Recent Progress on Tiling Proofs of q–Series Identities

David P. Little

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Weighted Tilings

Definition

A tiling is a covering of an infinitely long board:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

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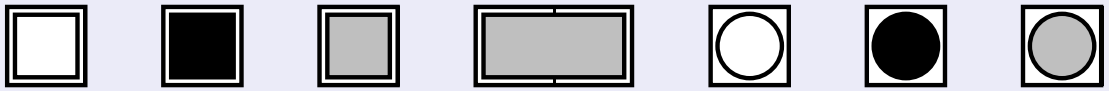

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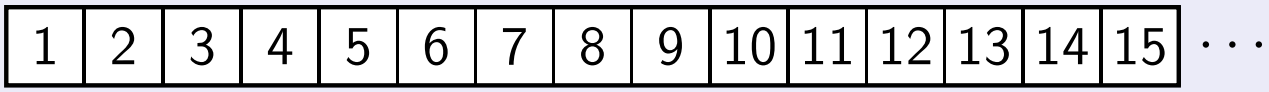

using different types of tiles:

The weight of a tiling T is given by

$$w(T) = \prod_{t \in T} w(t)$$

where $w(t)$ is the weight of the tile t. The weight of a white square will always be 1. Each tiling will have a finite number of weighted tiles.

Lebesgue Identities

The weight of tile t:

$$w(t) = \begin{cases} q^i & \text{if } t \text{ is a } \blacksquare \text{ in position } i \\ zq^i & \text{if } t \text{ is a } \boxed{} \text{ in position } i \\ 1 & \text{if } t \text{ is a } \square \text{ in position } i \end{cases}$$

Theorem

$$\sum_{n=0}^{\infty} \frac{(-z;q)_n}{(q;q)_n} q^{\binom{n+1}{2}} = \prod_{n=1}^{\infty} (1+q^n)(1+zq^{2n-1})$$

where $(z;q)_n = (1-z)(1-zq)\cdots(1-zq^{n-1})$.

Eight Identities of Rogers

$$\sum_{n\geq 0}\frac{z^n q^{n^2}}{(q;q)_n}=(-zq^2;q^2)_\infty\sum_{n\geq 0}\frac{z^n q^{n^2}}{(q^2;q^2)_n(-zq^2;q^2)_n}$$

$$\sum_{n\geq 0}\frac{z^n q^{n^2+n}}{(q;q)_n}=(-zq^2;q^2)_\infty\sum_{n\geq 0}\frac{z^n q^{n^2+2n}}{(q^2;q^2)_n(-zq^2;q^2)_n}$$

$$\sum_{n\geq 0}\frac{z^{2n} q^{4n^2+2n}}{(q^4;q^4)_n}=(zq^2;q^2)_\infty\sum_{n\geq 0}\frac{z^n q^{n^2+n}}{(q^2;q^2)_n(zq^2;q^2)_n}$$

$$\sum_{n\geq 0}\frac{z^n q^{2n^2+n}}{(q^2;q^2)_n}=(zq^2;q^2)_\infty\sum_{n\geq 0}\frac{z^n q^{(3n^2+n)/2}}{(q;q)_n(zq^2;q^2)_n}$$

Five More Identities

$$\sum_{n\geq 0} \frac{z^n q^{2n^2}}{(q^2;q^2)_n} = (zq;q^2)_\infty \sum_{n\geq 0} \frac{z^n q^{(3n^2+n)/2}}{(q;q)_n (zq;q^2)_{n+1}}$$

$$\sum_{n\geq 0} \frac{z^n q^{n^2}}{(q;q)_n} = (-zq;q)_\infty \sum_{n\geq 0} \frac{(-1)^n z^{2n} q^{3n^2}}{(q^2;q^2)_n (-zq;q)_{2n}}$$

$$\sum_{n\geq 0} \frac{z^n q^{n^2+n}(1-z^2q^{2n+3})}{(q;q)_n} = (-zq;q)_\infty \sum_{n\geq 0} \frac{(-1)^n z^{2n} q^{3n^2}}{(q^2;q^2)_n (-zq;q)_{2n+1}}$$

An Example

The weight of tile t:

$$w(t) = \begin{cases} -zq^i & \text{if } t \text{ is a } \blacksquare \text{ in position } i \\ zq^i & \text{if } t \text{ is a } \boxed{} \text{ in position } i \\ 1 & \text{if } t \text{ is a } \square \text{ in position } i \end{cases}$$

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Theorem (Cauchy)

$$\sum_{n=0}^{\infty} \frac{z^n q^{n^2}}{(q;q)_n (zq;q)_n} = \frac{1}{(zq;q)_\infty}$$

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Proof. Construct tilings in the following manner:

STEP 1: Place n in positions $1, 3, 5, \ldots, 2n-1$.

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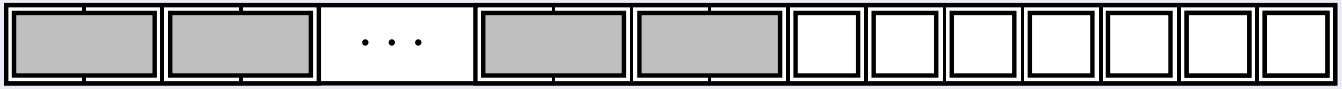

This accounts for a weight of

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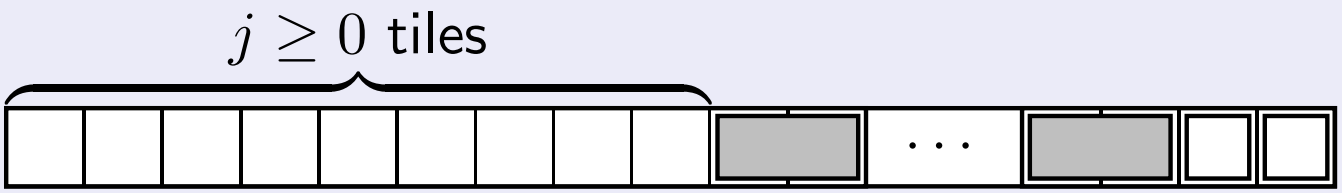
STEP 2: Pick $j \geq 0$ and insert ■ immediately after the jth tile.



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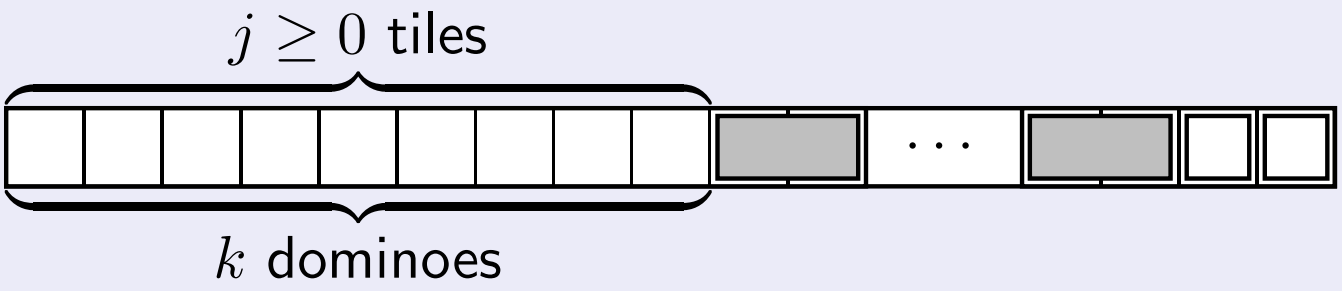

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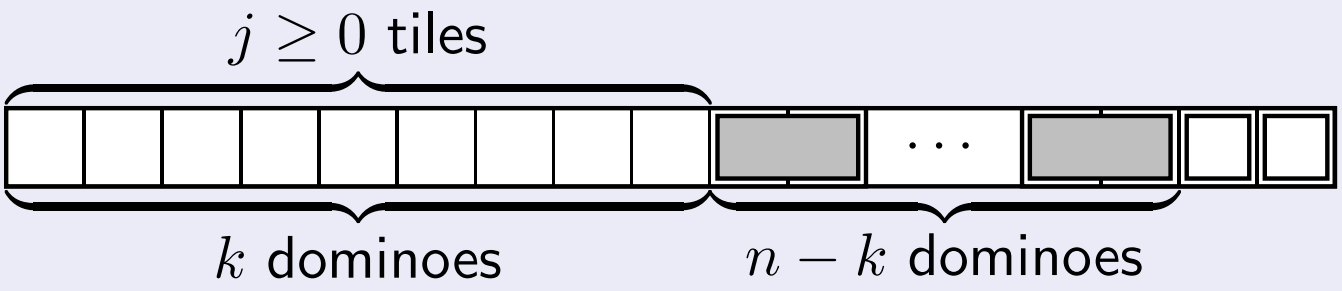

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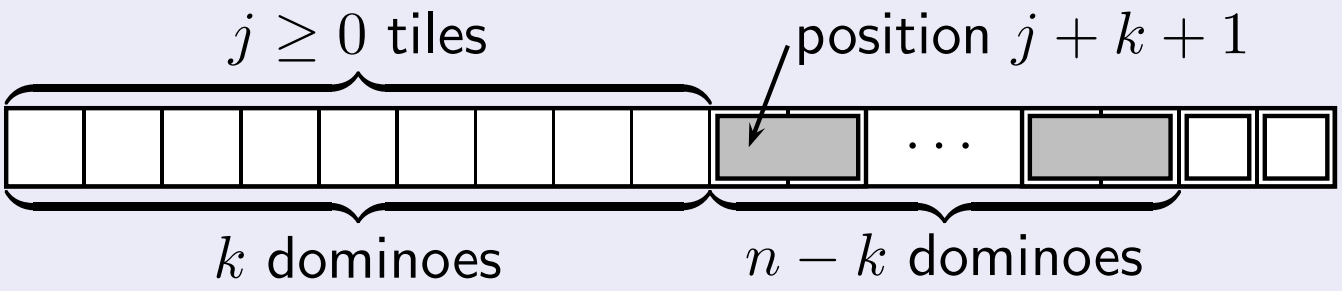

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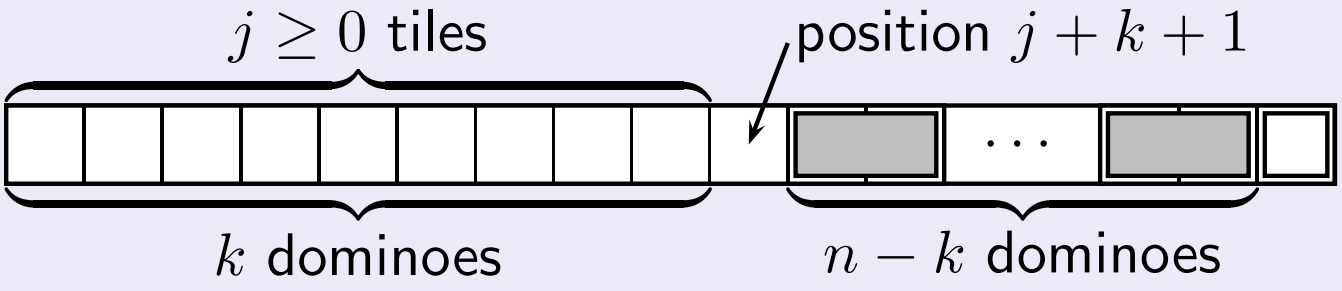

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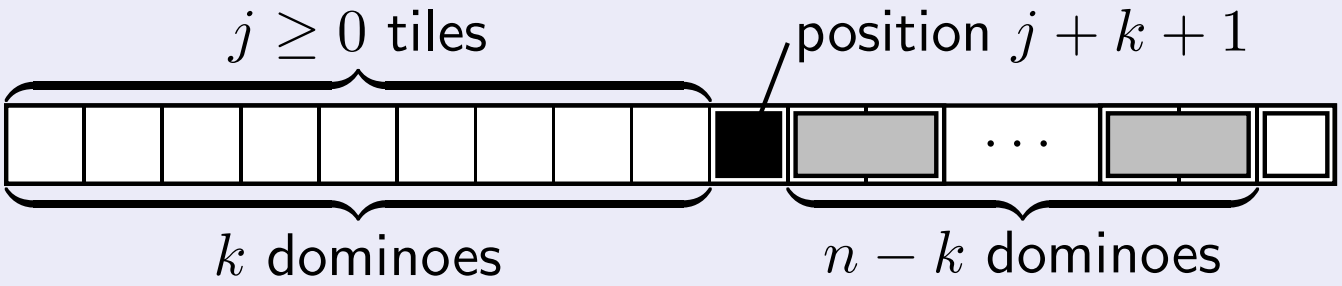

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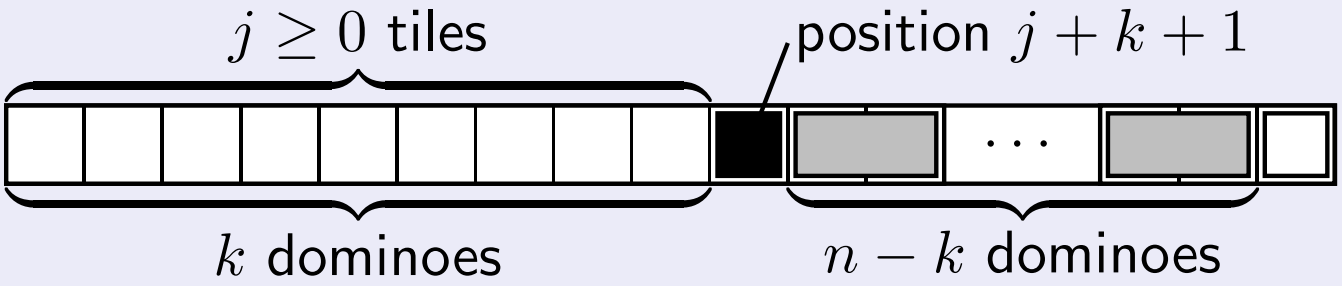

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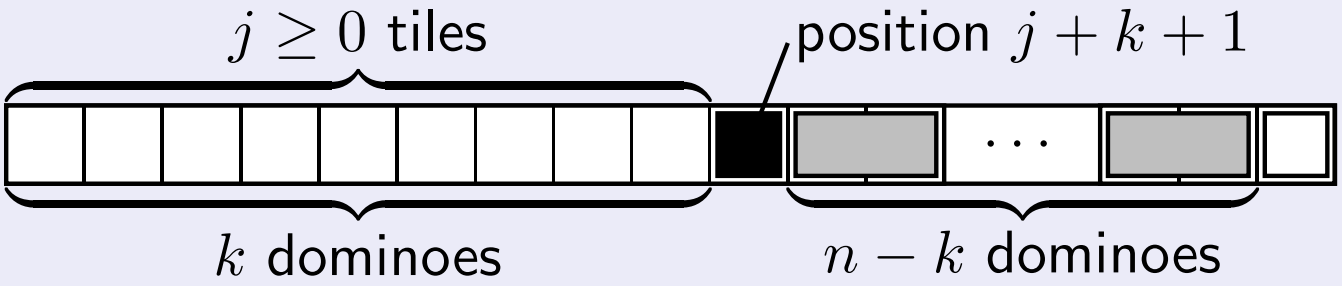

Increases the weight of the tiling by $-zq^{j+k+1}q^{n-k} = -zq^{n+j+1}$

$$\prod_{j\geq 0}(1 - zq^{n+j+1}) = \frac{(zq;q)_\infty}{(zq;q)_n}$$

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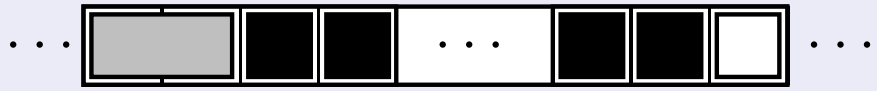
$$(zq;q)_\infty \sum_{n=0}^{\infty} \frac{z^n q^{n^2}}{(q;q)_n (zq;q)_n} = 1$$

Proof. Construct tilings in the following manner:

STEP 3: Project the dominoes



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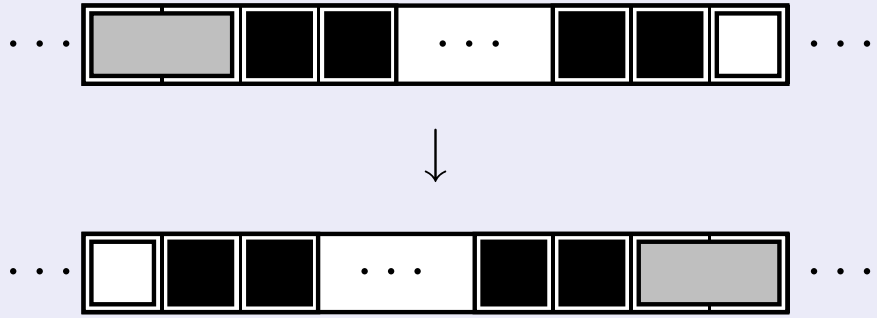

Increases the weight of the tiling by a factor of q.

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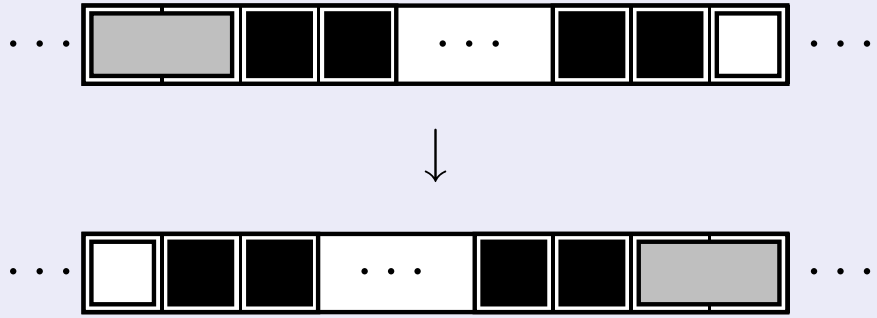

Increases the weight of the tiling by a factor of q. Therefore, the left–hand side is the generating function for all weighted tilings.

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When projecting tiles, always work in a right to left, weakly increasing manner. In other words, make sure that each domino is projected at least as many times as the dominoes to its left are projected.

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This process accounts for a weight of

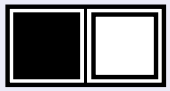
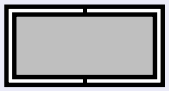
$$\frac{1}{(1-q)(1-q^2)(1-q^3)\cdots(1-q^n)} = \frac{1}{(q;q)_n}$$

Theorem (Cauchy)

$$(zq;q)_\infty \sum_{n=0}^{\infty} \frac{z^n q^{n^2}}{(q;q)_n (zq;q)_n} = 1$$

Proof. Cancel out all non-empty tilings:

STEP 4:

Find first occurrence of:  or 

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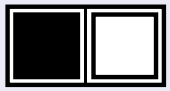
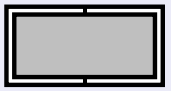
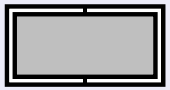
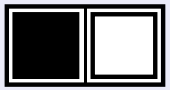
Find first occurrence of:  or 

and replace with:  or  (respectively)

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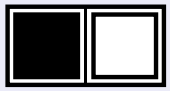
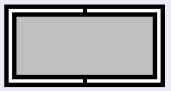
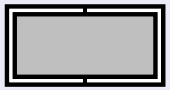
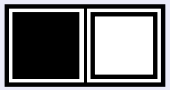
Find first occurrence of:  or 

and replace with:  or  (respectively)

The only remaining tiling is the empty tiling, which has weight 1.

q-ANALOG OF THE BINOMIAL SERIES

Weight tiles in the following manner:

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THEOREM (Cauchy)

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STEP 1: Place n black or gray squares in positions $1, 2, 3, \ldots, n$.

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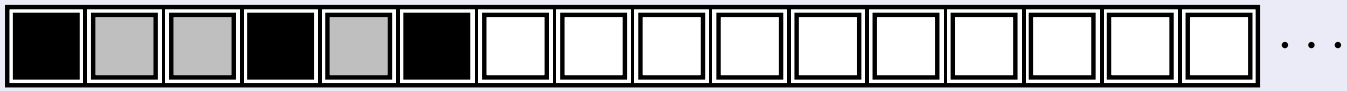
STEP 1: Place n black or gray squares in positions $1, 2, 3, \ldots, n$.



A ■ in position i accounts for a weight of a.

A □ in position i accounts for a weight of bq^{n-i}.

This process accounts for a weight of

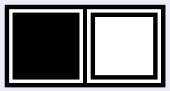
$$\prod_{i=1}^{n} (a + bq^{n-i}) = (-b/a;q)_n a^n$$

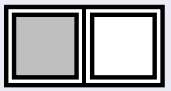
Theorem (Cauchy)

$$\sum_{n=0}^{\infty} \frac{(-b/a;q)_n a^n}{(q;q)_n} = \prod_{n=0}^{\infty} \frac{1+bq^n}{1-aq^n}$$

Proof. PART I: Interpret infinite series

STEP 2: Project the tiles.





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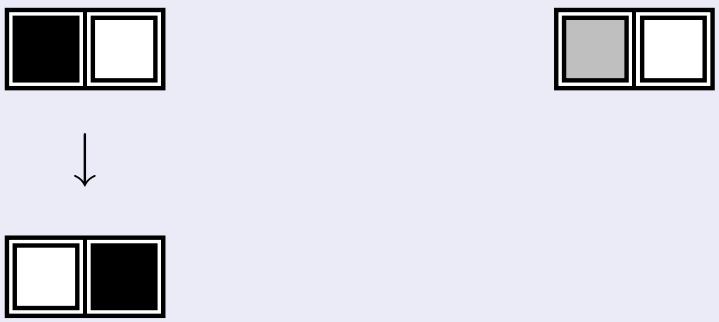

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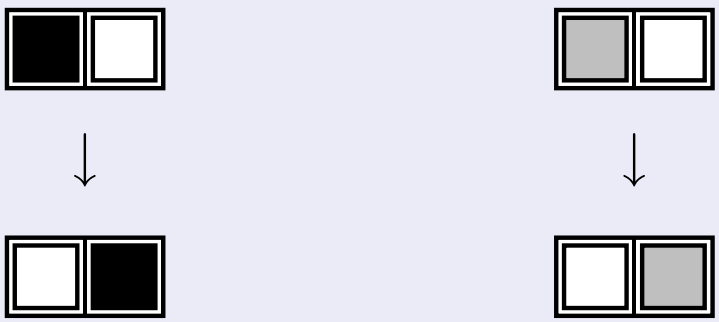


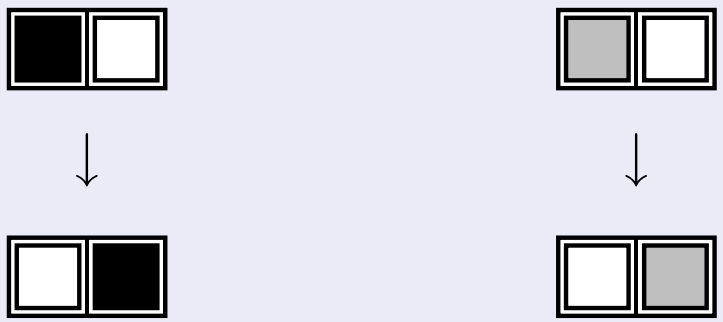
This process increases the weight of the tiling by a factor of q.

Theorem (Cauchy)

$$\sum_{n=0}^{\infty} \frac{(-b/a;q)_n a^n}{(q;q)_n} = \prod_{n=0}^{\infty} \frac{1+bq^n}{1-aq^n}$$

Proof. PART I: Interpret infinite series

STEP 2: Project the tiles.



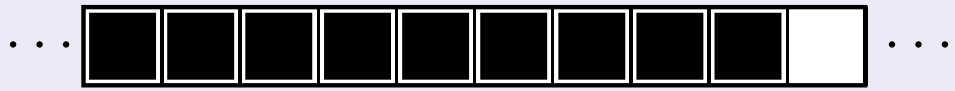
This process increases the weight of the tiling by a factor of q. Therefore, the left-hand side is the generating function for all weighted tilings.

Theorem (Cauchy)

$$\sum_{n=0}^{\infty} \frac{(-b/a;q)_n a^n}{(q;q)_n} = \prod_{n=0}^{\infty} \frac{1+bq^n}{1-aq^n}$$

Proof. PART II: Interpret infinite product

Each tiling can be broken up into segments:



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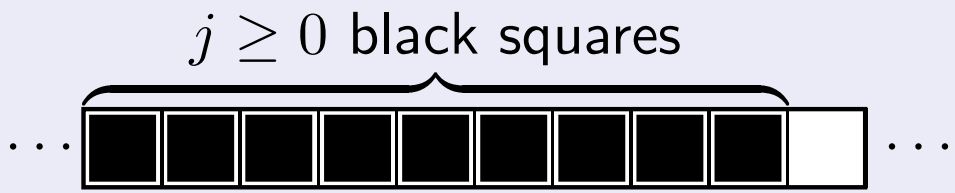

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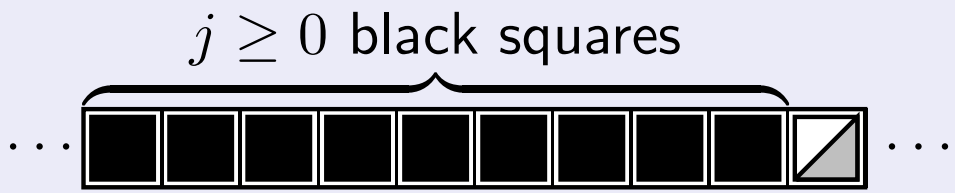

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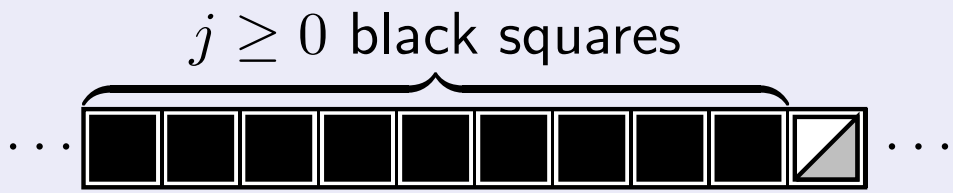

The weight of the nth segment for $n \geq 0$ is given by

$$(1+bq^n)\sum_{j=0}^{\infty} a^j q^{nj} = \frac{1+bq^n}{1-aq^n}$$

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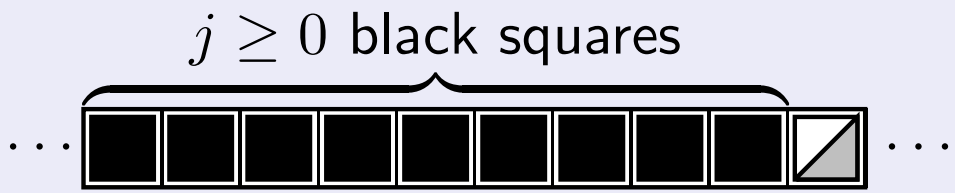

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Multiplying over $n \geq 0$ completes the construction.

A Few Observations

The generating function for all weighted tilings is given by

$$\sum_{n=0}^{\infty} \frac{(-b/a;q)_n a^n}{(q;q)_n}$$

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Adding the parameter c in the following manner allows us to count number of white squares before the last weighted tile.

$$\sum_{n=0}^{\infty} \frac{(-b/a;q)_n a^n}{(cq;q)_n}$$

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$$\sum_{n=0}^{\infty} \frac{(-b/a;q)_n a^n}{(cq;q)_n}$$

Multiplication by a produces the generating function for tilings that start with a black square.

$$\sum_{n=0}^{\infty} \frac{(-b/a;q)_n a^{n+1}}{(cq;q)_n}$$

Rogers-Fine Identity

Theorem

$$\sum_{n=0}^{\infty} \frac{(-b/a;q)_n a^n}{(cq;q)_n} = \sum_{n=0}^{\infty} \frac{(-b/a;q)_n(-b/c;q)_n a^n c^n q^{n^2}(1+bq^{2n})}{(a;q)_{n+1}(cq;q)_n}$$

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Multiplying both sides by $(1-a)$ yields:

$$(1-a)\sum_{n=0}^{\infty} \frac{(-b/a;q)_n a^n}{(cq;q)_n} = \sum_{n=0}^{\infty} \frac{(-b/a;q)_n(-b/c;q)_n a^n c^n q^{n^2}(1+bq^{2n})}{(aq;q)_n(cq;q)_n}$$

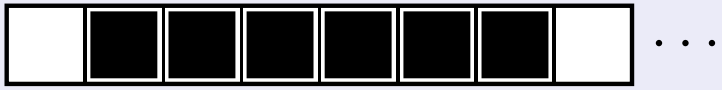
In this form, the left hand side counts tilings that do not start with a black square where the power of c keeps track of the number of white squares before the last weighted tile.

Theorem (Rogers-Fine)

$$(1-a)\sum_{n=0}^{\infty}\frac{(-b/a;q)_n a^n}{(cq;q)_n}=\sum_{n=0}^{\infty}\frac{(-b/a;q)_n(-b/c;q)_n a^n c^n q^{n^2}(1+bq^{2n})}{(aq;q)_n(cq;q)_n}$$

Proof. Interpret right–hand side

Front segments:



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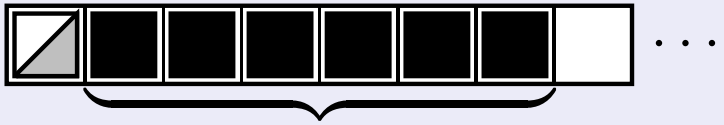

$j \geq 0$ black squares

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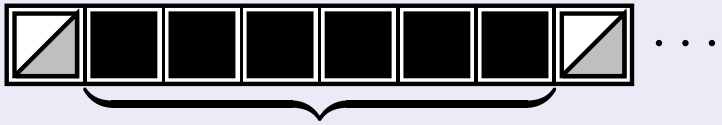

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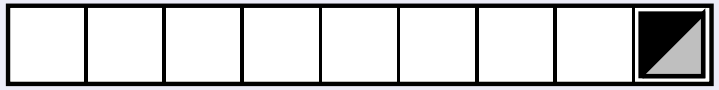

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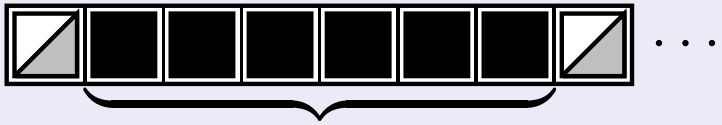

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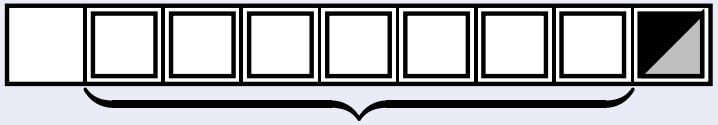

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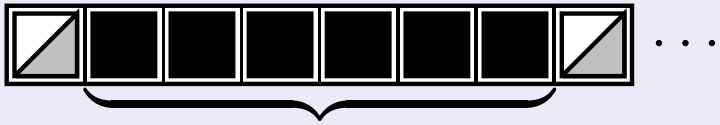

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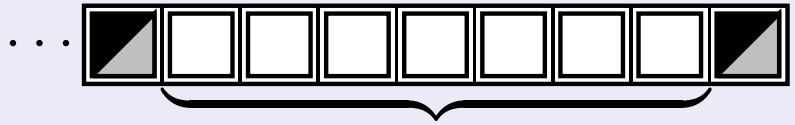

$k \geq 0$ white squares

The center of a tiling marks the transition between front segments and back segments. The center can either be empty or a gray square.

Theorem (Rogers-Fine)

$$(1-a)\sum_{n=0}^{\infty}\frac{(-b/a;q)_n a^n}{(cq;q)_n}=\sum_{n=0}^{\infty}\frac{(-b/a;q)_n(-b/c;q)_n a^n c^n q^{n^2}(1+bq^{2n})}{(aq;q)_n(cq;q)_n}$$

Proof. Interpret right–hand side

Tilings that start with □ or ■ and have n front/back segments.

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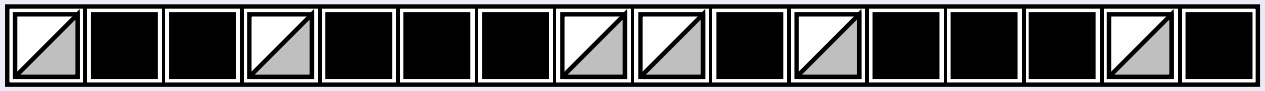
Proof. Interpret right–hand side

Tilings that start with □ or ■ and have n front/back segments.
Generating function for n front segments:



$$\frac{(c+b)}{(1-aq)}\frac{(c+bq)}{(1-aq^2)}\cdots\frac{(c+bq^{n-1})}{(1-aq^n)}=\frac{(-b/c;q)_n c^n}{(aq;q)_n}$$

Theorem (Rogers-Fine)

$$(1-a)\sum_{n=0}^{\infty}\frac{(-b/a;q)_n a^n}{(cq;q)_n}=\sum_{n=0}^{\infty}\frac{(-b/a;q)_n(-b/c;q)_n a^n c^n q^{n^2}(1+bq^{2n})}{(aq;q)_n(cq;q)_n}$$

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Tilings that start with □ or ■ and have n front/back segments.
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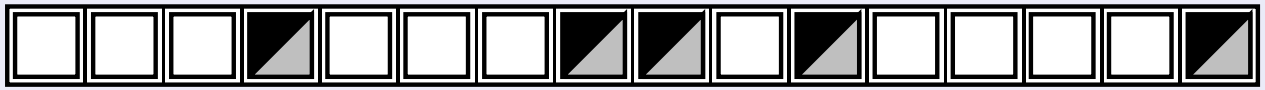

$$\frac{(aq^n+bq^{2n-1})}{(1-cq^n)}\cdots\frac{(aq^n+bq^{n+1})}{(1-cq^2)}\frac{(aq^n+bq^n)}{(1-cq)}=\frac{(-b/a;q)_n a^n q^{n^2}}{(cq;q)_n}$$

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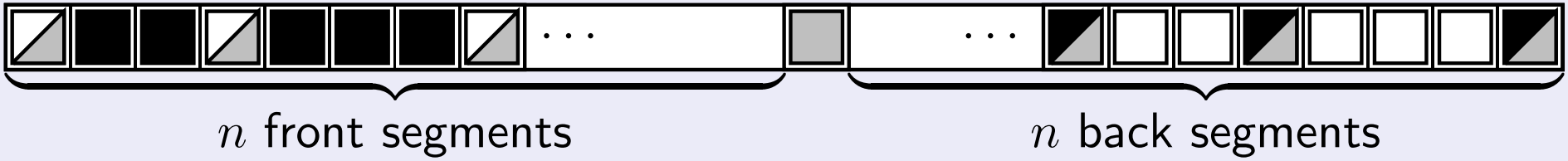
Proof. Interpret right–hand side

The center can either be empty or a gray square.



If the center is a gray square, then it has weight bq^n and increases the weight of the back segments by q^n.

Theorem (Rogers-Fine)

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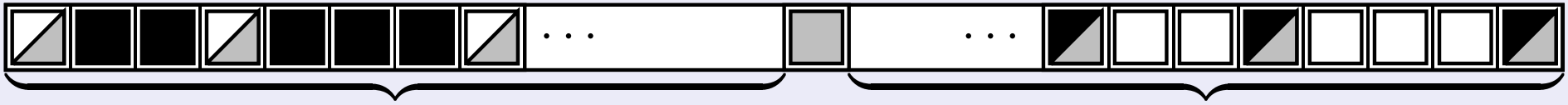

If the center is a gray square, then it has weight bq^n and increases the weight of the back segments by q^n.
In other words, the factor $(1+bq^{2n})$ represents the choice of the center.

Recall:

Theorem (Cauchy)

$$\sum_{n=0}^{\infty} \frac{(-b/a;q)_n a^n}{(q;q)_n} = \prod_{n=0}^{\infty} \frac{1+bq^n}{1-aq^n}$$

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Theorem

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$$\sum_{n=0}^{\infty} \frac{(a+c)(a+cq)\cdots(a+cq^{n-1})q^{\binom{n+1}{2}}}{(q;q)_n}(1-cq^{n+1})(1-cq^{n+2})\cdots$$

Theorem

$$(cq;q)_\infty \sum_{n=0}^{\infty} \frac{(-c/a;q)_n a^n q^{\binom{n+1}{2}}}{(q;q)_n (cq;q)_n} = \prod_{n=1}^{\infty} (1+aq^n)$$

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Weight tiles in the following manner:

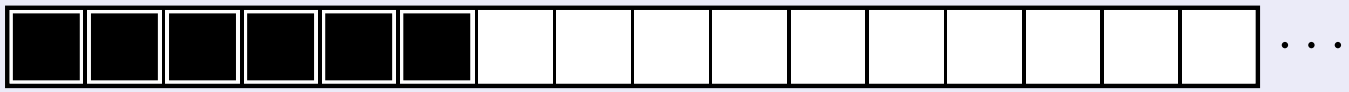
$$w(t) = \begin{cases} aq^i & \text{if } t \text{ is a } \blacksquare \text{ in position } i \\ cq^i & \text{if } t \text{ is a } \bullet \text{ in position } i \\ 1 & \text{if } t \text{ is a } \square \\ -cq^i & \text{if } t \text{ is a } \bigcirc \text{ in position } i \end{cases}$$

Theorem

$$(cq;q)_\infty \sum_{n=0}^{\infty} \frac{(-c/a;q)_n a^n q^{\binom{n+1}{2}}}{(q;q)_n (cq;q)_n} = \prod_{n=1}^{\infty} (1+aq^n)$$

Proof. PART I: Interpret infinite series

STEP 1: Place ■ in positions $1, 2, 3, \ldots, n$.



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This process accounts for a weight of

$$a^n q^{1+2+\cdots+n} = a^n q^{\binom{n+1}{2}}$$

Theorem

$$(cq;q)_\infty \sum_{n=0}^{\infty} \frac{(-c/a;q)_n a^n q^{\binom{n+1}{2}}}{(q;q)_n (cq;q)_n} = \prod_{n=1}^{\infty} (1+aq^n)$$

Proof. PART I: Interpret infinite series

STEP 2: Place $\square$ or $\bigcirc$ in positions $i > n$.

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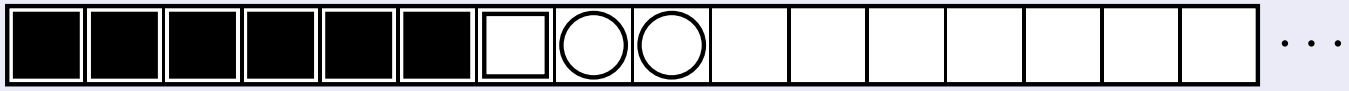

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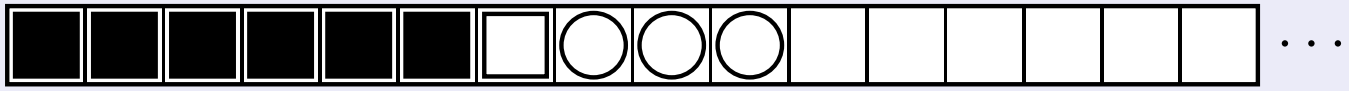

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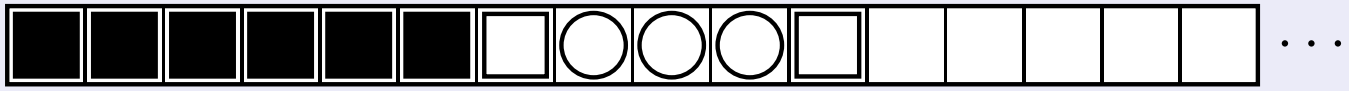

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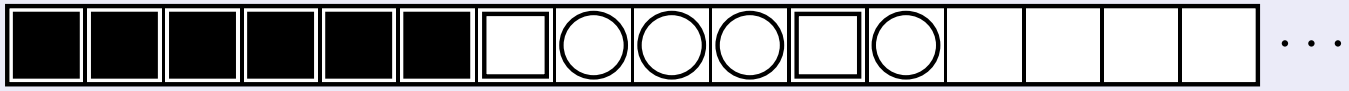

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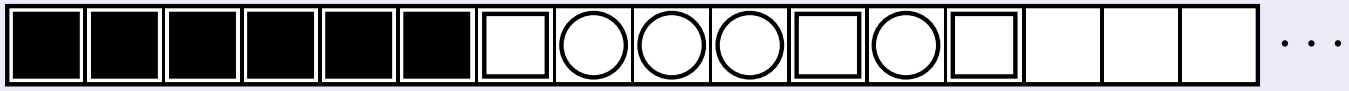

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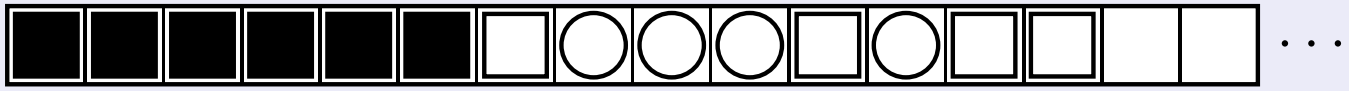

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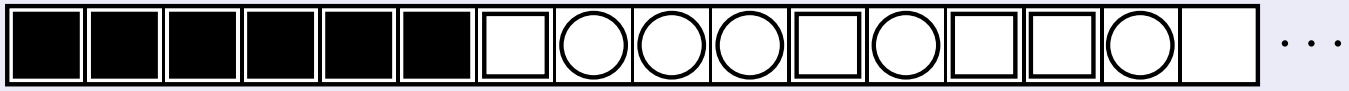

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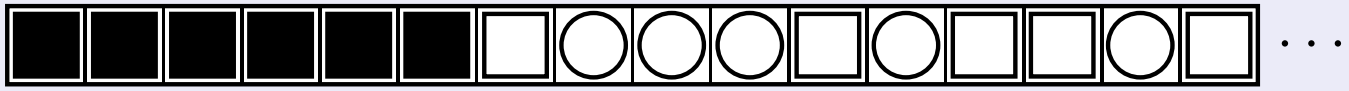


A □ in position i accounts for a weight of 1.
A ○ in position i accounts for a weight of $-cq^i$.
This process accounts for a weight of

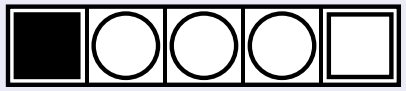
$$\prod_{i>n} (1-cq^i) = \frac{(cq;q)_\infty}{(cq;q)_n}$$

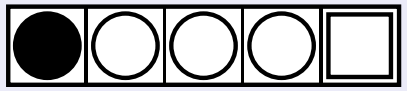
Theorem

$$(cq;q)_\infty \sum_{n=0}^{\infty} \frac{(-c/a;q)_n a^n q^{\binom{n+1}{2}}}{(q;q)_n (cq;q)_n} = \prod_{n=1}^{\infty} (1+aq^n)$$

Proof. PART I: Interpret infinite series

STEP 3: Project the black tiles.





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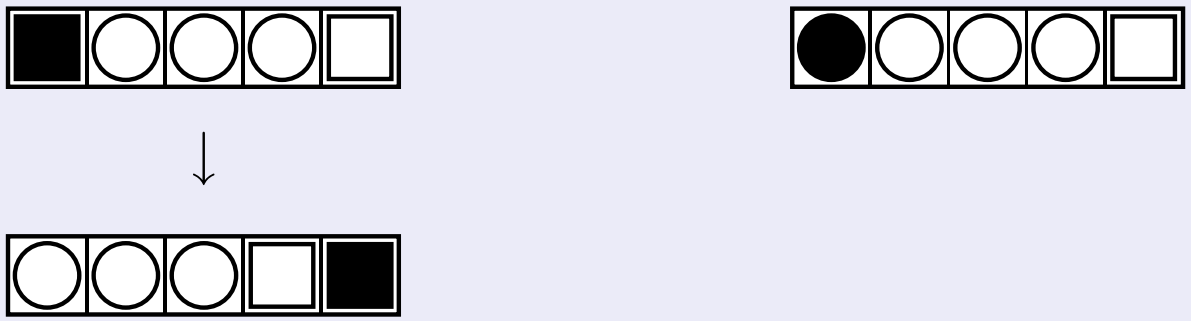

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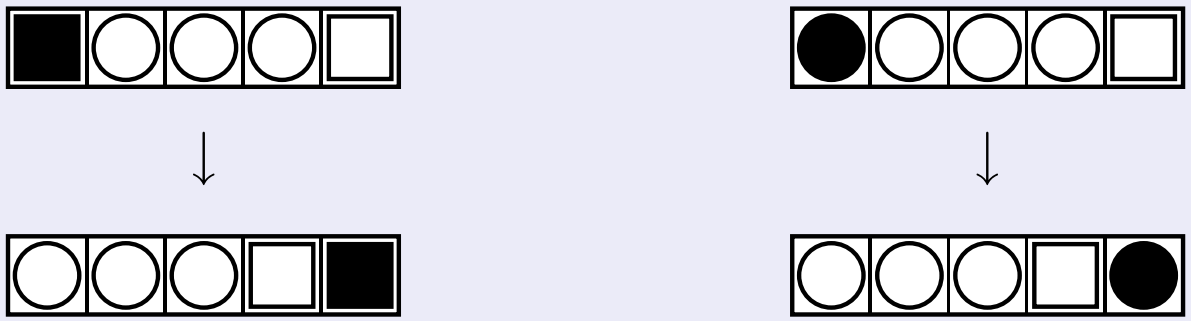


This process increases the weight of the tiling by a factor of q.

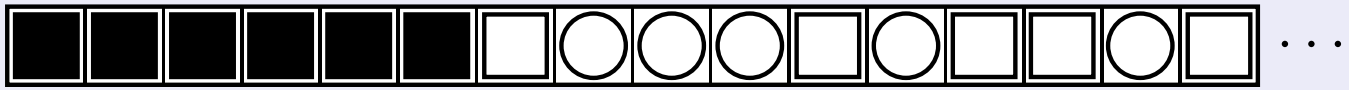
Theorem

$$(cq;q)_\infty \sum_{n=0}^{\infty} \frac{(-c/a;q)_n a^n q^{\binom{n+1}{2}}}{(q;q)_n (cq;q)_n} = \prod_{n=1}^{\infty} (1+aq^n)$$

Proof. PART I: Interpret infinite series

STEP 3a: Convert ■ into ●.

If a square is converted into a circle, project each of the tiles to its right.



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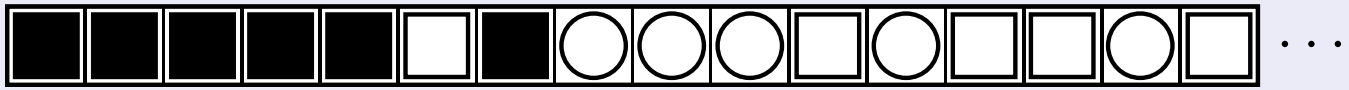

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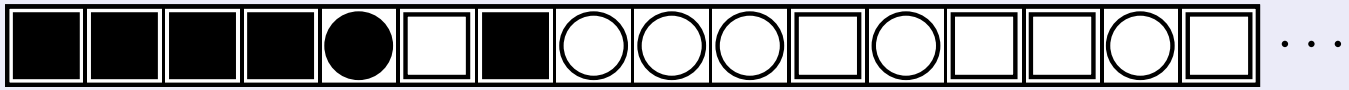

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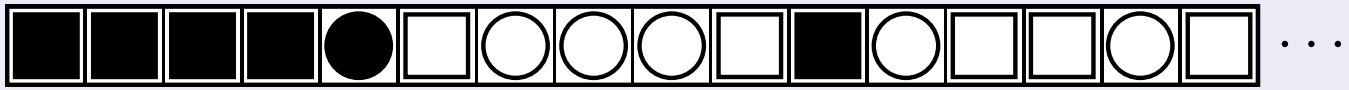

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Proof. PART I: Interpret infinite series

STEP 3a: Convert ■ into ●.
If a square is converted into a circle, project each of the tiles to its right.

The factor $(1+cq^{n-i}/a)$ represents the choice of converting the ith square into a circle.

$$\prod_{i=1}^{n} (1+cq^{n-i}/a) = (-c/a;q)_n$$

Theorem

$$(cq;q)_\infty \sum_{n=0}^{\infty} \frac{(-c/a;q)_n a^n q^{\binom{n+1}{2}}}{(q;q)_n (cq;q)_n} = \prod_{n=1}^{\infty} (1+aq^n)$$

Proof. PART I: Interpret infinite series

STEP 3b: Project the black tiles.

Constructs all tilings where every circle is followed by a white tile.

Theorem

$$(cq;q)_\infty \sum_{n=0}^{\infty} \frac{(-c/a;q)_n a^n q^{\binom{n+1}{2}}}{(q;q)_n (cq;q)_n} = \prod_{n=1}^{\infty} (1+aq^n)$$

Proof. PART II: Interpret infinite product

Cancel out any tilings with a ○ or ●

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$$(cq;q)_\infty \sum_{n=0}^{\infty} \frac{(-c/a;q)_n a^n q^{\binom{n+1}{2}}}{(q;q)_n (cq;q)_n} = \prod_{n=1}^{\infty} (1+aq^n)$$

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Cancel out any tilings with a ○ or ●

Find first occurrence of: $\boxed{\bigcirc}$ or $\boxed{\bullet}$

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Remaining tilings cannot have any circles.

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and replace with: $\boxed{\bullet}$ or $\boxed{\bigcirc}$ (respectively)

Remaining tilings cannot have any circles.

Therefore, each tiling can be constructed by simply deciding whether or not to place a black square in each position $n \geq 1$.

q-analog of Gauss's Theorem

Weight tiles in the following manner:

$$w(t) = \begin{cases} aq^i & \text{if } t \text{ is a } \blacksquare \text{ in position } i \\ cq^i & \text{if } t \text{ is a } \bullet \text{ in position } i \\ abq^i & \text{if } t \text{ is a } \blacksquare \text{ with } i \ \square \text{ or } \bigcirc \text{ to its left} \\ bcq^i & \text{if } t \text{ is a } \bullet \text{ with } i \ \square \text{ or } \bigcirc \text{ to its left} \\ 1 & \text{if } t \text{ is a } \square \\ -cq^i & \text{if } t \text{ is a } \bigcirc \text{ in position } i \end{cases}$$

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Theorem (Heine)

$$(cq;q)_\infty \sum_{n=0}^{\infty} \frac{(-c/a;q)_n(-q/b;q)_n a^n b^n}{(q;q)_n(cq;q)_n} = \prod_{n=1}^{\infty} \frac{(1+bcq^{n-1})(1+aq^n)}{(1-abq^{n-1})}$$

q-analog of Kummer's Theorem

Weight tiles in the following manner:

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q-analog of Kummer's Theorem

Weight tiles in the following manner:

$$w(t) = \begin{cases} q^i & \text{if } t \text{ is a } \blacksquare \text{ in position } i \\ cq^i & \text{if } t \text{ is a } \bullet \text{ in position } i \\ bq^i & \text{if } t \text{ is a } \blacksquare \text{ with } i \ \square \text{ or } \bigcirc \text{ to its left} \\ bcq^i & \text{if } t \text{ is a } \bullet \text{ with } i \ \square \text{ or } \bigcirc \text{ to its left} \\ -bcq^i & \text{if } t \text{ is a } \bigcirc \text{ in position } i+1 \\ 1 & \text{if } t \text{ is a } \square \end{cases}$$

Theorem (Bailey)

$$(bc;q)_\infty \sum_{n=0}^{\infty} \frac{(-c;q)_n(-q/b;q)_n b^n}{(q;q)_n(bc;q)_n} = \prod_{n=1}^{\infty} \frac{(1+q^n)(1+cq^{2n-1})(1+cb^2q^{2n-2})}{1-bq^{n-1}}$$

A q-Series Symmetry Result

Theorem (Ramanujan)

$$(-bq;q)_n \sum_{n=0}^{\infty} \frac{(-c/a;q)_n a^n q^{\binom{n+1}{2}}}{(q;q)_n(-bq;q)_n} = (-aq;q)_n \sum_{n=0}^{\infty} \frac{(-c/b;q)_n b^n q^{\binom{n+1}{2}}}{(q;q)_n(-aq;q)_n}$$

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Equivalently, the following function is symmetric in the variables a and b:

$$(-bq;q)_n \sum_{n=0}^{\infty} \frac{(-c/a;q)_n a^n q^{\binom{n+1}{2}}}{(q;q)_n(-bq;q)_n}$$

Labeled Tilings

Weight tiles in the following manner:

$$w(t) = \begin{cases} a_j q^i & \text{if } t \text{ is a } \square\, j \text{ in position } i,\ 1 \le j \le k+1 \\ c_j q^i & \text{if } t \text{ is a } \bigcirc\, j \text{ in position } i,\ 1 \le j \le k \\ 1 & \text{if } t \text{ is a } \square \text{ in position } i \end{cases}$$

Labeled Tilings

Weight tiles in the following manner:

$$w(t) = \begin{cases} a_j q^i & \text{if } t \text{ is a } \square j \text{ in position } i,\ 1 \le j \le k+1 \\ c_j q^i & \text{if } t \text{ is a } \bigcirc j \text{ in position } i,\ 1 \le j \le k \\ 1 & \text{if } t \text{ is a } \square \text{ in position } i \end{cases}$$

Consider the following generating function:

$$(-a_{k+1}q)_\infty \sum_{n_1,\ldots,n_k \ge 0} \frac{(-c_1/a_1)_{n_1} \cdots (-c_k/a_k)_{n_k} a_1^{n_1} \cdots a_k^{n_k}}{(q)_{n_1} \cdots (q)_{n_k} (-a_{k+1}q)_{n_1+\cdots+n_k}} q^{\binom{n_1+\cdots+n_k+1}{2}}.$$

$$(-a_{k+1}q)_\infty \sum_{n_1,\ldots,n_k\geq 0} \frac{(-c_1/a_1)_{n_1}\cdots(-c_k/a_k)_{n_k} a_1^{n_1}\cdots a_k^{n_k}}{(q)_{n_1}\cdots(q)_{n_k}(-a_{k+1}q)_{n_1+\cdots+n_k}} q^{\binom{n_1+\cdots+n_k+1}{2}}.$$

STEP 1: Place $n_1 \geq 0$ 1-squares in positions $1, 2, \ldots, n_1$, immediately followed by $n_2 \geq 0$ 2–squares, and so on.

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$\boxed{1}$	$\boxed{1}$	$\boxed{1}$																		$\cdots$

$$(-a_{k+1}q)_\infty \sum_{n_1,\ldots,n_k\geq 0} \frac{(-c_1/a_1)_{n_1}\cdots(-c_k/a_k)_{n_k} a_1^{n_1}\cdots a_k^{n_k}}{(q)_{n_1}\cdots(q)_{n_k}(-a_{k+1}q)_{n_1+\cdots+n_k}} q^{\binom{n_1+\cdots+n_k+1}{2}}.$$

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1	1	1	2	2																$\cdots$

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$\boxed{1}$	$\boxed{1}$	$\boxed{1}$	$\boxed{2}$	$\boxed{2}$	$\boxed{3}$	$\boxed{3}$	$\boxed{3}$	$\boxed{3}$												$\cdots$

$$(-a_{k+1}q)_\infty \sum_{n_1,\ldots,n_k\geq 0} \frac{(-c_1/a_1)_{n_1}\cdots(-c_k/a_k)_{n_k} a_1^{n_1}\cdots a_k^{n_k}}{(q)_{n_1}\cdots(q)_{n_k}(-a_{k+1}q)_{n_1+\cdots+n_k}} q^{\binom{n_1+\cdots+n_k+1}{2}}.$$

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1	1	1	2	2	3	3	3	3	4	4									$\cdots$

$$(-a_{k+1}q)_\infty \sum_{n_1,\ldots,n_k\geq 0} \frac{(-c_1/a_1)_{n_1}\cdots(-c_k/a_k)_{n_k} a_1^{n_1}\cdots a_k^{n_k}}{(q)_{n_1}\cdots(q)_{n_k}(-a_{k+1}q)_{n_1+\cdots+n_k}} q^{\binom{n_1+\cdots+n_k+1}{2}}.$$

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STEP 2: Arbitrarily place $k+1$-squares in positions $i > n_1+n_2+\cdots+n_k$.

1	1	1	2	2	3	3	3	3	4	4										$\cdots$

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1	1	1	2	2	3	3	3	3	4	4	5								$\cdots$

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1	1	1	2	2	3	3	3	3	4	4	5		5	5						$\cdots$

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1	1	1	2	2	3	3	3	3	4	4	5		5	5					

$\cdots$

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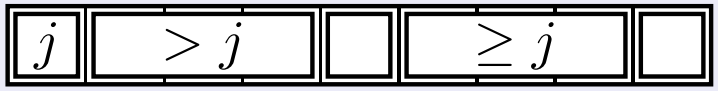
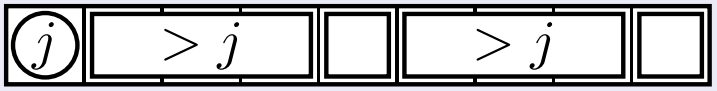
$$\prod_{i>n_1+n_2+\cdots+n_k} (1+a_{k+1}q^i) = \frac{(-a_{k+1}q;q)_\infty}{(-a_{k+1}q;q)_{n_1+n_2+\cdots+n_k}}$$

$$(-a_{k+1}q)_\infty \sum_{n_1,\ldots,n_k\geq 0} \frac{(-c_1/a_1)_{n_1}\cdots(-c_k/a_k)_{n_k}a_1^{n_1}\cdots a_k^{n_k}}{(q)_{n_1}\cdots(q)_{n_k}(-a_{k+1}q)_{n_1+\cdots+n_k}} q^{\binom{n_1+\cdots+n_k+1}{2}}.$$

STEP 3: Projectiles, $\square\, j$ and $\bigcirc\, j$.





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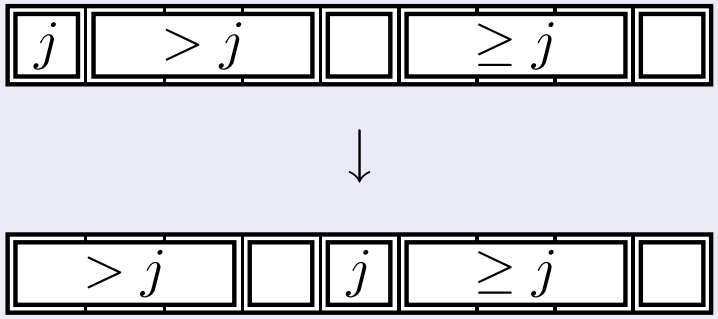

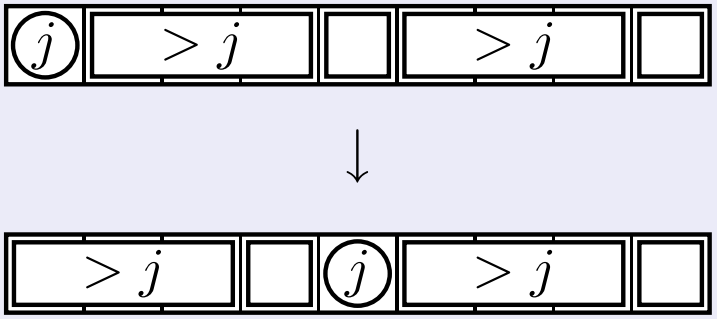

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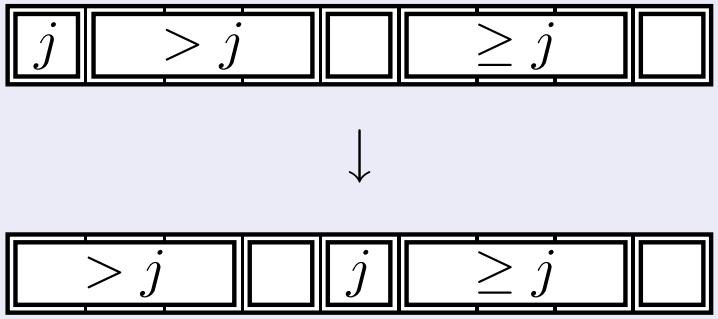

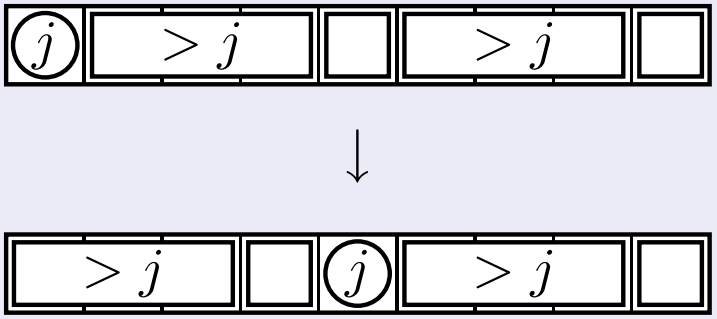

This process increases the weight of a tiling by a factor of q.

$$(-a_{k+1}q)_\infty \sum_{n_1,\ldots,n_k\geq 0} \frac{(-c_1/a_1)_{n_1}\cdots(-c_k/a_k)_{n_k} a_1^{n_1}\cdots a_k^{n_k}}{(q)_{n_1}\cdots(q)_{n_k}(-a_{k+1}q)_{n_1+\cdots+n_k}} q^{\binom{n_1+\cdots+n_k+1}{2}}.$$

STEP 3a: Decide whether or not to convert each $\square j$ into $\bigcirc j$ for each $j = k, \ldots, 2, 1$. If so, project every j-tile that appears to its right. This accounts for a weight of

$$\prod_{i=1}^{k}\prod_{j=1}^{n_j}(1 + c_i q^{j-1}/a_i) = (-c_1/a_1; q)_{n_1}\cdots(-c_k/a_k; q)_{n_k}$$

Note that every circle must be followed by a white square or a tile with a larger label.

$$(-a_{k+1}q)_\infty \sum_{n_1,\ldots,n_k\geq 0} \frac{(-c_1/a_1)_{n_1}\cdots(-c_k/a_k)_{n_k} a_1^{n_1}\cdots a_k^{n_k}}{(q)_{n_1}\cdots(q)_{n_k}(-a_{k+1}q)_{n_1+\cdots+n_k}} q^{\binom{n_1+\cdots+n_k+1}{2}}.$$

STEP 3b: Project the j-tiles for $j = k, \ldots, 2, 1$. As usual, work in a right to left manner and make sure to project each j-tile at least as many times as the j-tiles to its left are projected.

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$$\frac{1}{(q;q)_{n_1}\cdots(q;q)_{n_k}}$$

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Remark: We have constructed all labeled tilings that consist of weakly increasing sequences of weighted tiles separated by a white square where the label must strictly increase after a circle.

Labeled Tilings

Theorem

The generating function

$$(-a_{k+1}q)_\infty \sum_{n_1,\ldots,n_k\geq 0} \frac{(-c_1/a_1)_{n_1}\cdots(-c_k/a_k)_{n_k} a_1^{n_1}\cdots a_k^{n_k}}{(q)_{n_1}\cdots(q)_{n_k}(-a_{k+1}q)_{n_1+\cdots+n_k}} q^{\binom{n_1+\cdots+n_k+1}{2}}.$$

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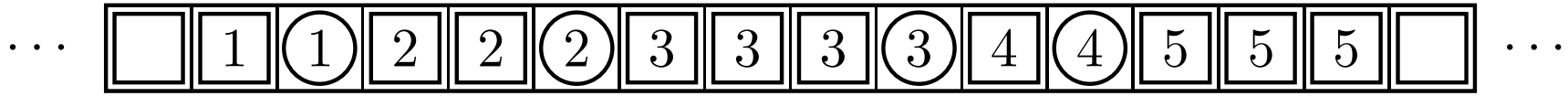
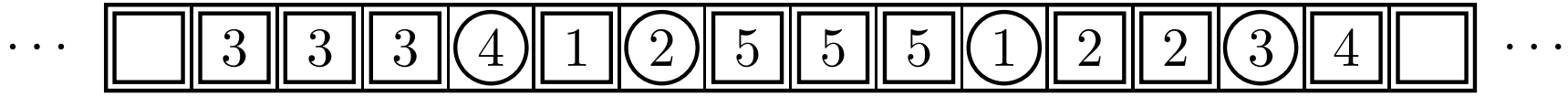
Note that the following sequences of tiles have the same weight:





Tiling Statistics

Let $\mathcal{P}_n$ denote the set of Pell tilings, (i.e., tilings with , ,) of a $1 \times n$ board.

Definition

The number of or that are immediately followed by a in tiling T is the number of *descents in* T, denoted $des(T)$.

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$$\sum_{T \in \mathcal{P}_n} x^{des(T)}$$

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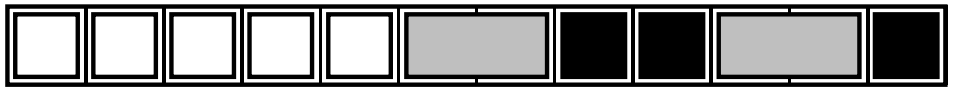
Find the generating function for descents

$$\sum_{T \in \mathcal{P}_n} x^{des(T)}$$

Weight tilings in the following manner:

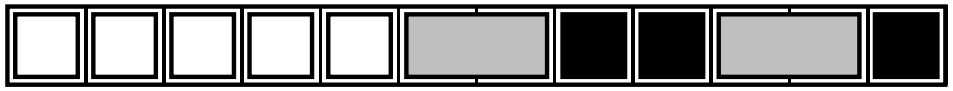
$$w(t) = \begin{cases} x & \text{if } t \text{ is the last tile on the board} \\ 1 \text{ or } -x & \text{otherwise} \end{cases}$$

Tilings with no Descents



Let $F_n(x)$ denote the G.F. for tilings of a $1 \times n$ board using and .

Tilings with no Descents



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$$F_0(x) = 1 \qquad \emptyset$$

$$F_1(x) = x$$

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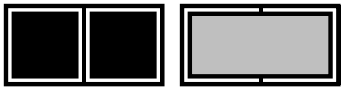
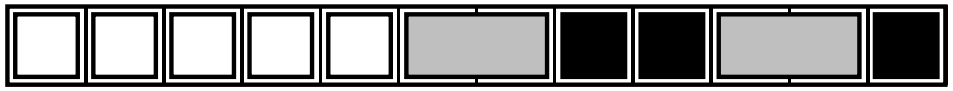


Tilings with no Descents



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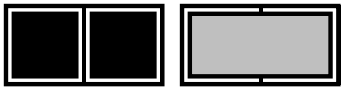
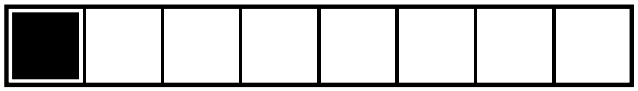
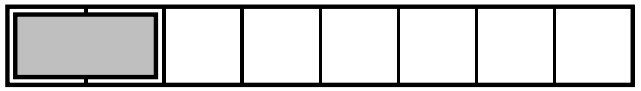


For $n \geq 3$,

$$F_n(x) = (1-x)F_{n-1}(x) + (1-x)F_{n-2}(x)$$





$$
\begin{aligned}
F(x,t) &= \sum_{n=0}^{\infty} F_n(x) t^n \\
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Let $G_n(x)$ denote the G.F. for tilings of a $1 \times n$ board using , and , where no or is followed by a .

$$F(x,t) = \sum_{n=0}^{\infty} F_n(x)t^n$$
$$= \frac{1 + (2x-1)t + (2x-1)t^2}{1 - (1-x)t + (1-x)t^2}$$

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$$G(x,t) = \sum_{n=0}^{\infty} G_n(x)t^n$$
$$= \frac{1}{1-(1-x)t}F(x,t) - \frac{(1-x)t}{1-(1-x)t} + \frac{xt}{1-(1-x)t}$$
$$= \frac{1 + 2(2x-1)t + x(2x-1)t^2 + (x-1)(2x-1)t^3}{(1-(1-x)t)(1-(1-x)t+(1-x)t^2)}$$

Symmetric Functions

Definition

Let $X_N = (x_1, x_2, \ldots, x_n)$. We say that $f(X_N)$ is symmetric if

$$f(x_1, x_2, \ldots, x_N) = f(x_{\sigma_1}, \ldots, x_{\sigma_N})$$

for all $\sigma \in S_N$.

Elementary Symmetric Functions:

$$e_k(X_N) = \sum_{1 \leq i_1 < \cdots < i_k \leq N} x_{i_1} \cdots x_{i_k}$$

$$e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_l}$$

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Homogeneous Symmetric Functions

$$h_k(X_N) = \sum_{1 \le i_1 \le \cdots \le i_k \le N} x_{i_1} \cdots x_{i_k}$$

$$h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_l}$$

Theorem

$$h_n(X) = \sum_{\lambda \vdash n} (-1)^{n-l(\lambda)} B_\lambda e_\lambda(X)$$

where B_λ is the number of compositions that are rearrangements of the parts of λ.

For example:

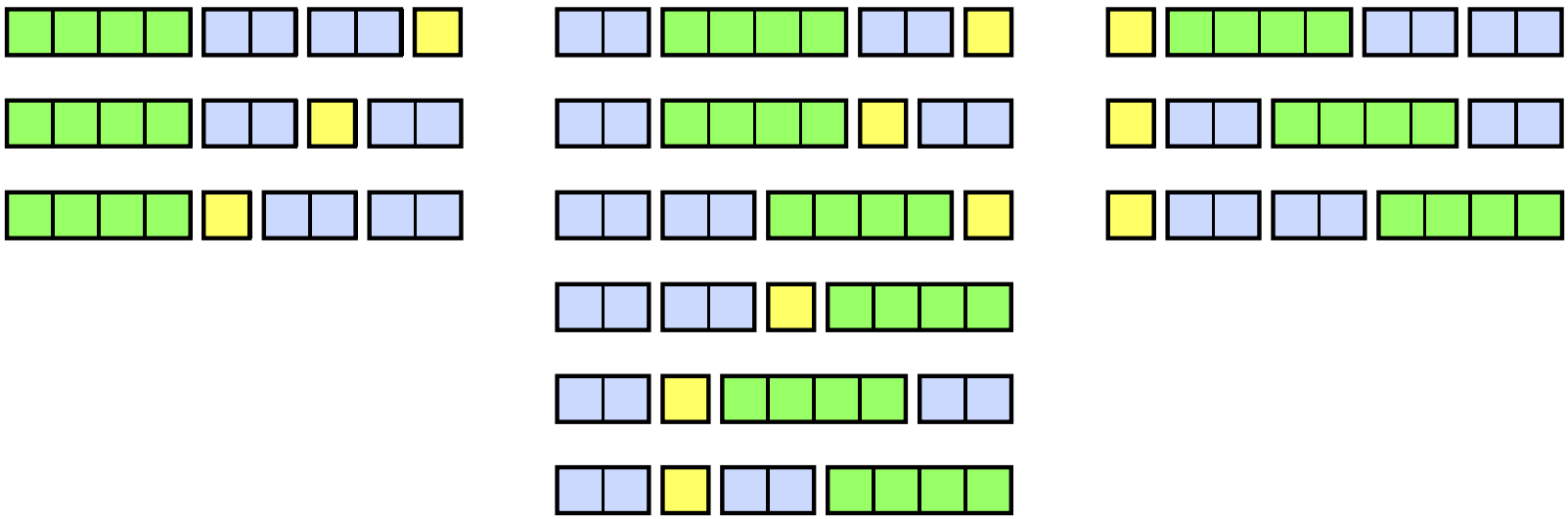
$$\begin{aligned}
h_3(X_3) &= x_1x_1x_1 + x_1x_1x_2 + x_1x_1x_3 + x_1x_2x_2 + x_1x_2x_3 \\
&\quad + x_1x_3x_3 + x_2x_2x_2 + x_2x_2x_3 + x_2x_3x_3 + x_3x_3x_3 \\
&= x_1x_2x_3 - 2(x_1x_2 + x_1x_3 + x_2x_3)(x_1 + x_2 + x_3) \\
&\quad + (x_1 + x_2 + x_3)^3 \\
&= e_3(X_3) - 2e_2(X_3)e_1(X_3) + e_1(X_3)^3 \\
&= e_3(X_3) - 2e_{2,1}(X_3) + e_{1,1,1}(X_3)
\end{aligned}$$

Theorem

$$h_n(X) = \sum_{\lambda \vdash n} (-1)^{n-l(\lambda)} B_\lambda e_\lambda(X)$$

where B_λ is the number of compositions that are rearrangements of the parts of λ.

Consider $B_{4,2,2,1} = 12$



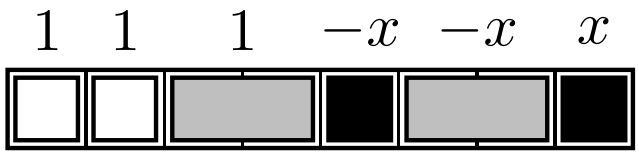
A Ring Homomorphism

Define $\zeta(e_n) = (-1)^{n-1} G(x,t)\Big|_{t^n}$.

$$\begin{aligned}
\zeta(h_n) &= \sum_{\lambda \vdash n} (-1)^{n-l(\lambda)} B_\lambda \zeta(e_\lambda(X)) \\
&= \sum_{\lambda \vdash n} (-1)^{n-l(\lambda)} B_\lambda \prod_{i=1}^{l(\lambda)} (-1)^{\lambda_i - 1} G(x,t)\Big|_{t^{\lambda_i}} \\
&= \sum_{\lambda \vdash n} B_\lambda \prod_{i=1}^{l(\lambda)} G(x,t)\Big|_{t^{\lambda_i}}
\end{aligned}$$

An Involution

Find first tile weighted by $-x$ or consecutive bricks with no descent.



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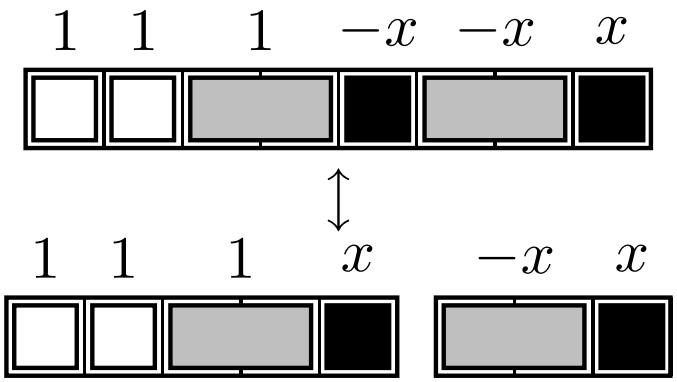

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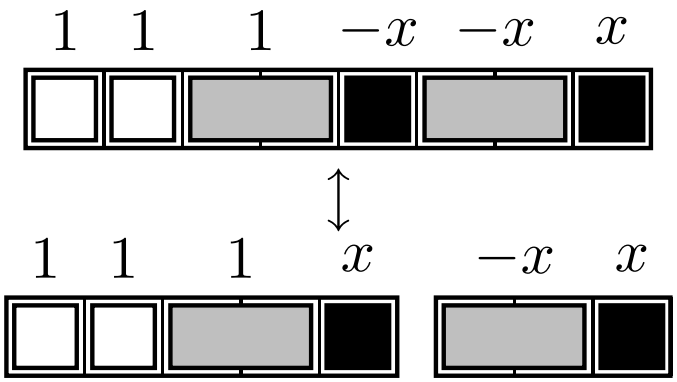
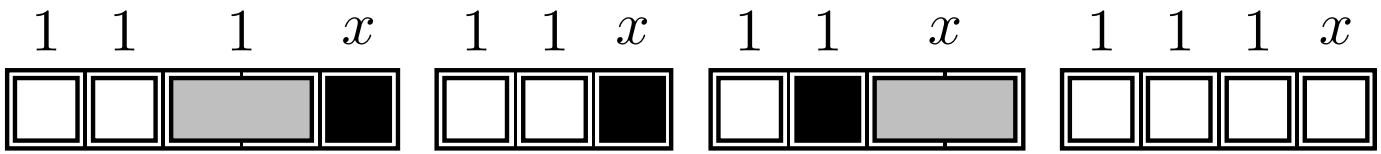


Fixed Points:



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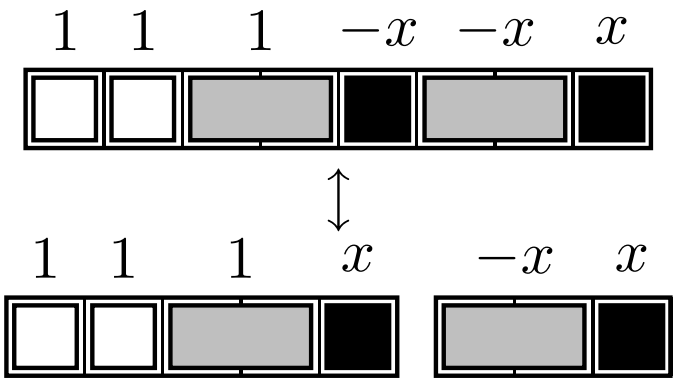

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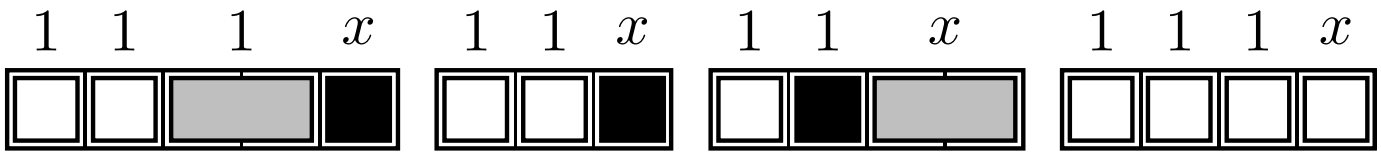

Conclusion:

$$\zeta(h_n) = \sum_{T \in \mathcal{P}_n} x^{des(T)+1}$$

Theorem

$$\sum_{n=0}^{\infty} h_n(X)t^n = \frac{1}{1+\sum\limits_{n\geq 1}(-1)^n e_n(X)t^n}$$

$$\begin{aligned}
\sum_{n=0}^{\infty} \zeta(h_n)t^n &= \frac{1}{1+\sum_{n=1}^{\infty}(-1)^n\zeta(e_n)t^n} \\
&= \frac{1}{1+\sum_{n=1}^{\infty}(-1)^n(-1)^{n-1}G(x,t)\big|_{t^n}t^n} \\
&= \frac{1}{1-\sum_{n=1}^{\infty}G(x,t)\big|_{t^n}t^n} \\
&= \frac{1}{1-(G(x,t)-1)} \\
&= \frac{1}{2-G(x,t)}
\end{aligned}$$

Recap

$$G(x,t) = \frac{1 + 2(2x-1)t + x(2x-1)t^2 + (x-1)(2x-1)t^3}{(1-(1-x)t)(1-(1-x)t+(1-x)t^2)}$$

$$\zeta(h_n) = \sum_{T \in \mathcal{P}_n} x^{des(T)+1}$$

$$\sum_{n=0}^{\infty} \zeta(h_n) t^n = \frac{1}{2 - G(x,t)}$$

Theorem

$$\sum_{T \in \mathcal{P}_n} x^{des(T)+1} = \frac{1}{2 - G(x,t)}\Big|_{t^n}$$