A New Combinatorial Approach to q–Series Identities

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Joint work with James Sellers

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Weighted Tilings

Definition

A tiling is a covering of an infinitely long board:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

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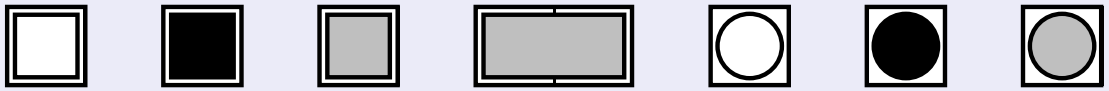

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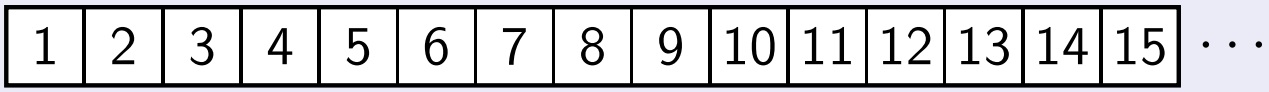

using different types of tiles:

The weight of a tiling T is given by

$$w(T) = \prod_{t \in T} w(t)$$

where $w(t)$ is the weight of the tile t. The weight of a white square will always be 1. Each tiling will have a finite number of weighted tiles.

Lebesgue Identites

The weight of tile t:

$$w(t) = \begin{cases} q^i & \text{if } t \text{ is a } \blacksquare \text{ in position } i \\ zq^i & \text{if } t \text{ is a } \text{▭} \text{ in position } i \\ 1 & \text{if } t \text{ is a } \square \text{ in position } i \end{cases}$$

Theorem

$$\sum_{n\geq 0} \frac{(-z;q)_n}{(q;q)_n} q^{\binom{n+1}{2}} = \prod_{n\geq 1} (1+q^n)(1+zq^{2n-1})$$

where $(z;q)_n = (1-z)(1-zq)\cdots(1-zq^{n-1})$.

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Proof. PART I: Interpret infinite series

STEP 1: Place n black squares in positions $1, 2, 3, \ldots, n$.

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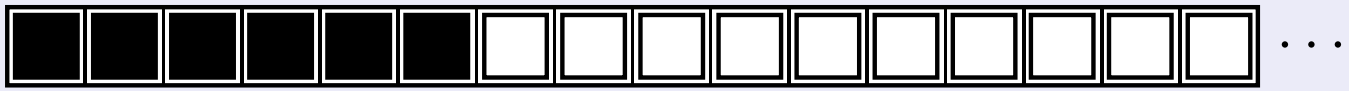

This accounts for a weight of

$$q^{1+2+3+\cdots+n} = q^{\binom{n+1}{2}}.$$

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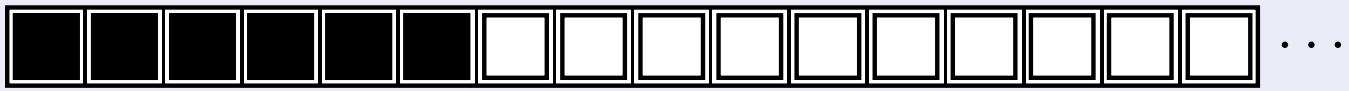

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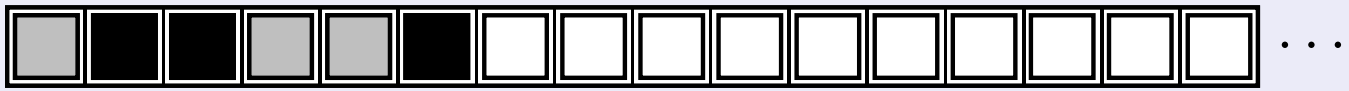

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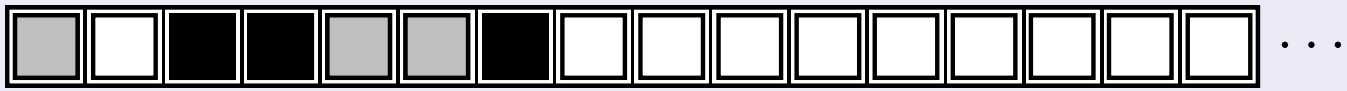

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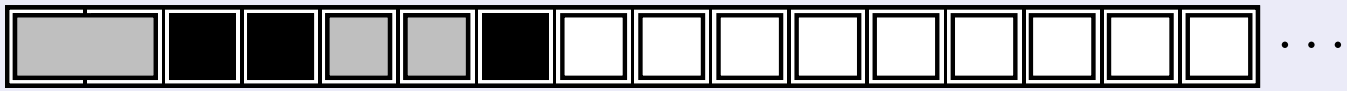

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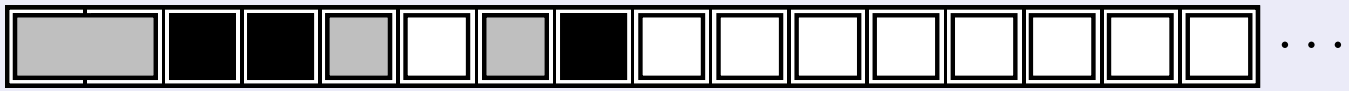

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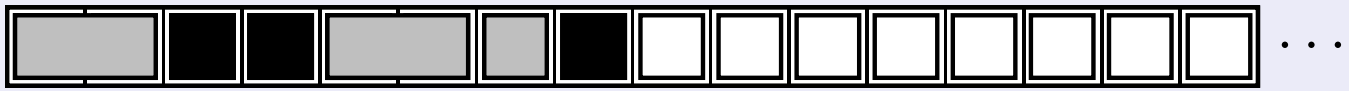

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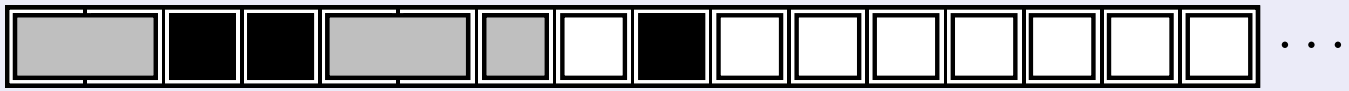

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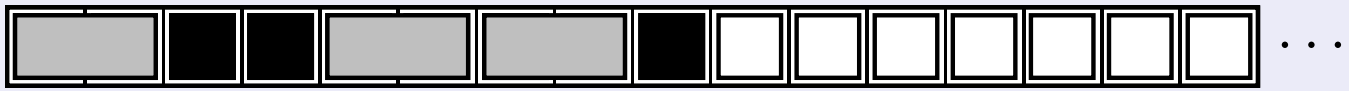

The choice of converting the ith black square into a domino is represented by the factor

$$(1+zq^{n-i}).$$

This accounts for

$$\prod_{i=1}^{n} (1+zq^{n-i}) = (-z;q)_n$$

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$$\sum_{n\geq 0} \frac{(-z;q)_n}{(q;q)_n} q^{\binom{n+1}{2}} = \prod_{n\geq 1} (1+q^n)(1+zq^{2n-1})$$

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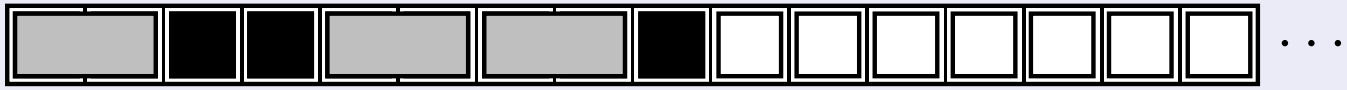

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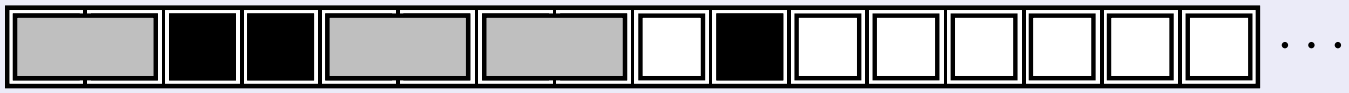

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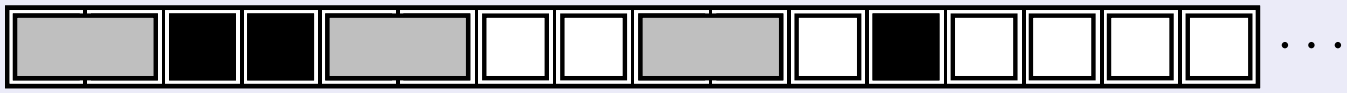

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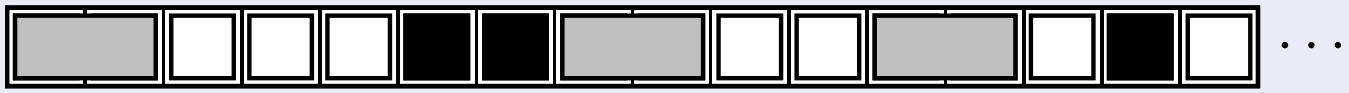

This process accounts for a weight of

$$\frac{1}{(1-q)(1-q^2)(1-q^3)\cdots(1-q^n)} = \frac{1}{(q;q)_n}$$

Thus, the infinite series is the generating function for all weighted tilings.

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STEP 1: Arbitrarily place ■.

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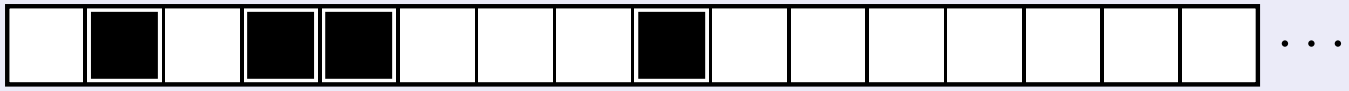

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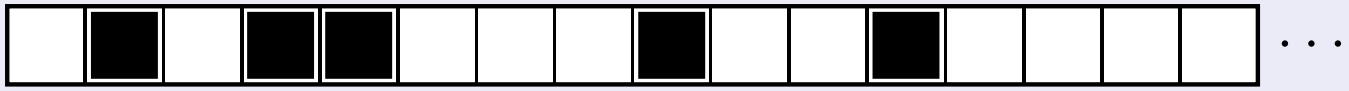

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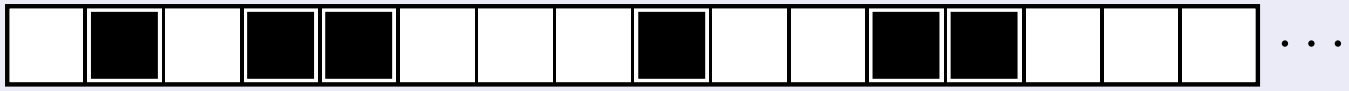

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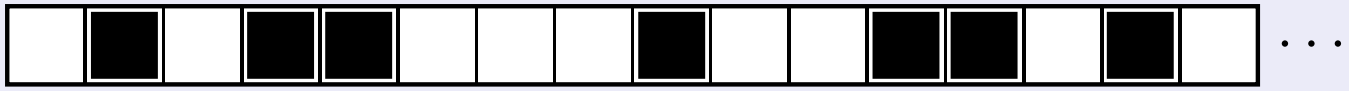

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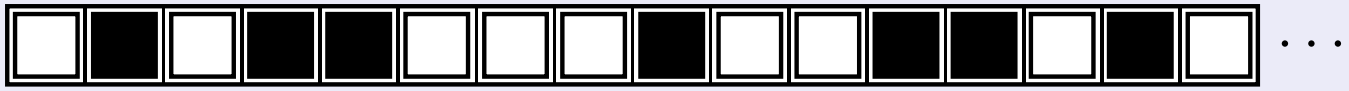

This accounts for a weight of

$$\prod_{n\geq 1} (1+q^n)$$

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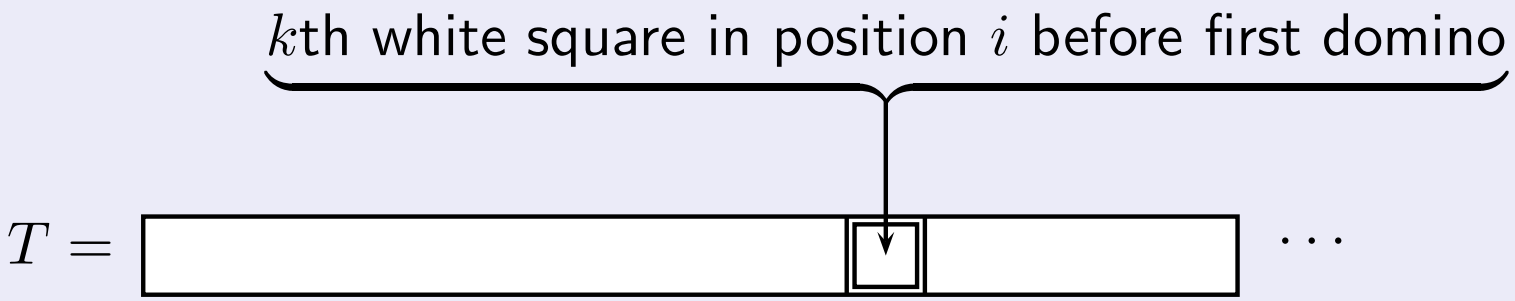
STEP 2: Insert ▭ at kth □.



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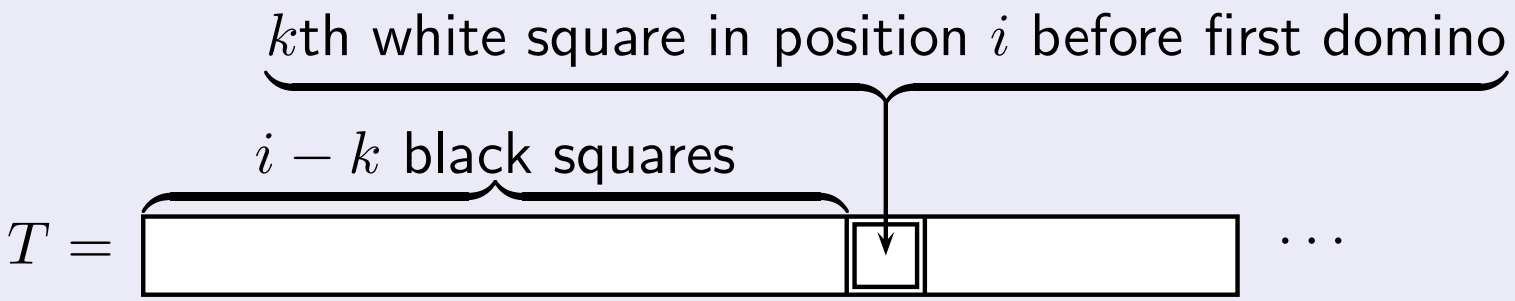

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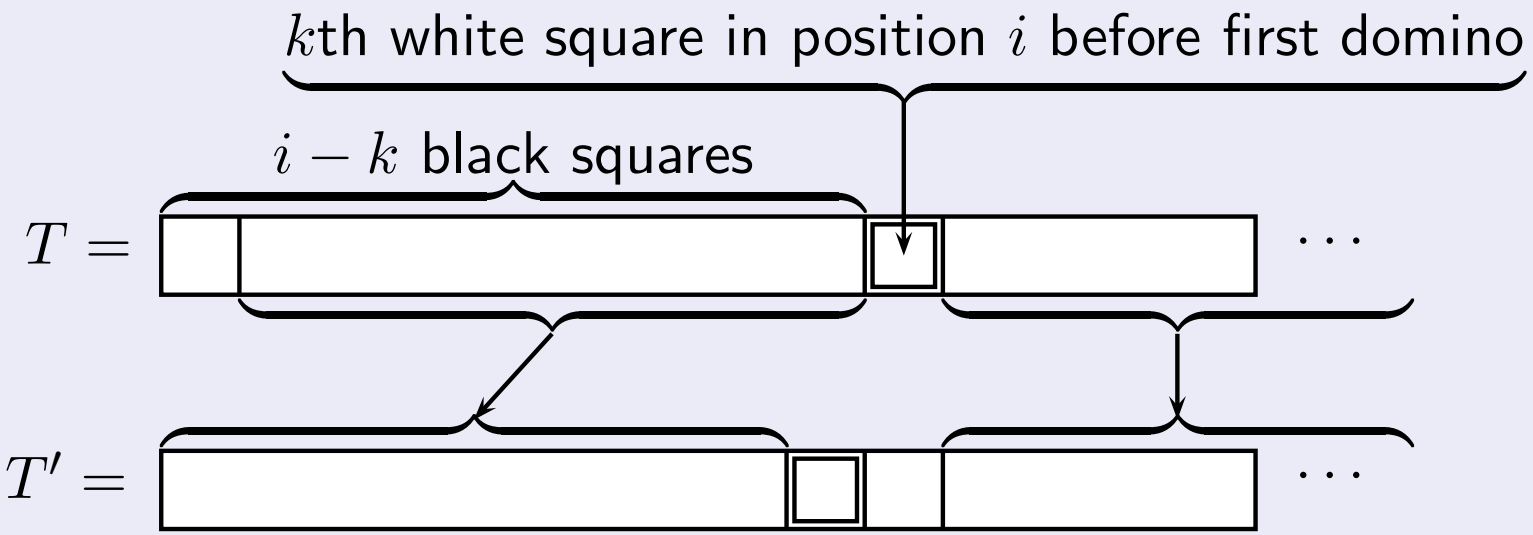

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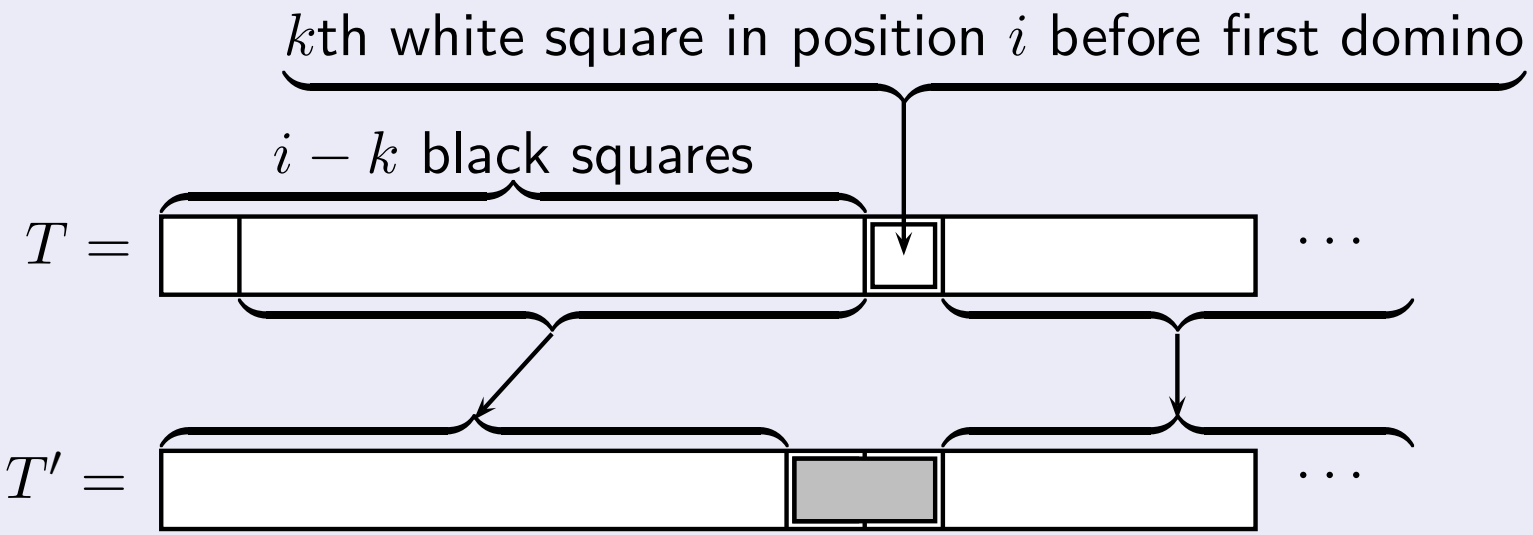

This increases the weight of the tiling by a factor of zq^{k-1}.

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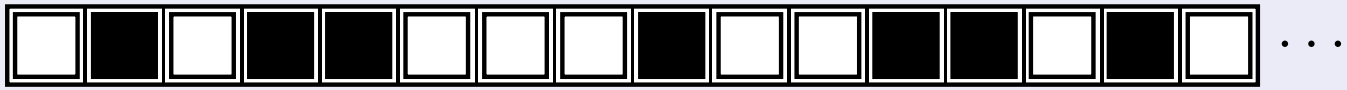

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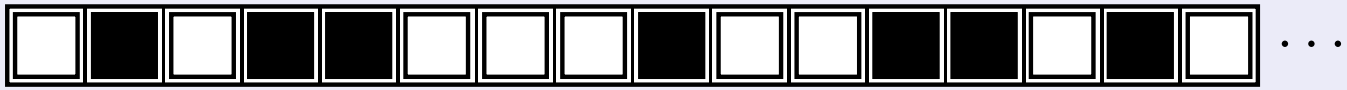

Insert three dominoes according to K $= \{8, 6, 2\}$.

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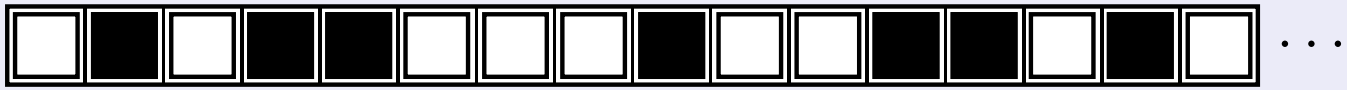

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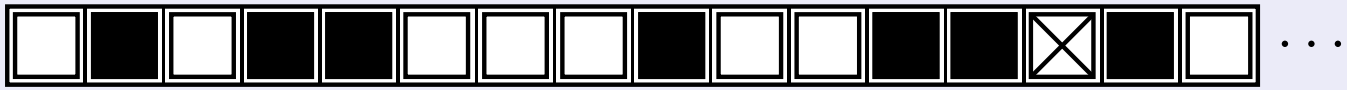

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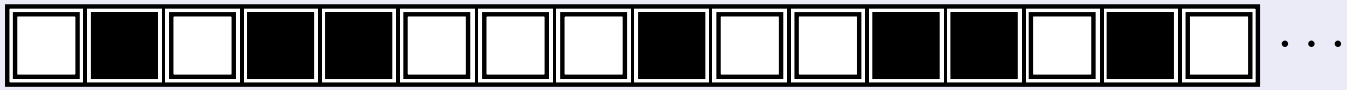

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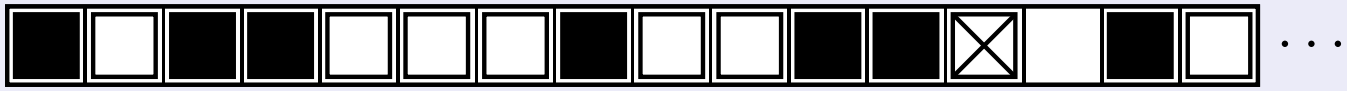

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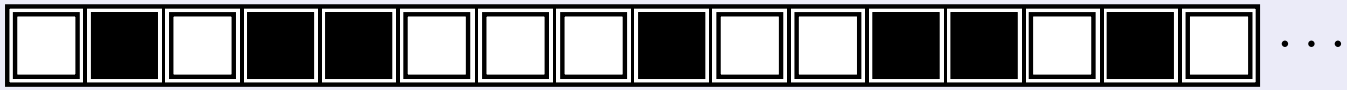
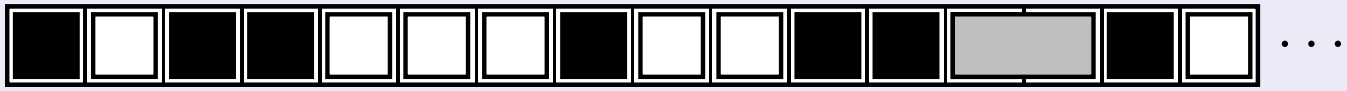


Insert three dominoes according to K $= \{8, 6, 2\}$.



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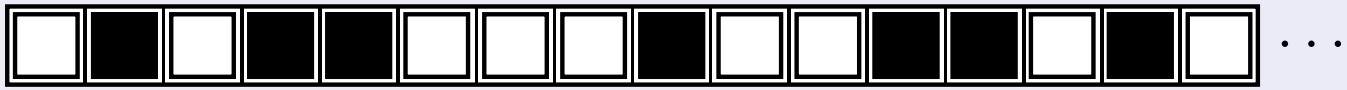

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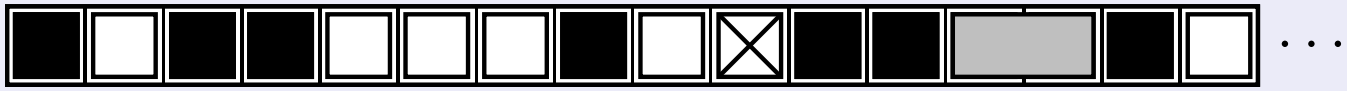

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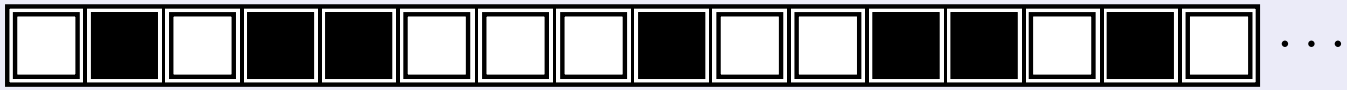

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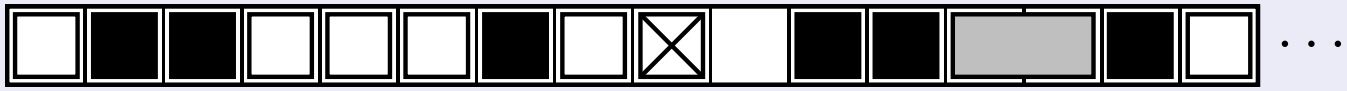

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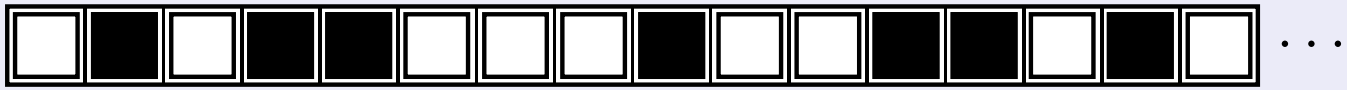

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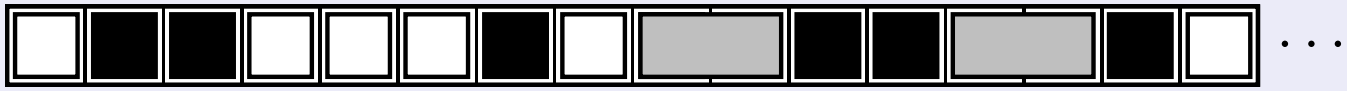

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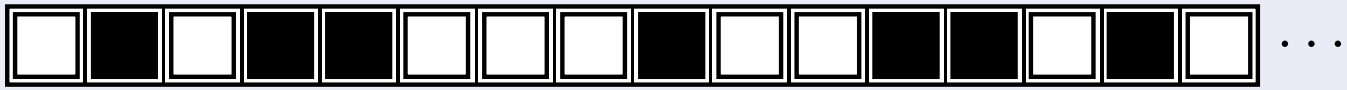

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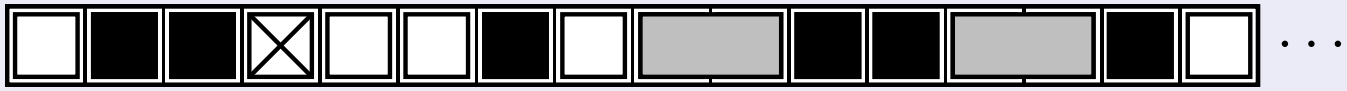

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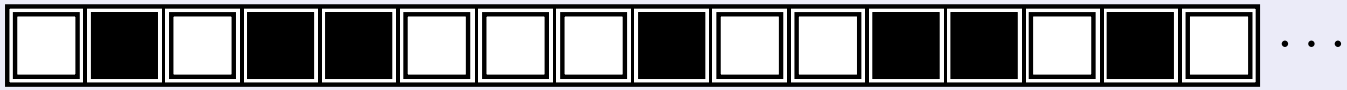

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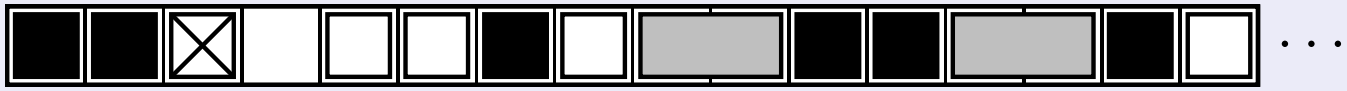

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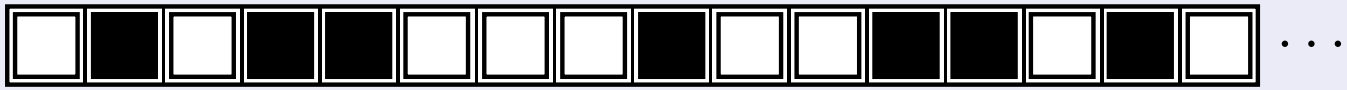

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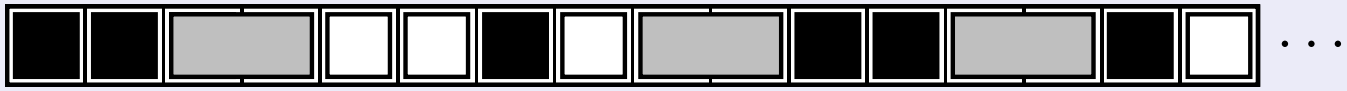

This accounts for a weight of

$$\prod_{n\geq 1} (1+q^{2n-1})$$

Classical Partition Identities

No dominoes (i.e., $z = 0$):

$$\sum_{n=0}^{\infty} \frac{q^{\binom{n+1}{2}}}{(q;q)_n} = \prod_{n=1}^{\infty} (1 + q^n)$$

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$$\prod_{n=1}^{\infty} (1 + q^n) = \prod_{n=1}^{\infty} \frac{1}{1 - q^{2n-1}}$$

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$$\prod_{n=1}^{\infty} (1 + q^n)(1 - q^{2n-1}) = 1$$

Classical Partition Identities

Odd dominoes and white squares only:

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Weight black squares by $-zq^i$.

$$\sum_{n=0}^{\infty} \frac{z^n q^{n^2}}{(q;q)_n (zq;q)_n} = \frac{1}{(zq;q)_\infty}$$

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$$(zq;q)_\infty \sum_{n=0}^{\infty} \frac{z^n q^{n^2}}{(q;q)_n (zq;q)_n} = 1$$

q-ANALOG OF THE BINOMIAL SERIES

Weight tiles in the following manner:

$$w(t) = \begin{cases} aq^i & \text{if } t \text{ is a } \blacksquare \text{ with } i \text{ } \square \text{ or } \square \text{ to its left} \\ bq^i & \text{if } t \text{ is a } \square \text{ with } i \text{ } \square \text{ or } \square \text{ to its left} \\ 1 & \text{if } t \text{ is a } \square \end{cases}$$

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THEOREM

$$\sum_{n=0}^{\infty} \frac{(-b/a;q)_n a^n}{(q;q)_n} = \prod_{n=0}^{\infty} \frac{1+bq^n}{1-aq^n}$$

Rogers-Fine Identity

Theorem

$$\sum_{n=0}^{\infty} \frac{(-b/a;q)_n a^n}{(cq;q)_n} = \sum_{n=0}^{\infty} \frac{(-b/a;q)_n(-b/c;q)_n a^n c^n q^{n^2}(1+bq^{2n})}{(a;q)_{n+1}(cq;q)_n}$$

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Multiplying both sides by $(1-a)$ yields:

$$(1-a)\sum_{n=0}^{\infty} \frac{(-b/a;q)_n a^n}{(cq;q)_n} = \sum_{n=0}^{\infty} \frac{(-b/a;q)_n(-b/c;q)_n a^n c^n q^{n^2}(1+bq^{2n})}{(aq;q)_n(cq;q)_n}$$

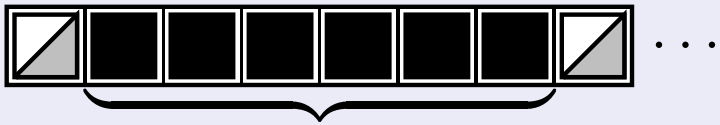
In this form, the left hand side counts tilings that do not start with a black square where the power of c keeps track of the number of white squares before the last weighted tile.

Theorem

$$(1-a)\sum_{n=0}^{\infty}\frac{(-b/a;q)_n a^n}{(cq;q)_n}=\sum_{n=0}^{\infty}\frac{(-b/a;q)_n(-b/c;q)_n a^n c^n q^{n^2}(1+bq^{2n})}{(aq;q)_n(cq;q)_n}$$

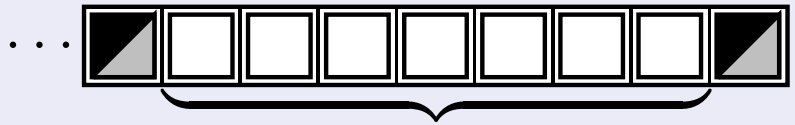
Proof. Interpret right–hand side

Front segments:



$j \geq 0$ black squares

Back segments:



$k \geq 0$ white squares

The center of a tiling marks the transition between front segments and back segments. The center can either be empty or a gray square.

Theorem

$$(1-a)\sum_{n=0}^{\infty}\frac{(-b/a;q)_n a^n}{(cq;q)_n}=\sum_{n=0}^{\infty}\frac{(-b/a;q)_n(-b/c;q)_n a^n c^n q^{n^2}(1+bq^{2n})}{(aq;q)_n(cq;q)_n}$$

Proof. Interpret right–hand side

Tilings that start with or and have n front/back segments.

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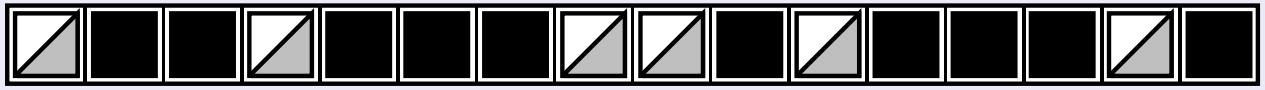
Proof. Interpret right–hand side

Tilings that start with □ or ■ and have n front/back segments.
Generating function for n front segments:



$$\frac{(c+b)}{(1-aq)}\frac{(c+bq)}{(1-aq^2)}\cdots\frac{(c+bq^{n-1})}{(1-aq^n)}=\frac{(-b/c;q)_n c^n}{(aq;q)_n}$$

Theorem

$$(1-a)\sum_{n=0}^{\infty}\frac{(-b/a;q)_n a^n}{(cq;q)_n}=\sum_{n=0}^{\infty}\frac{(-b/a;q)_n(-b/c;q)_n a^n c^n q^{n^2}(1+bq^{2n})}{(aq;q)_n(cq;q)_n}$$

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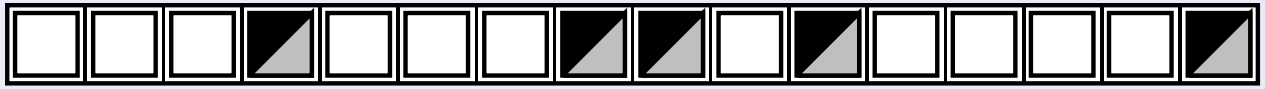
Generating function for n back segments:



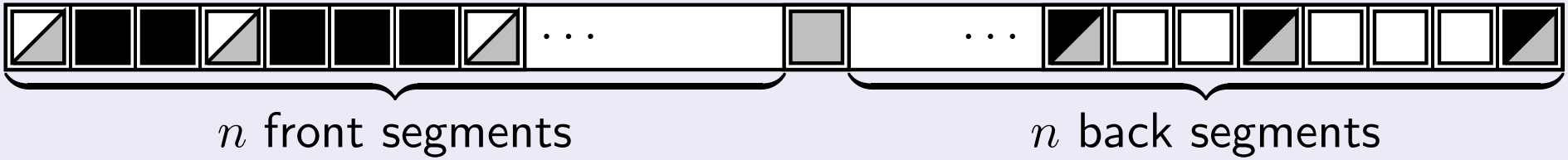
$$\frac{(aq^n+bq^{2n-1})}{(1-cq^n)}\cdots\frac{(aq^n+bq^{n+1})}{(1-cq^2)}\frac{(aq^n+bq^n)}{(1-cq)}=\frac{(-b/a;q)_n a^n q^{n^2}}{(cq;q)_n}$$

Theorem

$$(1-a)\sum_{n=0}^{\infty}\frac{(-b/a;q)_n a^n}{(cq;q)_n}=\sum_{n=0}^{\infty}\frac{(-b/a;q)_n(-b/c;q)_n a^n c^n q^{n^2}(1+bq^{2n})}{(aq;q)_n(cq;q)_n}$$

Proof. Interpret right–hand side

The center can either be empty or a gray square.



If the center is a gray square, then it has weight bq^n and increases the weight of the back segments by q^n.

Theorem

$$(1-a)\sum_{n=0}^{\infty}\frac{(-b/a;q)_n a^n}{(cq;q)_n}=\sum_{n=0}^{\infty}\frac{(-b/a;q)_n(-b/c;q)_n a^n c^n q^{n^2}(1+bq^{2n})}{(aq;q)_n(cq;q)_n}$$

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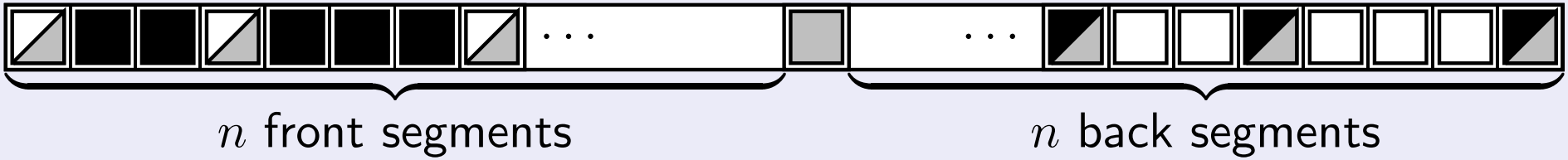

If the center is a gray square, then it has weight bq^n and increases the weight of the back segments by q^n.
In other words, the factor $(1+bq^{2n})$ represents the choice of the center.

Summary

Checklist:

- Lebesgue identities
- Classical partition identities
- q-binomial series
- Rogers-Fine identity
- Eight (plus five) identities of Rogers
- Limiting case of q-Gauss
- q-series symmetry result of Ramanujan

Future Work:

- Rogers-Ramanujan identities
- Tiling statistics

Eight Identities of Rogers

$$\sum_{n\geq 0}\frac{z^n q^{n^2}}{(q;q)_n}=(-zq^2;q^2)_\infty\sum_{n\geq 0}\frac{z^n q^{n^2}}{(q^2;q^2)_n(-zq^2;q^2)_n}$$

$$\sum_{n\geq 0}\frac{z^n q^{n^2+n}}{(q;q)_n}=(-zq^2;q^2)_\infty\sum_{n\geq 0}\frac{z^n q^{n^2+2n}}{(q^2;q^2)_n(-zq^2;q^2)_n}$$

$$\sum_{n\geq 0}\frac{z^{2n} q^{4n^2+2n}}{(q^4;q^4)_n}=(zq^2;q^2)_\infty\sum_{n\geq 0}\frac{z^n q^{n^2+n}}{(q^2;q^2)_n(zq^2;q^2)_n}$$

$$\sum_{n\geq 0}\frac{z^n q^{2n^2+n}}{(q^2;q^2)_n}=(zq^2;q^2)_\infty\sum_{n\geq 0}\frac{z^n q^{(3n^2+n)/2}}{(q;q)_n(zq^2;q^2)_n}$$

Five More Identities

$$\sum_{n\geq 0} \frac{z^n q^{2n^2}}{(q^2;q^2)_n} = (zq;q^2)_\infty \sum_{n\geq 0} \frac{z^n q^{(3n^2+n)/2}}{(q;q)_n (zq;q^2)_{n+1}}$$

$$\sum_{n\geq 0} \frac{z^n q^{n^2}}{(q;q)_n} = (-zq;q)_\infty \sum_{n\geq 0} \frac{(-1)^n z^{2n} q^{3n^2}}{(q^2;q^2)_n (-zq;q)_{2n}}$$

$$\sum_{n\geq 0} \frac{z^n q^{n^2+n}(1-z^2q^{2n+3})}{(q;q)_n} = (-zq;q)_\infty \sum_{n\geq 0} \frac{(-1)^n z^{2n} q^{3n^2}}{(q^2;q^2)_n (-zq;q)_{2n+1}}$$

Weight tiles in the following manner:

$$w(t) = \begin{cases} aq^i & \text{if } t \text{ is a } \blacksquare \text{ in position } i \\ cq^i & \text{if } t \text{ is a } \bullet \text{ in position } i \\ 1 & \text{if } t \text{ is a } \square \\ -cq^i & \text{if } t \text{ is a } \bigcirc \text{ in position } i \end{cases}$$

The following is a limiting case of q-Gauss

Theorem

$$(cq;q)_\infty \sum_{n=0}^{\infty} \frac{(-c/a;q)_n a^n q^{\binom{n+1}{2}}}{(q;q)_n (cq;q)_n} = \prod_{n=1}^{\infty} (1 + aq^n)$$

Labeled Tilings

Weight tiles in the following manner:

$$w(t) = \begin{cases} a_j z q^i & \text{if } t \text{ is a } \boxed{j} \text{ in position } i,\ 1 \le j \le k+1 \\ z_j q^i & \text{if } t \text{ is a } \textcircled{j} \text{ in position } i,\ 1 \le j \le k \\ 1 & \text{if } t \text{ is a } \square \text{ in position } i \end{cases}$$

Theorem

The following function is the generating function for labeled tilings that consist of weakly increasing sequences of weighted tiles followed by a single white square where the label must strictly increase after a circle.

$$(-a_{k+1}q)_\infty \sum_{n_1,\ldots,n_k \ge 0} \frac{(-z_1/a_1)_{n_1} \cdots (-z_k/a_k)_{n_k} a_1^{n_1} \cdots a_k^{n_k}}{(q)_{n_1} \cdots (q)_{n_k} (-a_{k+1}q)_{n_1+\cdots+n_k}} q^{\binom{n_1+\cdots+n_k+1}{2}}$$

Furthermore, this function is symmetric in the variables $(a_1, \ldots, a_{k+1})$ *as well as the variables* $(z_1, \ldots, z_k)$.