A New Combinatorial Approach to q–Series Identities

David P. Little
Penn State University
(Joint work with James Sellers)

August 14, 2008

www.math.psu.edu/dlittle

Pell Tilings

Definition

An (infinite) Pell tiling is a covering of an infinitely long board:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

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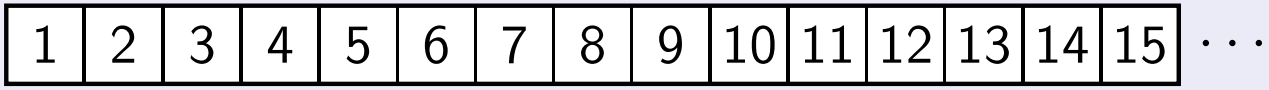

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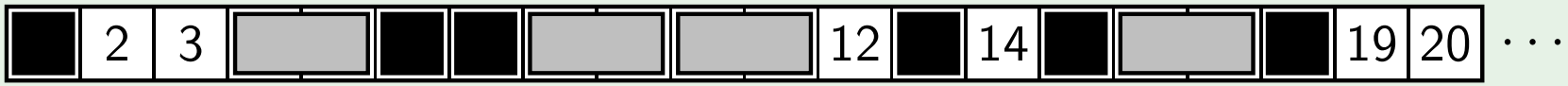

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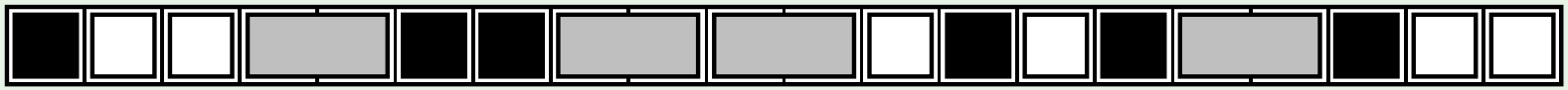

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Weighted Pell Tilings

The weight of tile t:

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The weight of tiling T:

$$w(T) = \prod_{t \in T} w(t)$$

Generating Function:

$$P(z; q) = \sum_{T \in \mathcal{P}} w(T)$$

An Operation on Tiles: Projection

Definition

Projection will refer to any invertible operation that moves tiles around on the board provided that

- the effect on the weight of the tiling does not depend on which tile was projected.
- the relative position of the projectiles (i.e., tiles that can be projected) does not change.

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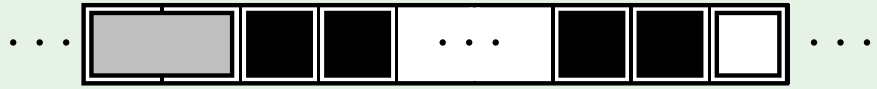

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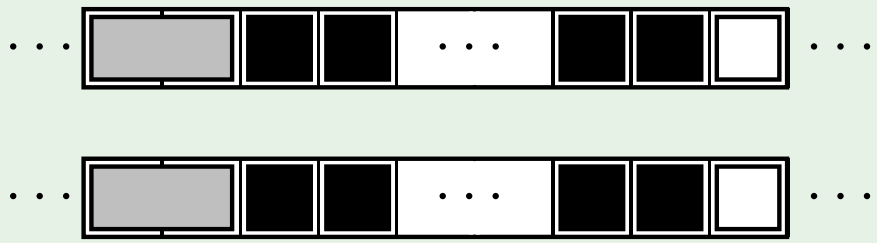

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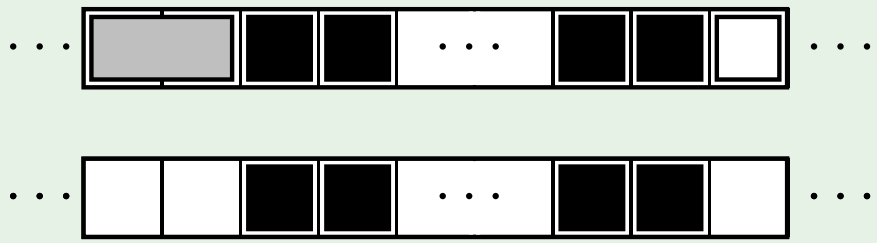

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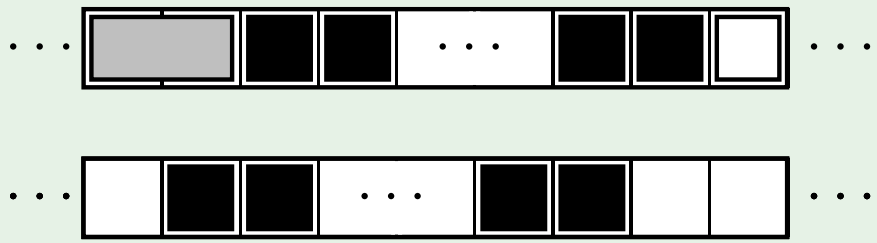

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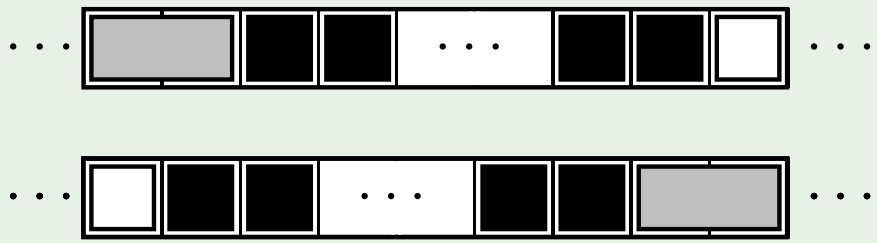

Projecting the (projec)Tiles

Goal: Construct all of the tilings that can be obtained by repeatedly applying the projection operation to the projectiles of a tiling T.

- always work in a right–to–left manner
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Lemma

Let T be a tiling that contains n projectiles. Then the generating function for all tilings that can be obtained from T by projecting the tiles is given by

$$\frac{w(T)}{(1-q^k)(1-q^{2k})(1-q^{3k})\cdots(1-q^{nk})} = \frac{w(T)}{(q^k;q^k)_n}$$

if each projection increases the weight of a tiling by a factor of q^k.

Recall

The weight of tile t:

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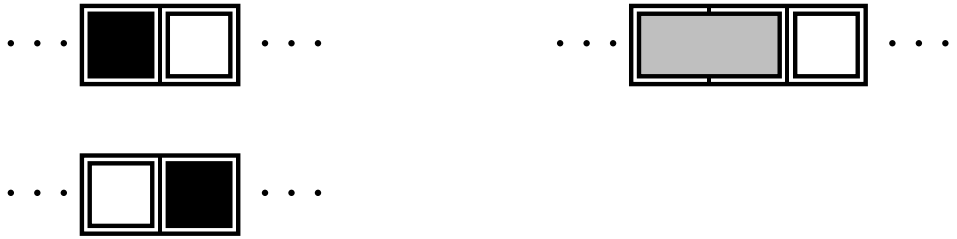

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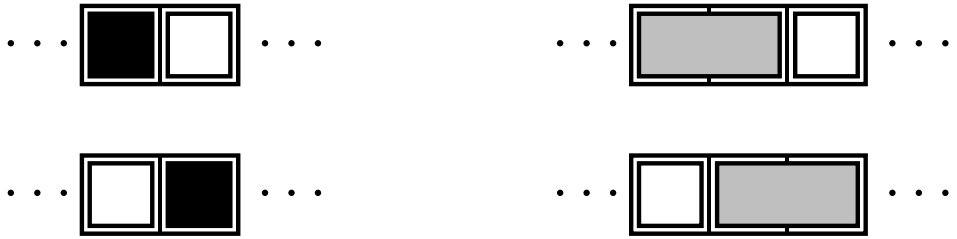

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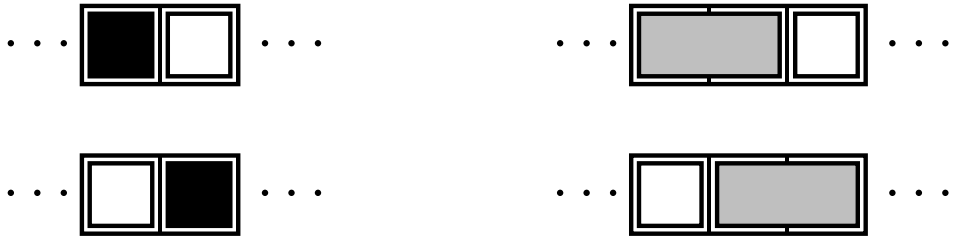


Projecting a black square or a domino increases the weight of a tiling by a factor of q.

Series Expansion of $P(z;q)$

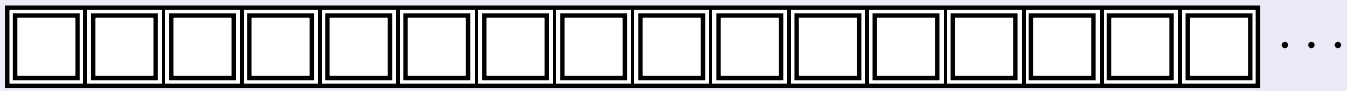
Theorem

$$P(z;q) = \sum_{n=0}^{\infty} \frac{(-z;q)_n}{(q;q)_n} q^{\binom{n+1}{2}}$$

where $(z;q)_n = (1-z)(1-zq)\cdots(1-zq^{n-1})$.

Proof.

STEP I: Place n black squares in positions $1, 2, 3, \ldots, n$.



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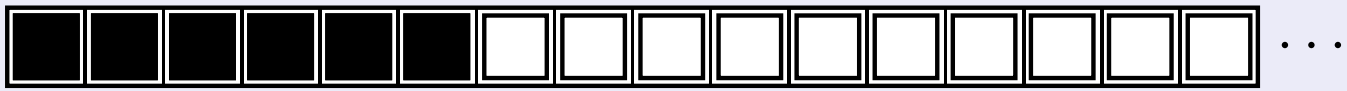

This accounts for a weight of $q^{\binom{n+1}{2}}$.

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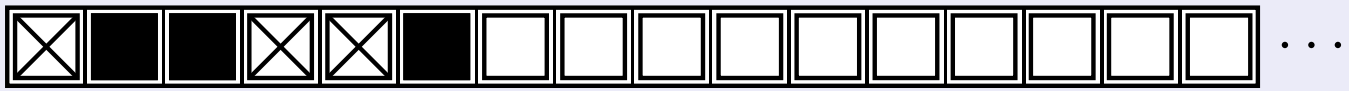

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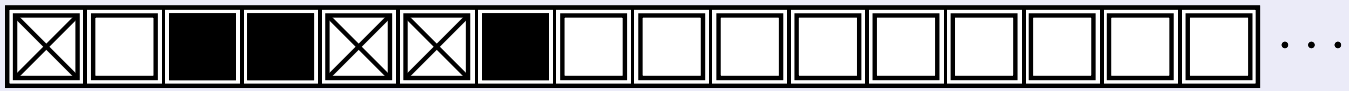

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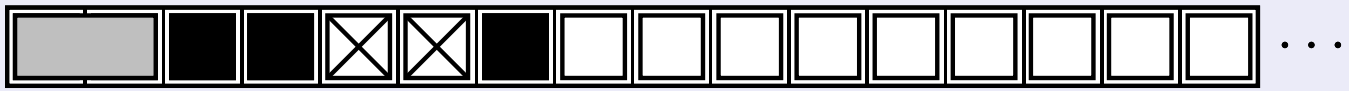

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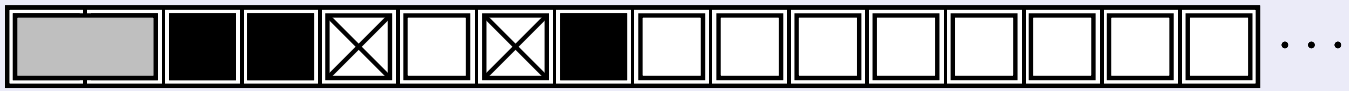

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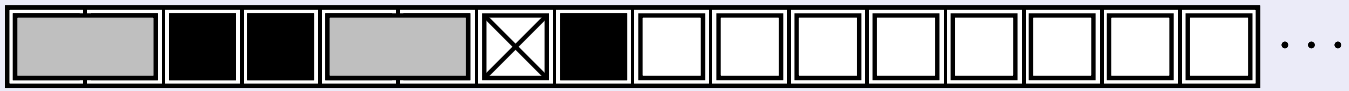

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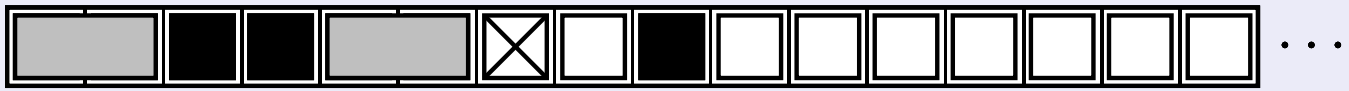

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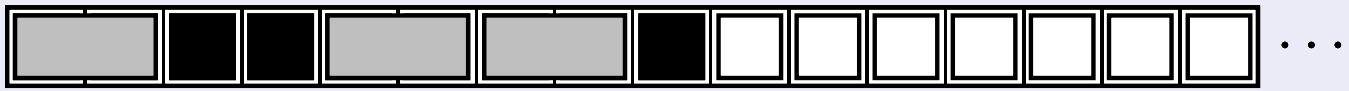

The choice of converting the ith black square into a domino is represented by the factor $(1+zq^{n-i})$.

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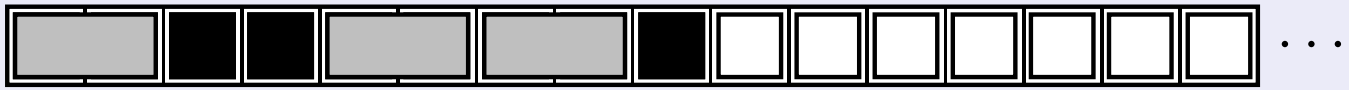

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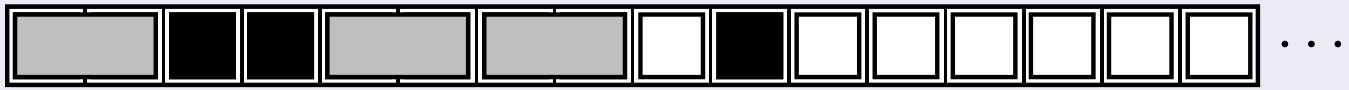

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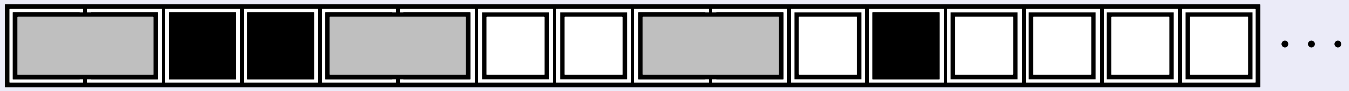

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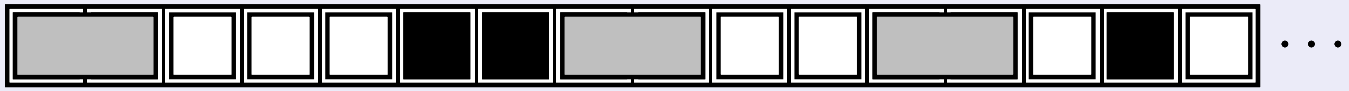

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Proof.

Setting $z = 0$ eliminates any tiling that contains a domino.

Tilings can then be constructed by just deciding whether each position should be covered by a white square or a black square.

The factor $(1+q^n)$ represents the choice of covering position n with a white square or a black square.

$\square$

Lebesgue Identity

Theorem (Lebesgue, 1840)

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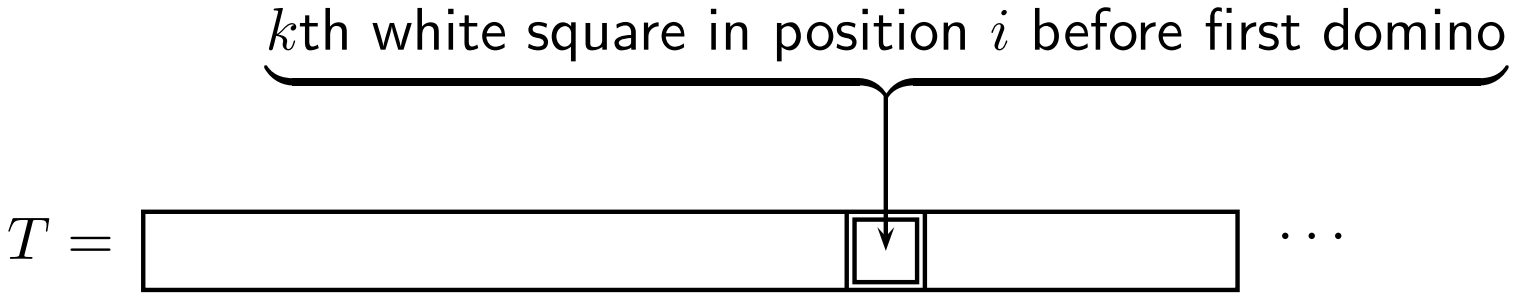
Can the factors of $(1+zq^{2n-1})$ be interpreted as inserting dominoes?

Inserting Dominoes



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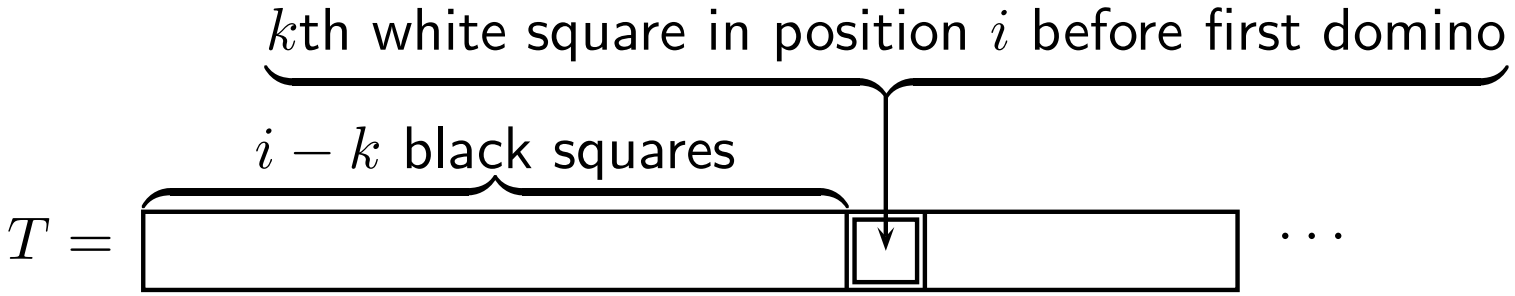

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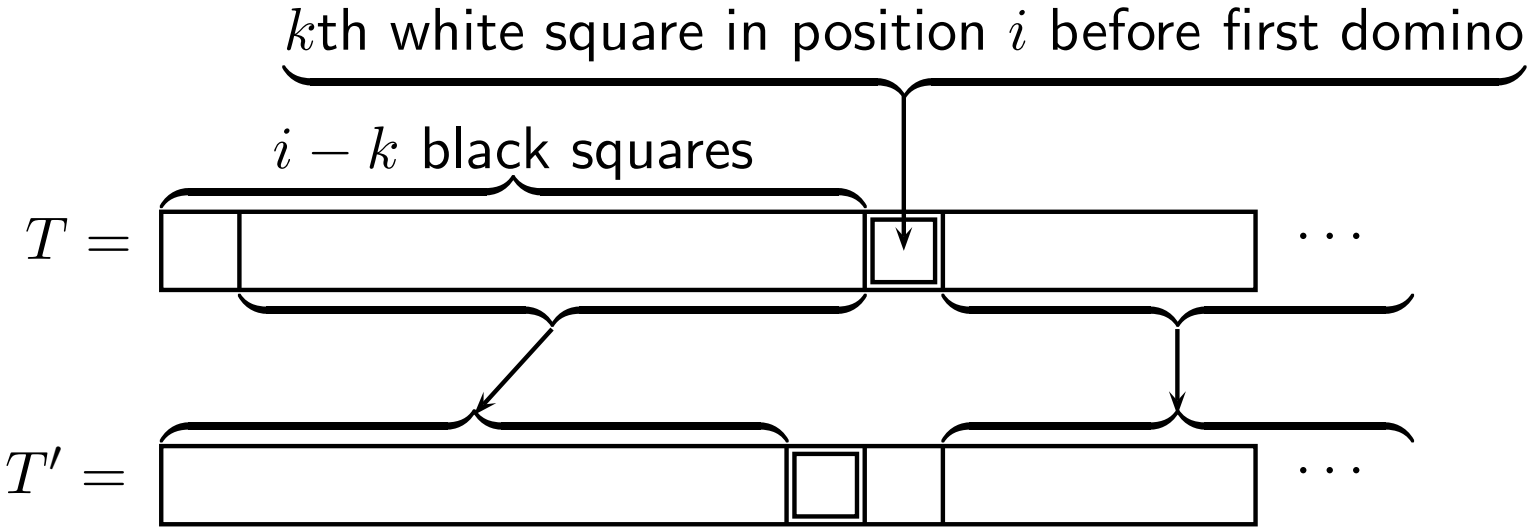

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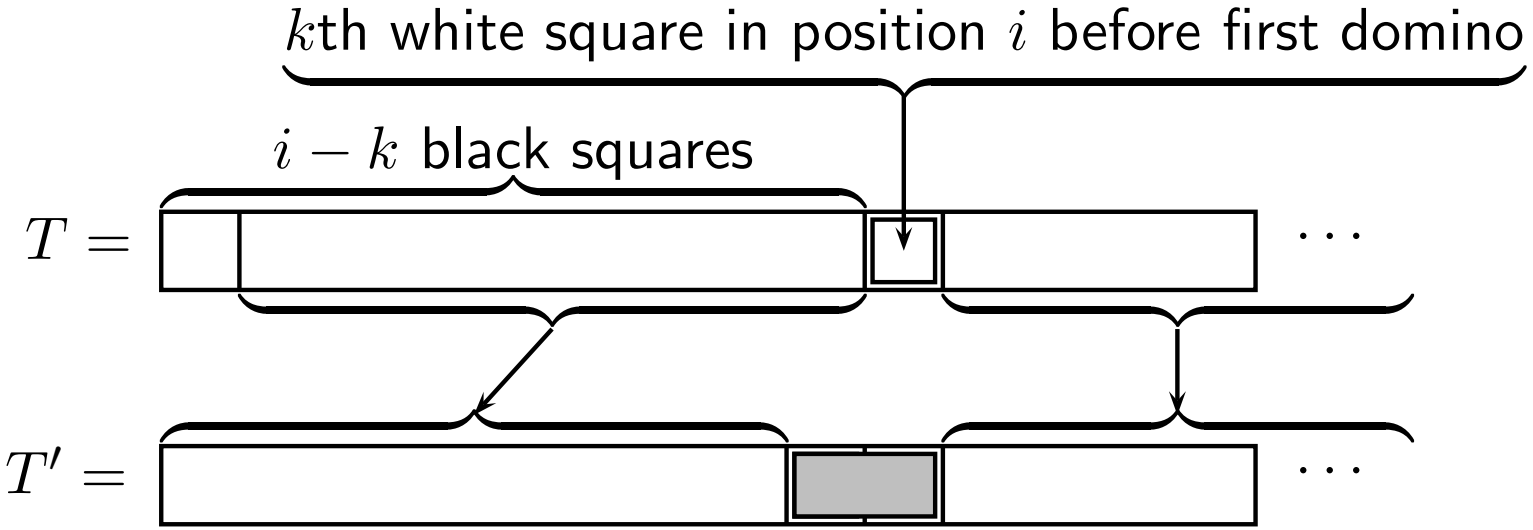

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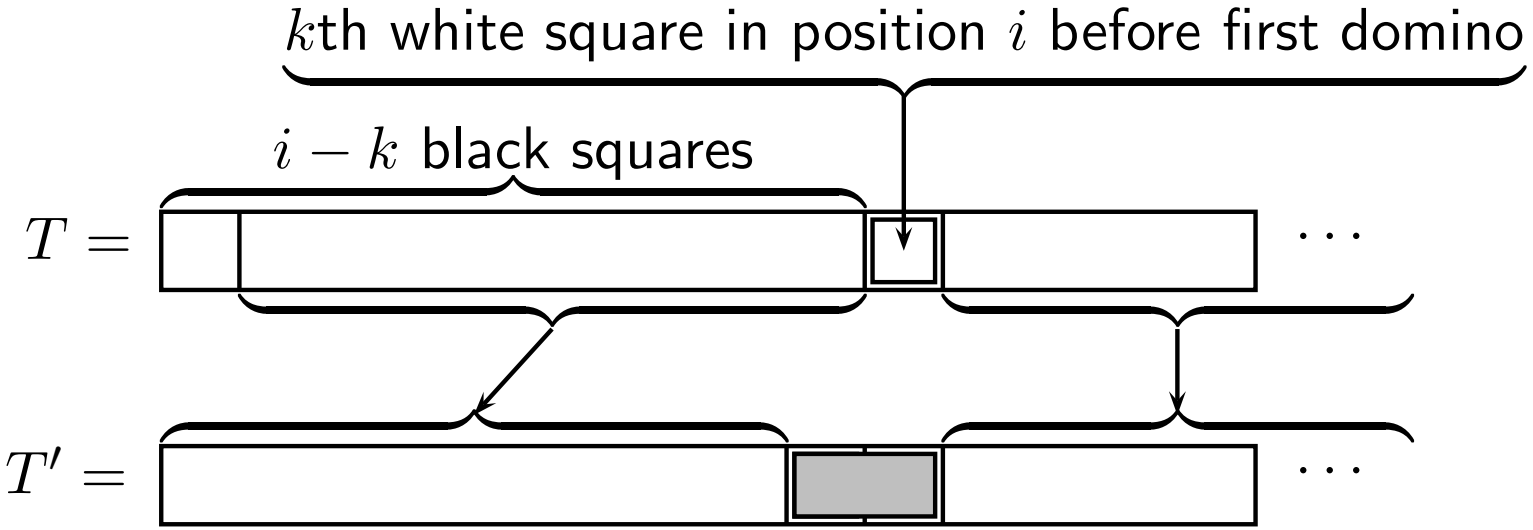

Lemma If the kth white square appears in position $i > 1$ of T, then

$$w(T') = zq^{-(i-k)+(i-1)}w(T) = zq^{k-1}w(T).$$

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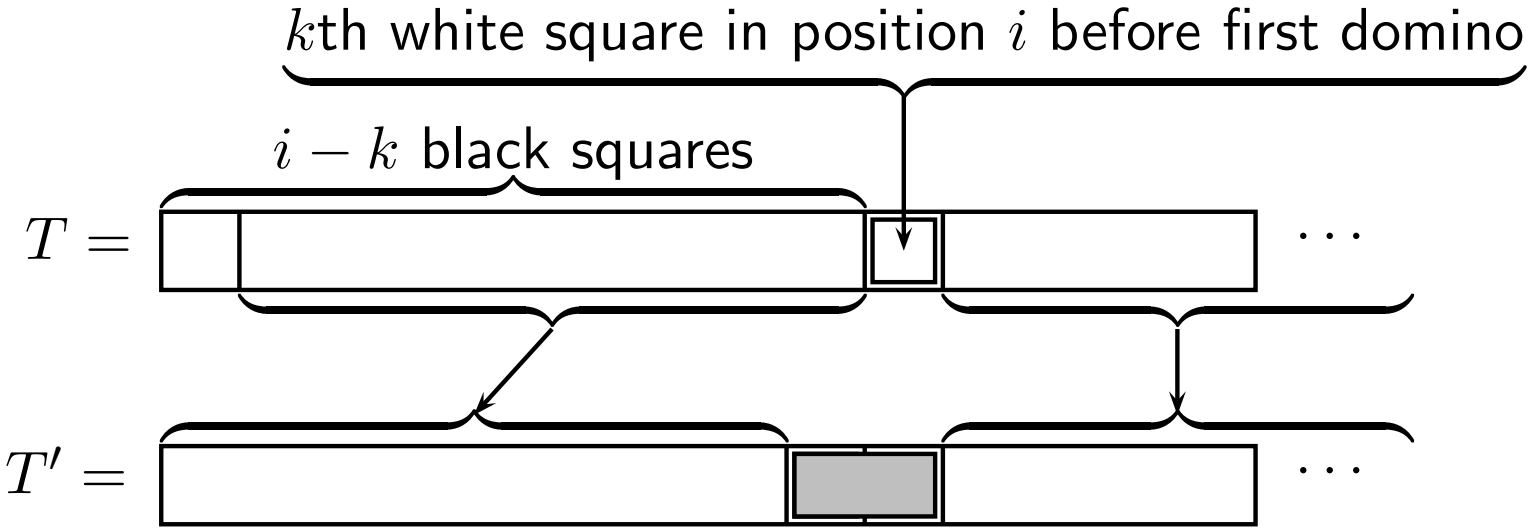

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Observation:

If the above operation is always performed when k is of the same parity, then the operation can be reversed.

Theorem

$$P(z;q) = \prod_{n=1}^{\infty}(1+q^n)(1+zq^{2n-1}).$$

Proof.

Construct tiling $T^{(0)}$ by arbitrarily covering the board with squares. This explains each factor of $(1+q^n)$.

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Thus, each factor of $(1+zq^{2n-1})$ simply expresses the decision of whether or not to include $2n$ in I. The tiling $T^{(l)}$ is the final result of our construction.

Example

Consider the following term taken from $\prod_{n=1}^{\infty}(1+q^n)(1+zq^{2n-1})$.

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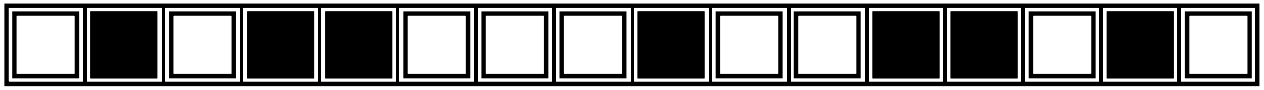
$T^{(0)} =$  $\cdots$

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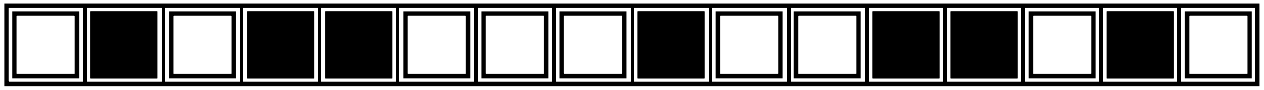
$T^{(0)} =$  $\cdots$

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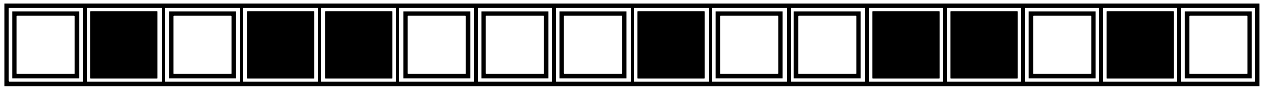
$T^{(0)} =$  $\cdots$

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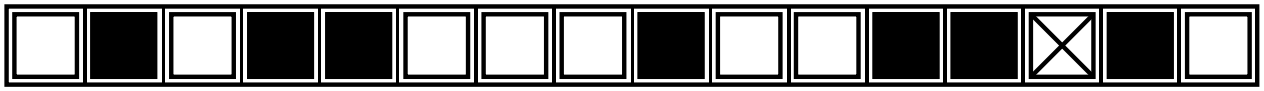
$T^{(1)} =$ 

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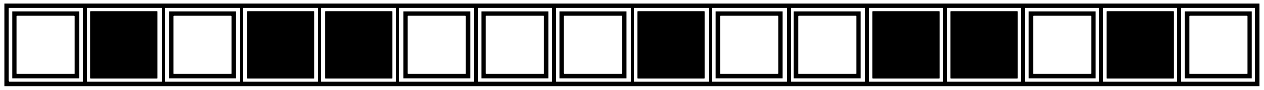
$T^{(0)} =$  $\cdots$

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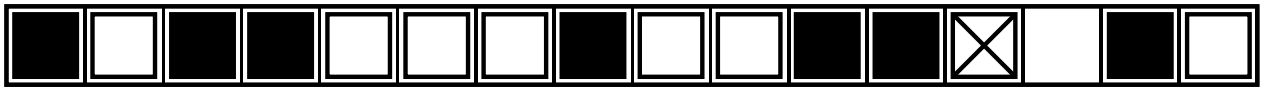
$T^{(1)} =$ 

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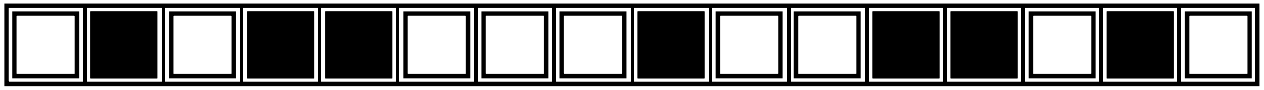
$T^{(0)} =$  $\cdots$

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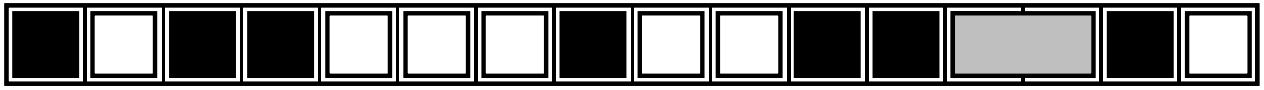
$T^{(1)} =$  $\cdots$

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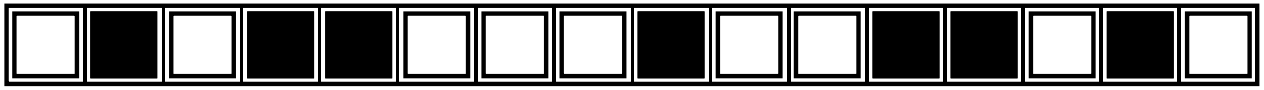
$T^{(0)} =$  $\cdots$

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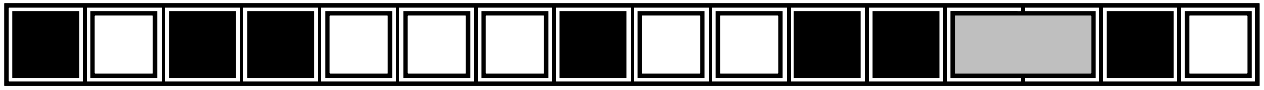
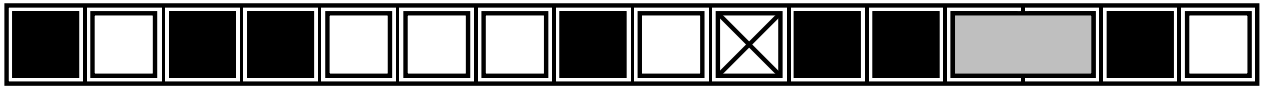
$T^{(1)} =$  $\cdots$

$T^{(2)} =$ 

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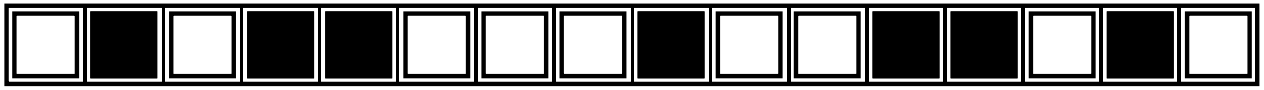
$T^{(0)} =$  $\cdots$

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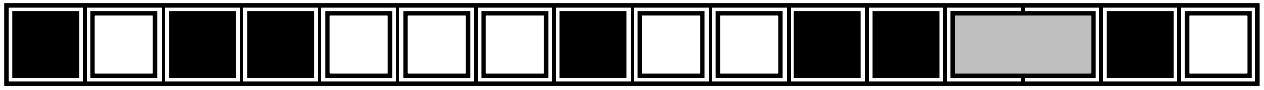
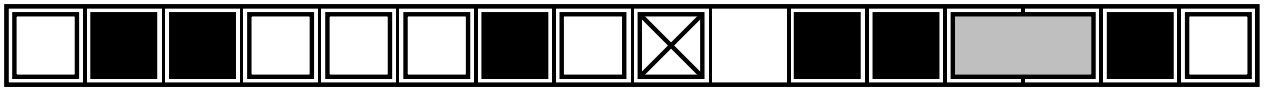
$T^{(1)} =$  $\cdots$

$T^{(2)} =$ 

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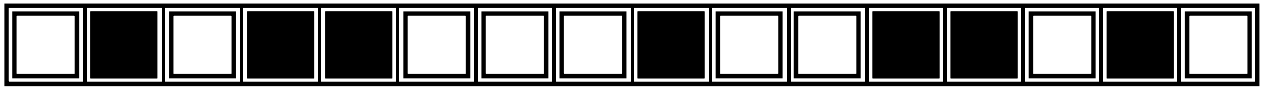
$T^{(0)} =$  $\cdots$

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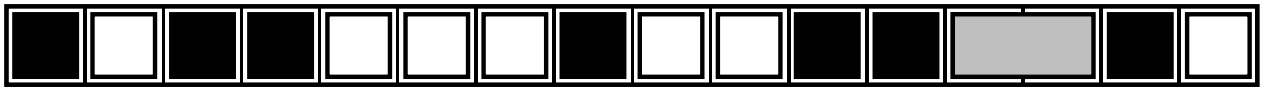
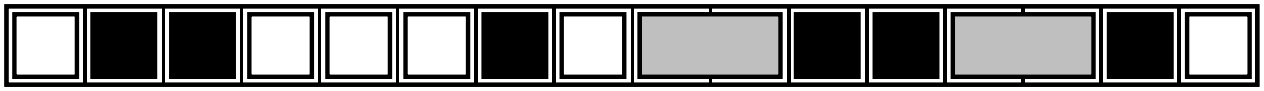
$T^{(1)} =$  $\cdots$

$T^{(2)} =$  $\cdots$

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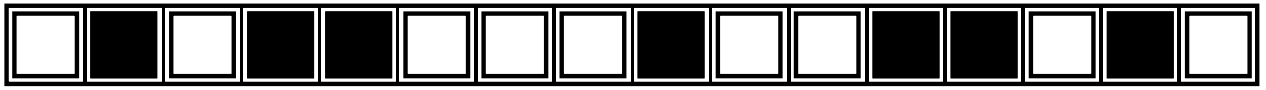
$T^{(0)} =$  $\cdots$

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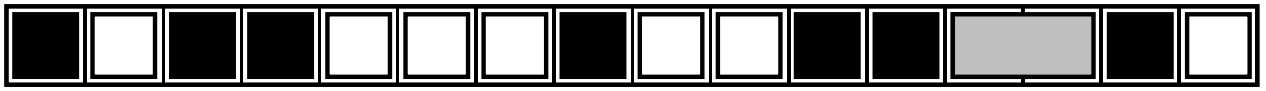
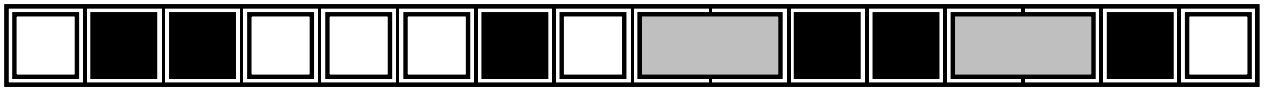
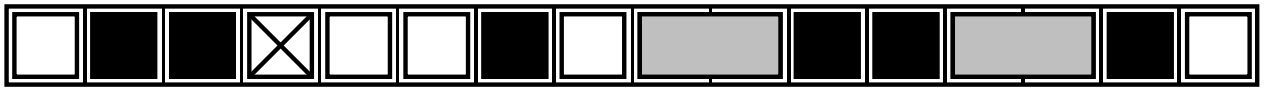
$T^{(1)} =$  $\cdots$

$T^{(2)} =$  $\cdots$

$T^{(3)} =$ 

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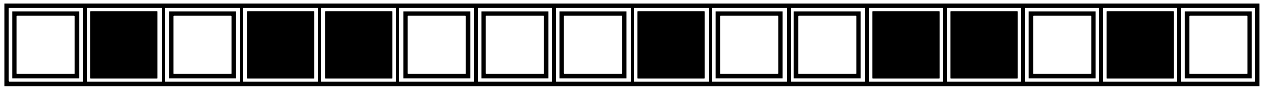
$T^{(0)} =$  $\cdots$

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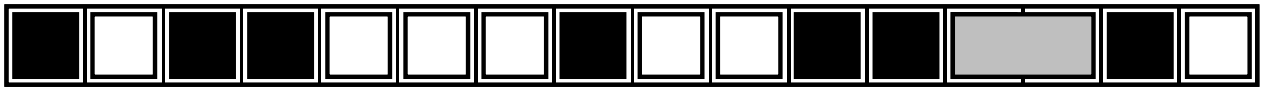
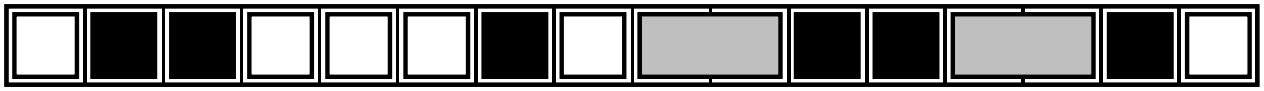
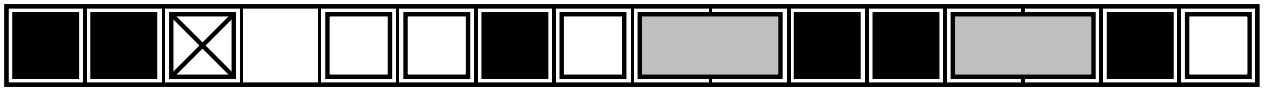
$T^{(1)} =$  $\cdots$

$T^{(2)} =$  $\cdots$

$T^{(3)} =$ 

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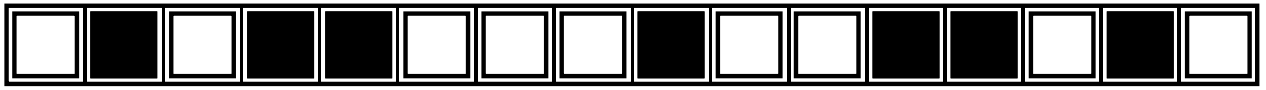
$T^{(0)} =$  $\cdots$

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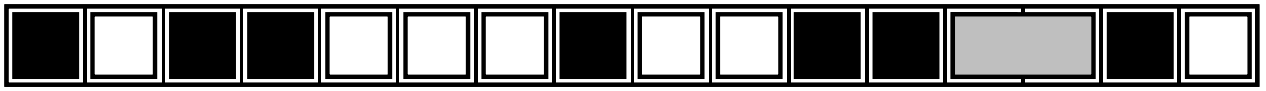
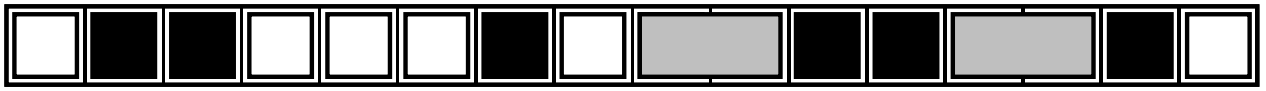
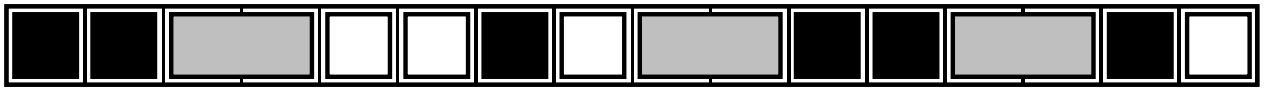
$T^{(1)} =$  $\cdots$

$T^{(2)} =$  $\cdots$

$T^{(3)} =$  $\cdots$

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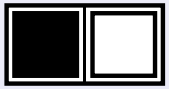
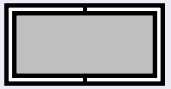
Find first occurrence of:  or 

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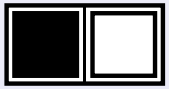
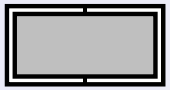
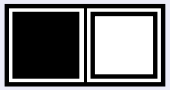
Find first occurrence of:  or

and replace with:  or  (respectively)

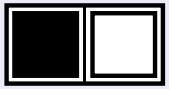
Another Theorem of Euler

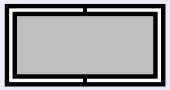
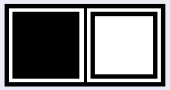
Theorem

$$\prod_{n=1}^{\infty}(1+q^n)(1-q^{2n-1}) = 1$$

Proof.

Let $z = -1$ in the product side of Lebesgue: $\displaystyle\prod_{n=1}^{\infty}(1+q^n)(1+zq^{2n-1})$

Find first occurrence of:  or

and replace with:  or  (respectively)

Only remaining tiling consists of all white squares, which has weight 1.

□

Rogers–Ramanujan Identities

Theorem Rogers (1894), Ramanujan (1913), Schur (1916)

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \prod_{n=1}^{\infty} \frac{1}{(1-q^{5n-1})(1-q^{5n-4})}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \prod_{n=1}^{\infty} \frac{1}{(1-q^{5n-2})(1-q^{5n-3})}$$

Partition theoretic interpretation of first identity:

the number of partitions of n into parts that differ by at least 2 is equal to the number of partitions of n into parts that are congruent to ± 1 mod 5.

Rogers–Ramanujan Type Identities

Theorem Rogers (1894,1917)

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^4;q^4)_n} = \prod_{n=1}^{\infty} \frac{1}{(1-q^{5n-1})(1-q^{5n-4})(1+q^{2n})}$$

$$\sum_{n=0}^{\infty} \frac{q^{(3n^2-n)/2}}{(q;q)_n(q;q^2)_n} = \prod_{n=1}^{\infty} \frac{(1-q^{10n-4})(1-q^{10n-6})(1-q^{10n})}{(1-q^n)}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_{2n}} = \prod_{n=1}^{\infty} \frac{(1-q^{20n-8})(1-q^{20n-12})(1-q^{20n})(1+q^{2n-1})}{(1-q^{2n})}$$

Rogers–Ramanujan Type Identities

Theorem Rogers (1894,1917)

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = (-q^2;q^2)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^4;q^4)_n}$$

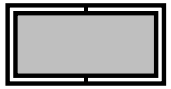
$$\sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q^2;q^2)_n} = (q;q^2)_\infty \sum_{n=0}^{\infty} \frac{q^{(3n^2-n)/2}}{(q;q)_n(q;q^2)_n}$$

$$\sum_{n=0}^{\infty} \frac{q^{4n^2}}{(q^4;q^4)_n} = (q;q^2)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_{2n}}$$

Fibonacci Tilings

Weight tiles in the following manner:

$$w(t) = \begin{cases} zq^i & \text{if } t \text{ is a } \\ 1 & \text{if } t \text{ is a } \end{cases}$$


in position i

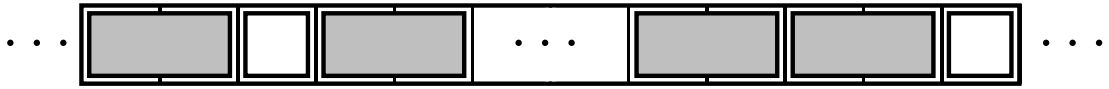
in position i

Fibonacci Tilings

Weight tiles in the following manner:

$$w(t) = \begin{cases} zq^i & \text{if } t \text{ is a } \boxed{} \text{ in position } i \\ 1 & \text{if } t \text{ is a } \boxed{} \text{ in position } i \end{cases}$$

Projectiles: odd dominoes (i.e., dominoes that appear in an odd position)



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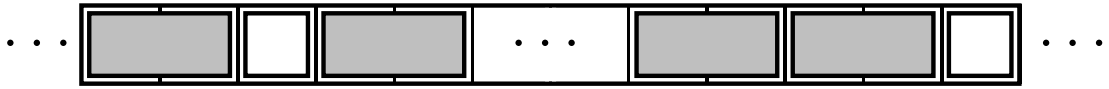

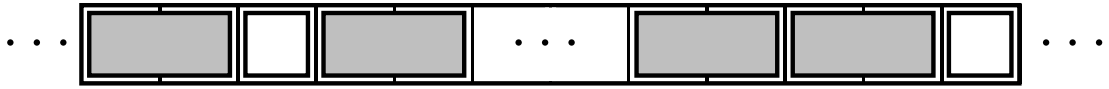

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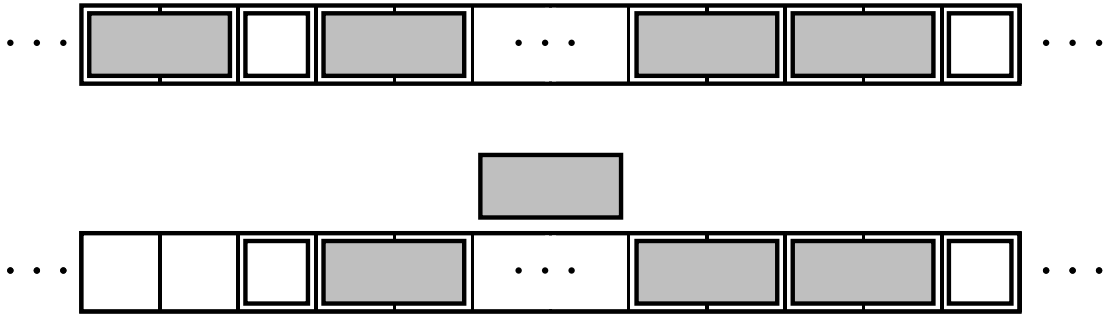

Fibonacci Tilings

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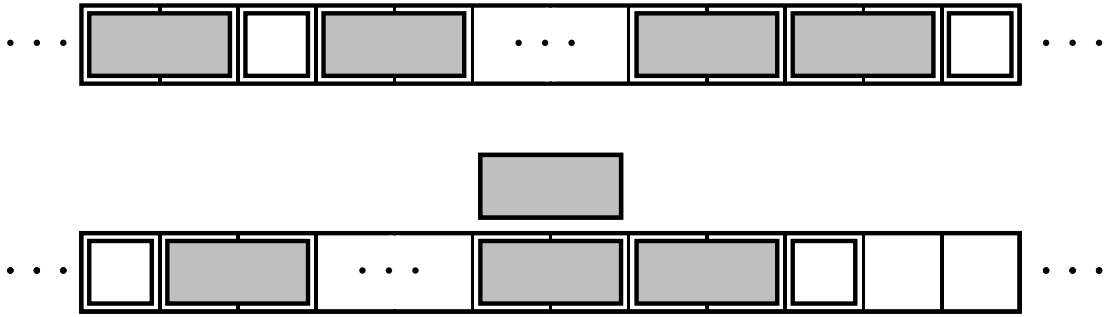

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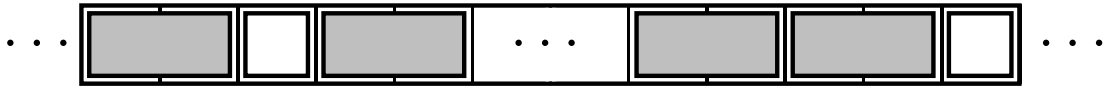

Fibonacci Tilings

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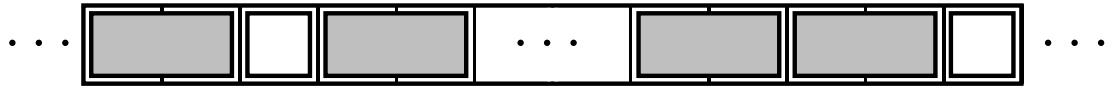
Projectiles: odd dominoes (i.e., dominoes that appear in an odd position)



Projecting an odd domino increases the weight of a tiling by a factor of q^2.

Theorem (Sylvester)

$$\sum_{n=0}^{\infty} \frac{z^n q^{n^2}}{(q^2;q^2)_n} = \prod_{n=1}^{\infty} (1 + zq^{2n-1})$$

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$$\sum_{n=0}^{\infty} \frac{z^n q^{n^2}}{(q^2;q^2)_n} = \prod_{n=1}^{\infty} (1 + zq^{2n-1})$$

Proof.

RHS: Generating function for Fibonacci tilings that use only white squares and odd dominoes.

Theorem (Sylvester)

$$\sum_{n=0}^{\infty} \frac{z^n q^{n^2}}{(q^2;q^2)_n} = \prod_{n=1}^{\infty} (1 + zq^{2n-1})$$

Proof.

RHS: Generating function for Fibonacci tilings that use only white squares and odd dominoes.

LHS: We can construct the same objects in the following manner:

1. Place n odd dominoes in positions $1, 3, 5, \ldots, 2n-1$.
2. Project the odd dominoes.

$\square$

Theorem (Rogers)

$$\sum_{n=0}^{\infty} \frac{z^n q^{n^2}}{(q;q)_n} = (-zq^2;q^2)_\infty \sum_{n=0}^{\infty} \frac{z^n q^{n^2}}{(q^2;q^2)_n(-zq^2;q^2)_n}$$

Proof.

LHS: Count all Fibonacci tilings according to number of dominoes.

Theorem (Rogers)

$$\sum_{n=0}^{\infty} \frac{z^n q^{n^2}}{(q;q)_n} = (-zq^2;q^2)_\infty \sum_{n=0}^{\infty} \frac{z^n q^{n^2}}{(q^2;q^2)_n(-zq^2;q^2)_n}$$

Proof.

LHS: Count all Fibonacci tilings according to number of dominoes.
RHS: Count all Fibonacci tilings according to number of odd dominoes.

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$$\sum_{n=0}^{\infty} \frac{z^n q^{n^2}}{(q;q)_n} = (-zq^2;q^2)_\infty \sum_{n=0}^{\infty} \frac{z^n q^{n^2}}{(q^2;q^2)_n(-zq^2;q^2)_n}$$

Proof.

LHS: Count all Fibonacci tilings according to number of dominoes.
RHS: Count all Fibonacci tilings according to number of odd dominoes.

1. Place n odd dominoes in positions $1, 3, 5, \ldots, 2n-1$.

Theorem (Rogers)

$$\sum_{n=0}^{\infty} \frac{z^n q^{n^2}}{(q;q)_n} = (-zq^2;q^2)_\infty \sum_{n=0}^{\infty} \frac{z^n q^{n^2}}{(q^2;q^2)_n(-zq^2;q^2)_n}$$

Proof.

LHS: Count all Fibonacci tilings according to number of dominoes.
RHS: Count all Fibonacci tilings according to number of odd dominoes.

1. Place n odd dominoes in positions $1, 3, 5, \ldots, 2n-1$.
2. Arbitrarily place even dominoes in positions $2n+2, 2n+4, 2n+6, \ldots$

$$z^n q^{n^2} \prod_{j \geq n+1} (1 + zq^{2j}) = (-zq^2;q^2)_\infty \frac{z^n q^{n^2}}{(-zq^2;q^2)_n}$$

Theorem (Rogers)

$$\sum_{n=0}^{\infty} \frac{z^n q^{n^2}}{(q;q)_n} = (-zq^2;q^2)_\infty \sum_{n=0}^{\infty} \frac{z^n q^{n^2}}{(q^2;q^2)_n(-zq^2;q^2)_n}$$

Proof.

LHS: Count all Fibonacci tilings according to number of dominoes.
RHS: Count all Fibonacci tilings according to number of odd dominoes.

1. Place n odd dominoes in positions $1, 3, 5, \ldots, 2n-1$.
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$$z^n q^{n^2} \prod_{j \geq n+1} (1 + zq^{2j}) = (-zq^2;q^2)_\infty \frac{z^n q^{n^2}}{(-zq^2;q^2)_n}$$

3. Project the odd dominoes.

$\square$

Even Weighted Fibonacci Tilings

Weight tiles in the following manner:

$$w(t) = \begin{cases} -zq^{2i} & \text{if } t \text{ is an even } \boxed{} \text{ in position } 2i \\ zq^{2i} & \text{if } t \text{ is an odd } \boxed{} \text{ in position } 2i-1 \\ 1 & \text{if } t \text{ is a } \boxed{} \text{ in position } i \end{cases}$$

Even Weighted Fibonacci Tilings

Weight tiles in the following manner:

$$w(t) = \begin{cases} -zq^{2i} & \text{if } t \text{ is an even } \text{[tile]} \text{ in position } 2i \\ zq^{2i} & \text{if } t \text{ is an odd } \text{[tile]} \text{ in position } 2i-1 \\ 1 & \text{if } t \text{ is a } \text{[tile]} \text{ in position } i \end{cases}$$

How does this affect projectiles?

- Increasing position of odd domino by one does not change its weight.

Even Weighted Fibonacci Tilings

Weight tiles in the following manner:

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How does this affect projectiles?

- Increasing position of odd domino by one does not change its weight.
- Increasing position of even domino by one increases its weight by q^2.

Even Weighted Fibonacci Tilings

Weight tiles in the following manner:

$$w(t) = \begin{cases} -zq^{2i} & \text{if } t \text{ is an even } \boxed{} \text{ in position } 2i \\ zq^{2i} & \text{if } t \text{ is an odd } \boxed{} \text{ in position } 2i-1 \\ 1 & \text{if } t \text{ is a } \boxed{} \text{ in position } i \end{cases}$$

How does this affect projectiles?

- Increasing position of odd domino by one does not change its weight.
- Increasing position of even domino by one increases its weight by q^2.

Thus, projecting a domino still increases the weight of a tiling by q^2.

Theorem (Rogers)

$$\sum_{n=0}^{\infty} \frac{z^{2n} q^{4n^2+2n}}{(q^4;q^4)_n} = (zq^2;q^2)_\infty \sum_{n=0}^{\infty} \frac{z^n q^{n^2+n}}{(q^2;q^2)_n (zq^2;q^2)_n}$$

Proof.

Step I: RHS counts even weighted Fibonacci tilings.

1. Place n odd dominoes in positions $1, 3, 5, \ldots, 2n-1$.
2. Arbitrarily place even dominoes in positions $2n+2, 2n+4, 2n+6, \ldots$

$$z^n q^{2+4+\cdots+2n} \prod_{j \geq n+1} (1 - zq^{2j}) = (zq^2;q^2)_\infty \frac{z^n q^{n^2+n}}{(zq^2;q^2)_n}$$

3. Project the odd dominoes

Theorem (Rogers)

$$\sum_{n=0}^{\infty} \frac{z^{2n} q^{4n^2+2n}}{(q^4;q^4)_n} = (zq^2;q^2)_\infty \sum_{n=0}^{\infty} \frac{z^n q^{n^2+n}}{(q^2;q^2)_n (zq^2;q^2)_n}$$

Proof.

Step II: Cancel out terms via involution

Find first occurrence of:

Theorem (Rogers)

$$\sum_{n=0}^{\infty} \frac{z^{2n} q^{4n^2+2n}}{(q^4;q^4)_n} = (zq^2;q^2)_\infty \sum_{n=0}^{\infty} \frac{z^n q^{n^2+n}}{(q^2;q^2)_n (zq^2;q^2)_n}$$

Proof.

Step II: Cancel out terms via involution

Find first occurrence of:

and replace with:

and vice versa.

Theorem (Rogers)

$$\sum_{n=0}^{\infty} \frac{z^{2n} q^{4n^2+2n}}{(q^4;q^4)_n} = (zq^2;q^2)_\infty \sum_{n=0}^{\infty} \frac{z^n q^{n^2+n}}{(q^2;q^2)_n (zq^2;q^2)_n}$$

Proof.

Step II: Cancel out terms via involution

Find first occurrence of:

and replace with:

and vice versa.

Remaining tilings cannot have any even dominoes nor an odd number of consecutive odd dominoes.

Theorem (Rogers)

$$\sum_{n=0}^{\infty} \frac{z^{2n} q^{4n^2+2n}}{(q^4;q^4)_n} = (zq^2;q^2)_\infty \sum_{n=0}^{\infty} \frac{z^n q^{n^2+n}}{(q^2;q^2)_n (zq^2;q^2)_n}$$

Proof.

Step III: Count fixed points of involution

Fixed points consist of sequences of an even number of consecutive odd dominoes.

Theorem (Rogers)

$$\sum_{n=0}^{\infty} \frac{z^{2n} q^{4n^2+2n}}{(q^4;q^4)_n} = (zq^2;q^2)_\infty \sum_{n=0}^{\infty} \frac{z^n q^{n^2+n}}{(q^2;q^2)_n (zq^2;q^2)_n}$$

Proof.

Step III: Count fixed points of involution

Fixed points consist of sequences of an even number of consecutive odd dominoes.

1. Place $2n$ odd dominoes in positions $1, 3, 5, \ldots, 4n-1$.

$$z^{2n} q^{2+4+6+\cdots+4n} = z^{2n} q^{4n^2+2n}$$

2. Project the odd dominoes *in pairs*

$$\frac{1}{(1-q^4)(1-q^8)\cdots(1-q^{4n})} = \frac{1}{(q^4;q^4)_n}$$

$\square$

Signed Jacobsthal Tilings

Weight tiles in the following manner:

$$w(t) = \begin{cases} zq^i & \text{if } t \text{ is a } \text{[black tile]} \text{ in position } i \\ -zq^i & \text{if } t \text{ is a } \text{[grey tile]} \text{ in position } i \\ 1 & \text{if } t \text{ is a } \text{[white tile]} \text{ in position } i \end{cases}$$

Signed Jacobsthal Tilings

Weight tiles in the following manner:

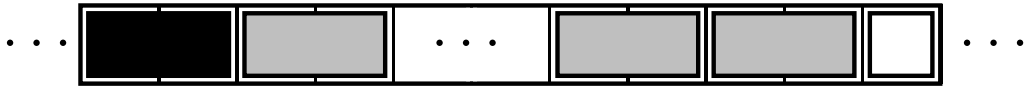
$$w(t) = \begin{cases} zq^i & \text{if } t \text{ is a } \\ -zq^i & \text{if } t \text{ is a } \\ 1 & \text{if } t \text{ is a } \end{cases}$$

in position i

in position i

in position i

Projectiles: black dominoes



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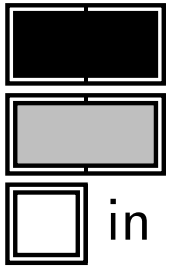

in position i

in position i

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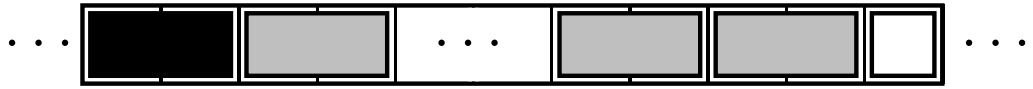
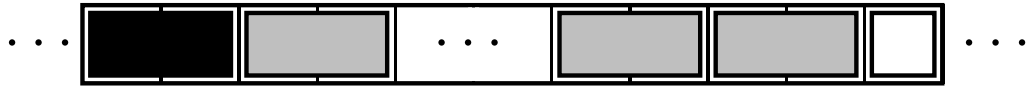
Projectiles: black dominoes





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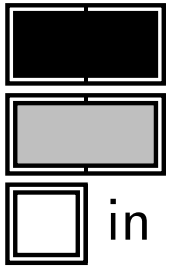
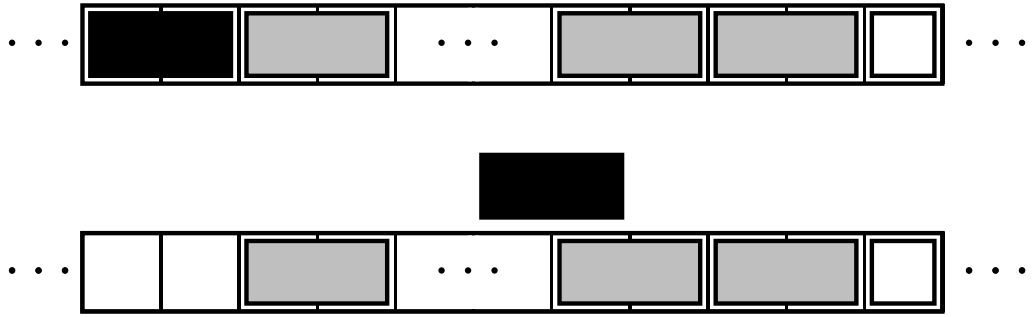


in position i

in position i

in position i

Projectiles: black dominoes



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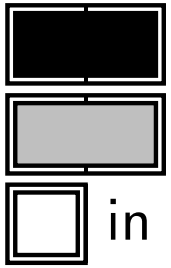
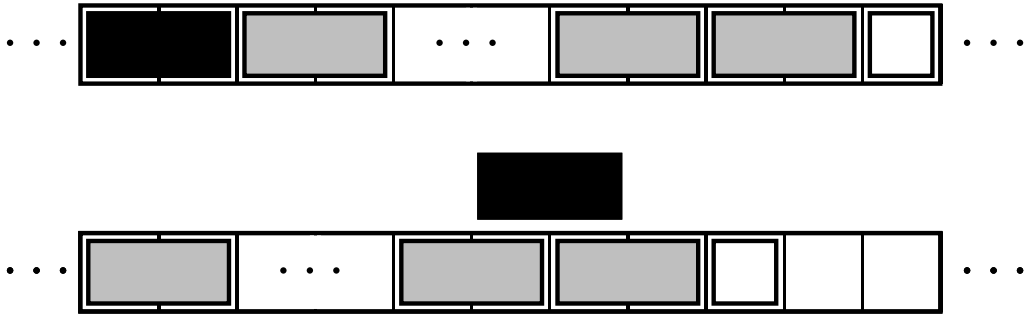


in position i

in position i

in position i

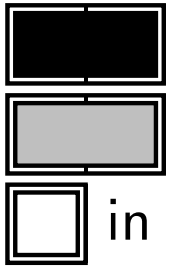
Projectiles: black dominoes



Signed Jacobsthal Tilings

Weight tiles in the following manner:

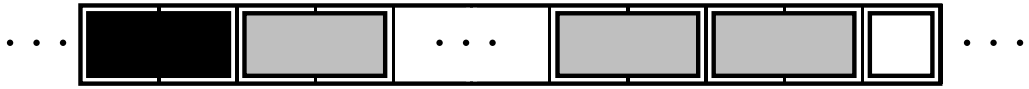
$$w(t) = \begin{cases} zq^i & \text{if } t \text{ is a } \\ -zq^i & \text{if } t \text{ is a } \\ 1 & \text{if } t \text{ is a } \end{cases}$$

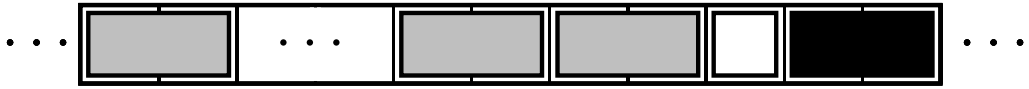

in position i

in position i

in position i

Projectiles: black dominoes





Signed Jacobsthal Tilings

Weight tiles in the following manner:

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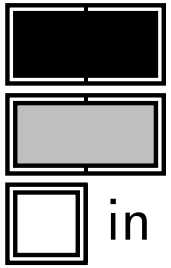
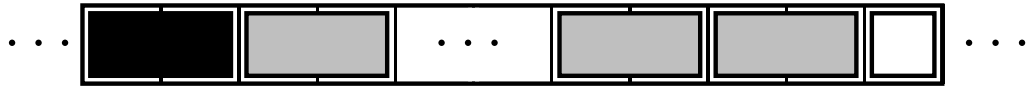


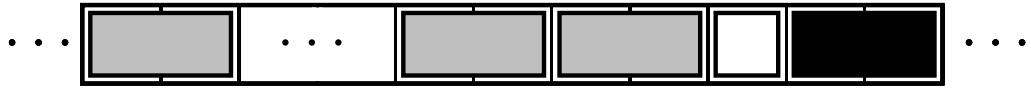
in position i

in position i

in position i

Projectiles: black dominoes





Projecting a black domino increases the weight of a tiling by a factor of q.

Theorem (Rogers)

$$\sum_{n=0}^{\infty} \frac{z^n q^{2n^2+n}}{(q^2;q^2)_n} = (zq^2;q^2)_\infty \sum_{n=0}^{\infty} \frac{z^n q^{(3n^2+n)/2}}{(q;q)_n (zq^2;q^2)_n}$$

Proof.

Step I: RHS counts signed Jacobsthal tilings where each black domino is preceded by a white square and there are no odd gray dominoes.

1. Place n black dominoes in positions $1, 3, 5, \ldots, 2n-1$.
2. Arbitrarily place even gray dominoes in positions $2j$ for $j \geq n+1$
3. Project the ith black domino exactly i times.

$$z^n q^{1+3+\cdots+2n-1} q^{\binom{n+1}{2}} \prod_{j \geq n+1} (1 - zq^{2j}) = (zq^2;q^2)_\infty \frac{z^n q^{(3n^2+n)/2}}{(zq^2;q^2)_n}$$

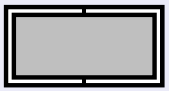
4. Project the odd dominoes

Theorem (Rogers)

$$\sum_{n=0}^{\infty} \frac{z^n q^{2n^2+n}}{(q^2;q^2)_n} = (zq^2;q^2)_\infty \sum_{n=0}^{\infty} \frac{z^n q^{(3n^2+n)/2}}{(q;q)_n (zq^2;q^2)_n}$$

Proof.

Step II: Cancel out terms via involution

Find first occurrence of an even domino: or 

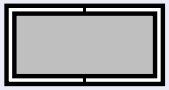
Theorem (Rogers)

$$\sum_{n=0}^{\infty} \frac{z^n q^{2n^2+n}}{(q^2;q^2)_n} = (zq^2;q^2)_\infty \sum_{n=0}^{\infty} \frac{z^n q^{(3n^2+n)/2}}{(q;q)_n (zq^2;q^2)_n}$$

Proof.

Step II: Cancel out terms via involution

Find first occurrence of an even domino: or

and replace with:  or (respectively)

Remaining tilings cannot have any even dominoes.

Theorem (Rogers)

$$\sum_{n=0}^{\infty} \frac{z^n q^{2n^2+n}}{(q^2;q^2)_n} = (zq^2;q^2)_\infty \sum_{n=0}^{\infty} \frac{z^n q^{(3n^2+n)/2}}{(q;q)_n (zq^2;q^2)_n}$$

Proof.

Step III: Count fixed points of involution

Fixed points consist of white squares and odd black dominoes.

1. Place n odd black dominoes in positions $3, 7, 11 \ldots, 4n-1$.

$$z^n q^{3+7+11+\cdots+4n-1} = z^n q^{2n^2+n}$$

2. Project the odd dominoes. Each one must be projected an even number of times.

$\square$

Other Weight Functions

Weight tiles in the following manner:

$$w(t) = \begin{cases} aq^i & \text{if } t \text{ is a } \blacksquare \text{ with } i \text{ non-black squares to its left} \\ bq^i & \text{if } t \text{ is a } \square \text{ with } i \text{ non-black squares to its left} \\ 1 & \text{if } t \text{ is a } \square \text{ in position } i \end{cases}$$

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Theorem (Cauchy)

$$\sum_{n=0}^{\infty} \frac{(-b/a;q)_n a^n}{(q;q)_n} = \prod_{n=0}^{\infty} \frac{1+bq^n}{1-aq^n}$$

Weight tiles in the following manner:

$$w(t) = \begin{cases} aq^i & \text{if } t \text{ is a } \blacksquare \text{ in position } i \\ cq^i & \text{if } t \text{ is a } \bullet \text{ in position } i \\ abq^i & \text{if } t \text{ is a } \square \text{ with } i \text{ white tiles to its left} \\ bcq^i & \text{if } t \text{ is a } \bigcirc \text{ with } i \text{ white tiles to its left} \\ -cq^i & \text{if } t \text{ is a } \bigcirc \text{ in position } i \\ 1 & \text{if } t \text{ is a } \square \text{ in position } i. \end{cases}$$

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Theorem (Heine) q-analog of Gauss's Theorem:

$$(cq;q)_\infty \sum_{n=0}^{\infty} \frac{(-c/a;q)_n(-q/b;q)_n a^n b^n}{(q;q)_n(cq;q)_n} = \prod_{i=1}^{\infty} \frac{(1+bcq^{n-1})(1+aq^n)}{(1-abq^{n-1})}$$

Labeled Tilings

Weight tiles in the following manner:

$$w(t) = \begin{cases} a_j z q^i & \text{if } t \text{ is a } \boxed{j} \text{ in position } i,\ 1 \leq j \leq k+1 \\ z_j q^i & \text{if } t \text{ is a } \textcircled{j} \text{ in position } i,\ 1 \leq j \leq k \\ 1 & \text{if } t \text{ is a } \square \text{ in position } i \end{cases}$$

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Theorem

The following function is the generating function for labeled tilings that consist of weakly increasing sequences of weighted tiles followed by a single white square where the label must strictly increase after a circle.

$$(-a_{k+1}q)_\infty \sum_{n_1,\ldots,n_k \ge 0} \frac{(-z_1/a_1)_{n_1} \cdots (-z_k/a_k)_{n_k} a_1^{n_1} \cdots a_k^{n_k}}{(q)_{n_1} \cdots (q)_{n_k} (-a_{k+1}q)_{n_1+\cdots+n_k}} q^{\binom{n_1+\cdots+n_k+1}{2}}$$

Labeled Tilings

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Furthermore, this function is symmetric in the variables $(a_1, \ldots, a_{k+1})$ *as well as the variables* $(z_1, \ldots, z_k)$.