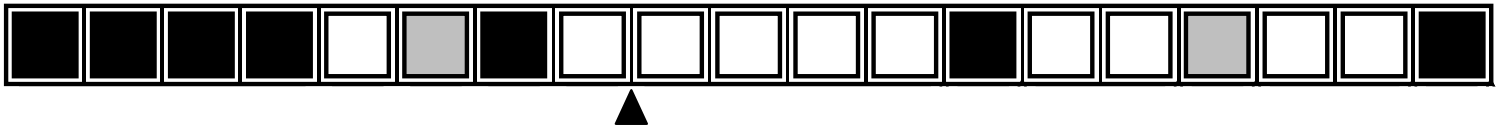

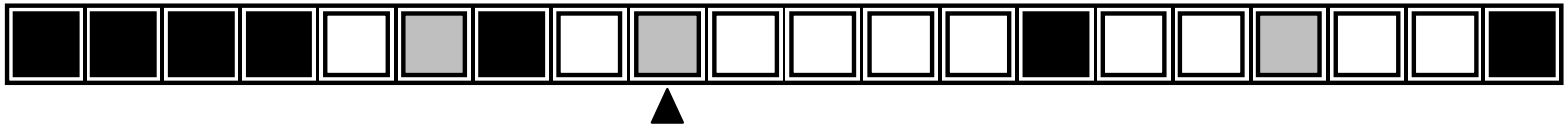

The *degree* of a tiling is the number of gray or white squares strictly to the left of its weighted center.

Proof Of Rogers-Fine, Part II

$$G(a,b,c;q) = \sum_{n=0}^{\infty} \frac{(-b/a)_n(-b/c)_n a^n c^n q^{n^2}(1+bq^{2n})}{(a)_{n+1}(cq)_n}$$

Consider all tilings of degree n.

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Consider all tilings of degree n.
Place n gray and/or white squares in positions 1 through n.

$$\prod_{j=1}^{n}(c+bq^{j-1}) = (-b/c)_n c^n$$

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Place n black and/or gray squares in positions $n+1$ through $2n$.

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Weighted center may or may not coincide with a gray square.

$$(1+bq^{2n})$$

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Insert black squares *before* weighted center

$$\frac{1}{1-a} \cdot \frac{1}{1-aq} \cdots \frac{1}{1-aq^n} = \frac{1}{(a)_{n+1}}$$

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Insert white squares *after* weighted center

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Generating function for all tilings of degree n is

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Functional Equations

Each of the following functional equations follow from the observation that

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$$G(a, b, c; q) = \frac{1-c}{1-a} + \frac{b+c}{1-a} G(aq, bq, c; q)$$

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$$G(a, b, c; q) = \frac{1-c}{1-a} + \frac{b+c}{1-a} G(aq, bq, c; q)$$

$$G(a, b, c; q) = \frac{1+b}{1-a} + \frac{(a+b)(b+c)q}{(1-a)(1-cq)} G(aq, bq^2, cq; q)$$

Specialization: $c = 1$

q-analog of binomial series

$$\sum_{n=0}^{\infty} \frac{(-b/a)_n a^n}{(q)_n} = \prod_{i=0}^{\infty} \frac{1 + bq^i}{1 - aq^i}$$

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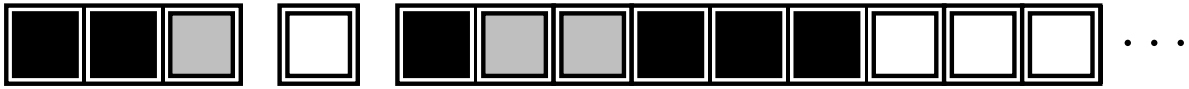

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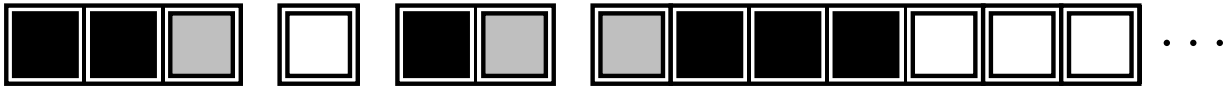

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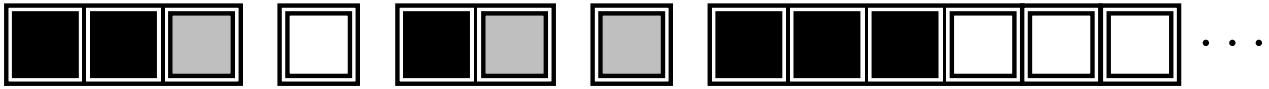

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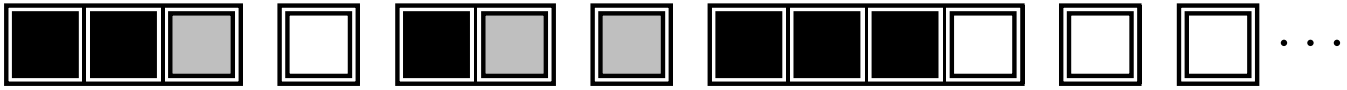

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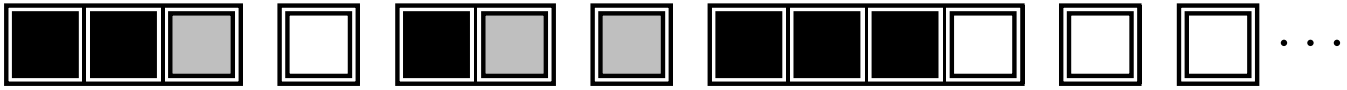

Weight of the ith segment of tiles:

$$(1+bq^i)\sum_{j=0}^{\infty} a^j q^{ij} = \frac{1+bq^i}{1-aq^i}$$

Symmetry

The series $(1-a)G(a,b,c;q)$ is symmetric in the variables a and c.

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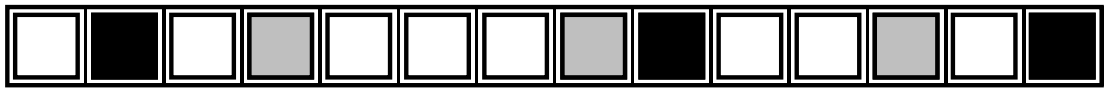

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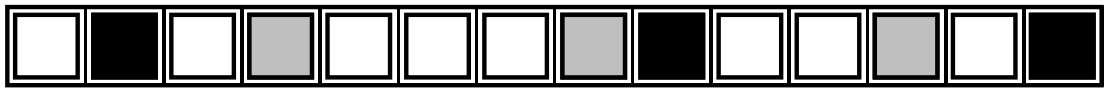

Reverse order of tiles, convert black squares into white squares and vice versa.

Other Results Using Similar Techniques

- Numerous classical partition identities
- Lebesgue identities
- Rogers-Fine
- q-binomial series
- Eight (plus five) q-series identities of Rogers
- q-series symmetry results
- Future results?

q-analog of Gauss's Theorem

Weight tiles in the following manner:

$$w(t) = \begin{cases} aq^i & \text{if } t \text{ is a } \blacksquare \text{ in position } i \\ cq^i & \text{if } t \text{ is a } \bullet \text{ in position } i \\ abq^i & \text{if } t \text{ is a } \blacksquare \text{ with } i \ \square \text{ or } \bigcirc \text{ to its left} \\ bcq^i & \text{if } t \text{ is a } \bullet \text{ with } i \ \square \text{ or } \bigcirc \text{ to its left} \\ -cq^i & \text{if } t \text{ is a } \bigcirc \text{ in position } i \\ 1 & \text{if } t \text{ is a } \square \end{cases}$$

Consider tilings where each circle is followed by a white tile.

Theorem

$$(cq)_\infty \sum_{n=0}^{\infty} \frac{(-c/a)_n(-q/b)_n a^n b^n}{(q)_n (cq)_n} = \prod_{n=1}^{\infty} \frac{(1+bcq^{n-1})(1+aq^n)}{(1-abq^{n-1})}$$

q-analog of Kummer's Theorem

Weight tiles in the following manner:

$$w(t) = \begin{cases} q^i & \text{if } t \text{ is a } \blacksquare \text{ in position } i \\ cq^i & \text{if } t \text{ is a } \bullet \text{ in position } i \\ bq^i & \text{if } t \text{ is a } \blacksquare \text{ with } i \ \square \text{ or } \bigcirc \text{ to its left} \\ bcq^i & \text{if } t \text{ is a } \bigcirc \text{ with } i \ \square \text{ or } \bigcirc \text{ to its left} \\ -bcq^i & \text{if } t \text{ is a } \bigcirc \text{ in position } i+1 \\ 1 & \text{if } t \text{ is a } \square \end{cases}$$

Consider tilings where each circle is followed by a white tile.

Theorem

$$(bc)_\infty \sum_{n=0}^{\infty} \frac{(-c)_n(-q/b)_n b^n}{(q)_n(bc)_n} = \prod_{n=1}^{\infty} \frac{(1+q^n)(1+cq^{2n-1})(1+cb^2q^{2n-2})}{1-bq^{n-1}}$$