Combinatorial Analysis of the Geometric Series

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Analytic Convergence of a Series

The series

$$\sum_{i=0}^{\infty} a_i$$

converges analytically if and only if the sequence of partial sums,

$$s_n = a_0 + a_1 + \cdots + a_n$$

converges.

In other words, an infinite sum is defined to be the limit of a finite sum:

$$\sum_{i=0}^{\infty} a_i = \lim_{n\to\infty} \sum_{i=0}^{n} a_i$$

The Geometric Series

The series

$$\sum_{n=0}^{\infty} a_n$$

is **geometric** if there exists $r \in \mathbb{C}$ such that for all integers $n \geq 0$,

$$\frac{a_{n+1}}{a_n} = r.$$

All geometric series are of the form

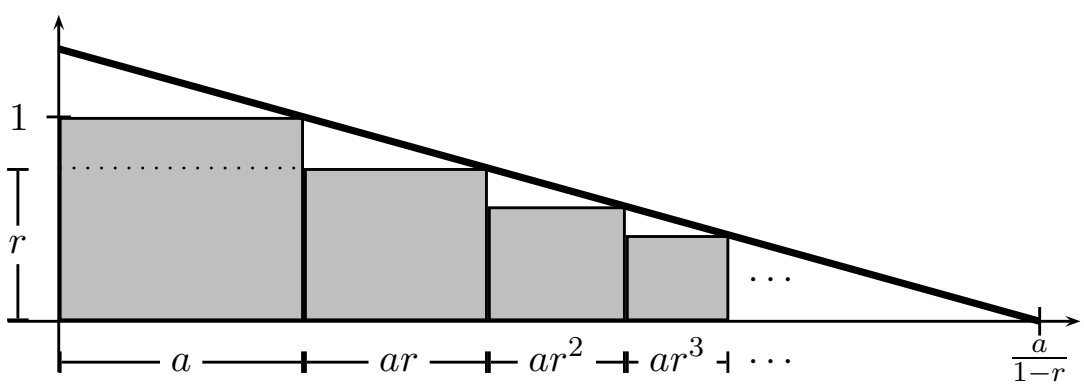
$$\sum_{n=0}^{\infty} a \cdot r^n = a + ar + ar^2 + ar^3 + \cdots$$

and converge to

$$\frac{a}{1-r}$$

if and only if $|r| < 1$.

A Real Geometric Series

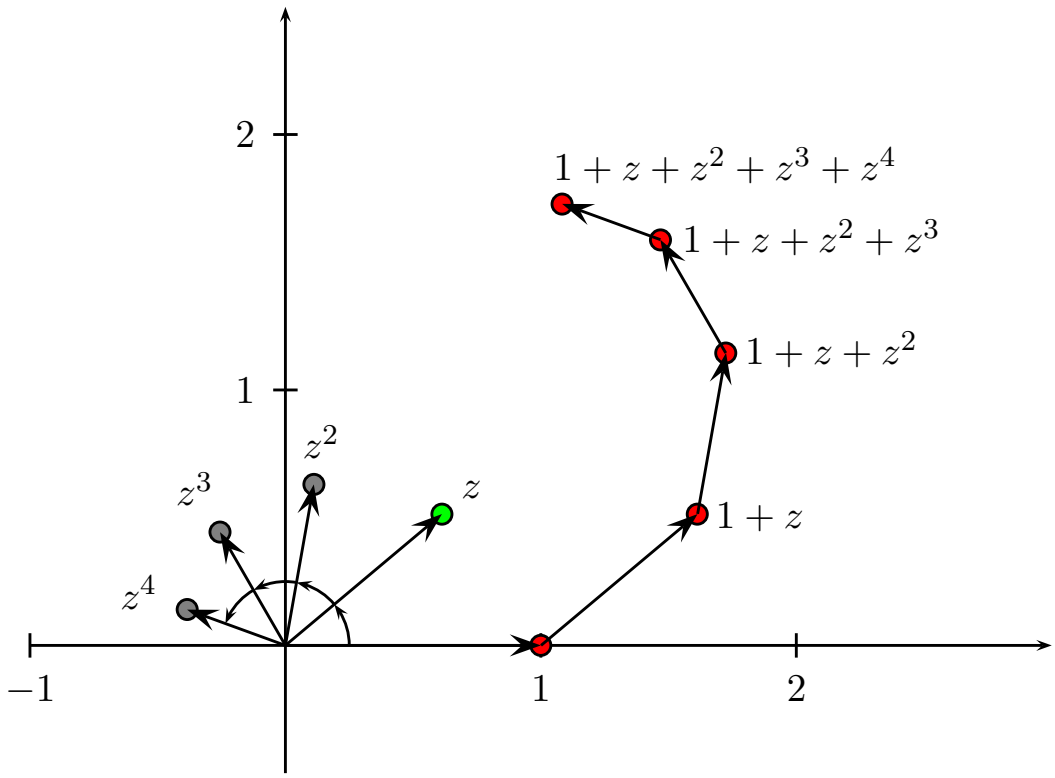


The sum of the widths of the rectangles is given by the geometric series

$$a + ar + ar^2 + ar^3 + ar^4 + \cdots$$

And if $0 < r < 1$, it converges by the monotonic sequence theorem.

A Complex Geometric Series



Formal Power Series

Given a sequence $c_0, c_1, c_2, \ldots$, the corresponding **formal power series** is given by

$$\sum_{n=0}^{\infty} c_n q^n$$

For a formal power series, convergence comes down to the computability of the coefficients c_n, and not the values of q that result in a convergent series.

The formal power series

$$\sum_{n=0}^{\infty} n! q^n = 1 + q + 2q^2 + 6q^3 + 24q^4 + 120q^5 + \cdots$$

is combinatorially significant since the sequence of coefficients is the number of permutations, but yet it has no analytic significance because its radius of convergence is 0.

Convergence of a Formal Power Series

Convergence = Computability

The formal power series of the function $F(q)$ exists if and only if for every integer $n \geq 0$, the coefficient of q^n can be computed in a finite number of operations.

Example

$$F(q) = \sum_{n=1}^{\infty} \frac{1}{1-q^n} = \frac{1}{1-q} + \frac{1}{1-q^2} + \frac{1}{1-q^3} + \cdots$$

has no formal power series expansion since the constant term 1 appears in every term. In other words, the coefficient of q^0 cannot be computed in a finite number of operations.

Example

The following function has a well-defined formal power series.

$$\sum_{n=1}^{\infty} \frac{q^n}{1-q^n} = \frac{q}{1-q} + \frac{q^2}{1-q^2} + \frac{q^3}{1-q^3} + \cdots$$

The coefficient of q^4 can be computed in the following manner

$$\begin{aligned}
\left(\sum_{n=1}^{\infty} \frac{q^n}{1-q^n}\right)\Big|_{q^4} &= \left(\frac{q}{1-q} + \frac{q^2}{1-q^2} + \frac{q^3}{1-q^3} + \frac{q^4}{1-q^4}\right)\Big|_{q^4} \\
&= \quad q(1+q+q^2+q^3+q^4+\cdots) \\
&\quad + q^2(1+q^2+q^4+q^6+q^8+\cdots) \\
&\quad + q^3(1+q^3+q^6+q^9+q^{12}+\cdots) \\
&\quad + q^4(1+q^4+q^8+q^{12}+q^{16}+\cdots)\Big|_{q^4} \\
&= (q+q^2+q^3+q^4) + (q^2+q^4) + (0) + (q^4)\Big|_{q^4} \\
&= 3
\end{aligned}$$

Convergence of a Formal Power Series

In general, the coefficient of q^N in

$$\sum_{n=1}^{\infty} \frac{q^n}{1-q^n}$$

is the number of divisors of N.

$$\sum_{n=1}^{\infty} \frac{q^n}{1-q^n} = q + 2q^2 + 2q^3 + 3q^4 + 2q^5 + 4q^6 + 2q^7 + \cdots$$

Algebra of Formal Power Series

$$\sum_{n=0}^{\infty} a_n q^n \pm \sum_{n=0}^{\infty} b_n q^n = \sum_{n=0}^{\infty} (a_n \pm b_n)\, q^n$$

$$\left(\sum_{r=0}^{\infty} a_r q^r\right)\left(\sum_{s=0}^{\infty} b_s q^s\right) = \sum_{n=0}^{\infty} \left(\sum_{r=0}^{n} a_r b_{n-r}\right) q^n$$

- The collection of formal power series with the operations of addition and multiplication defined above forms a **ring**.
- Series with nonzero constant term are the elements that have a multiplicative inverse.

Generating Functions

The function $F(q)$ is the **generating function** of the sequence $\{c_n\}$ if its power series representation is given by

$$\sum_{n=0}^{\infty} c_n q^n$$

Example

$(1+q)^n$ is the generating function for $\binom{n}{0}, \binom{n}{1}, \cdots, \binom{n}{n}$.

$\dfrac{1}{1-q}$ is the generating function for $1, 1, 1, 1, \ldots$.

$\dfrac{1}{(1-q)^k}$ is the generating function for $\binom{k-1}{k-1}, \binom{k}{k-1}, \binom{k+1}{k-1}, \ldots$.

Combinatorial Interpretations

Let A and B be **disjoint multisets** and let a_n be the number of ways to select n objects from A and let b_n be the number of ways to select n objects from B.

If $A(q)$ and $B(q)$ are the corresponding generating functions, then

- $A(q) + B(q)$ is the generating function for $\{a_n + b_n\}_{n \geq 0}$, the number of ways to select n things from A **or** n things from B **but not both**.
- $A(q)B(q)$ is the generating function for $\left\{\sum_{r=0}^{n} a_r b_{n-r}\right\}_{n \geq 0}$, the number of ways to select n objects from $A \cup B$.

An Example

$$\frac{1}{1-q} = 1 + q + q^2 + q^3 + \cdots$$

is the G.F. for the number of ways to write n as a sum of ones.

$$\frac{1}{1-q^2} = 1 + q^2 + q^4 + q^6 + \cdots$$

is the G.F. for the number of ways to write n as a sum of twos.

$$\frac{1}{(1-q)(1-q^2)} = 1 + q + 2q^2 + 2q^3 + 3q^4 + 3q^5 + 4q^6 + \cdots$$

is the G.F. for the number of ways to write n as an unordered sum of ones and twos.

Partitions

$$\frac{1}{(1-q)(1-q^2)\cdots(1-q^N)}$$

is the G.F. for the number of ways to write n as an unordered sum of positive integers less than or equal to N.

Definition

An **integer partition** of n is a weakly decreasing sequence of positive integers that sum to n.

$$\prod_{i=1}^{\infty} \frac{1}{1-q^i} = 1 + 1q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + 11q^6 + \cdots$$

is the G.F. for the number of integer partitions.

Hypergeometric Series

The series

$$\sum_{n=0}^{\infty} c_n$$

is said to be **hypergeometric** if $c_0 = 1$ and for all integers $n \geq 0$, $\dfrac{c_{n+1}}{c_n}$ is a rational function of n.

Example

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$\tan^{-1}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)} \qquad \ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}$$

Suppose

$$\frac{c_{n+1}}{c_n} = \frac{(a+n)z}{b+n}$$

Then

$$\begin{aligned} c_{n+1} = \frac{(a+n)z}{b+n} \cdot c_n &= \frac{(a+n)z}{b+n} \cdot \frac{(a+n-1)z}{b+n-1} \cdot c_{n-1} \\ &\vdots \\ &= \frac{(a+n)z}{b+n} \cdot \frac{(a+n-1)z}{b+n-1} \cdots \frac{az}{b} \cdot c_0 \\ &= \frac{a(a+1)(a+2)\cdots(a+n)z^{n+1}}{b(b+1)(b+2)\cdots(b+n)} = \frac{(a)_{n+1} z^{n+1}}{(b)_{n+1}} \end{aligned}$$

where

$$(z)_n = \begin{cases} 1 & \text{if } n = 0 \\ z(z+1)(z+2)\cdots(z+n-1) & \text{otherwise} \end{cases}$$

is called a **shifted factorial**.

Generalized Hypergeometric Series

Ratio of consecutive terms is a rational function of n:

$$ {}_rF_s\left[\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix}; z\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_r)_n}{(b_1)_n \cdots (b_s)_n} \frac{z^n}{n!} $$

$$ \begin{aligned} (1+z)^a &= {}_1F_0\left[\begin{matrix} -a \\ - \end{matrix}; -z\right] \\ \ln(1+z) &= z\ {}_2F_1\left[\begin{matrix} 1, 1 \\ 2 \end{matrix}; -z\right] \\ \sin^{-1}(z) &= z\ {}_2F_1\left[\begin{matrix} 1/2, 1/2 \\ 3/2 \end{matrix}; z^2\right] \\ \tan^{-1}(z) &= z\ {}_2F_1\left[\begin{matrix} 1/2, 1 \\ 3/2 \end{matrix}; -z^2\right] \\ e^z &= {}_0F_0\left[\begin{matrix} - \\ - \end{matrix}; z\right] \end{aligned} $$

Basic Hypergeometric Series

Ratio of consecutive terms is a rational function of q^n:

$$ {}_r\phi_s\left[\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix}; q, z\right] = \sum_{n=0}^{\infty} \frac{(a_1;q)_n (a_2;q)_n \cdots (a_r;q)_n}{(b_1;q)_n \cdots (b_s;q)_n} \frac{z^n}{(q;q)_n} $$

where the symbol $(z;q)_n$ is called a q-**shifted factorial** and defined by

$$ (z;q)_n = \begin{cases} 1 & \text{if } n = 1 \\ (1-z)(1-zq)(1-zq^2)\cdots(1-zq^{n-1}) & \text{otherwise} \end{cases} $$

q-analog of the binomial series

Theorem (Cauchy)

$$\sum_{n=0}^{\infty} \frac{(-a/z;q)_n z^n}{(q;q)_n} = \prod_{n=0}^{\infty} \frac{1+aq^n}{1-zq^n}$$

$$\sum_{n=0}^{\infty} \frac{(z+a)(z+aq)\cdots(z+aq^{n-1})}{(1-q)(1-q^2)\cdots(1-q^n)} = \frac{(1+a)(1+aq)(1+aq^2)\cdots}{(1-z)(1-zq)(1-zq^2)\cdots}$$

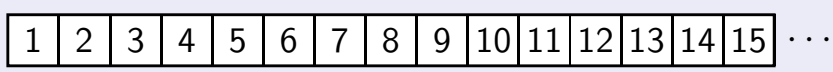
Combinatorial Proof

Show that both sides have the same formal power series expansion. Specifically, we will show that the coefficient of q^n on both sides of the equation counts the same set of combinatorial objects.

Weighted Tilings

Definition

A tiling is a covering of an infinitely long board:



using different types of tiles:

The weight of a tiling T is given by

$$w(T) = \prod_{t \in T} w(t)$$

where $w(t)$ is the weight of the tile t. The weight of a white square will always be 1. Each tiling will have a finite number of non-white square tiles.

q-ANALOG OF THE BINOMIAL SERIES

Weight tiles in the following manner:

$$w(t) = \begin{cases} zq^i & \text{if } t \text{ is a } \blacksquare \text{ with } i \text{ } \square \text{ or } \square \text{ to its left} \\ aq^i & \text{if } t \text{ is a } \square \text{ with } i \text{ } \square \text{ or } \square \text{ to its left} \\ 1 & \text{if } t \text{ is a } \square \end{cases}$$

THEOREM (Cauchy)

$$\sum_{n=0}^{\infty} \frac{(-a/z;q)_n z^n}{(q;q)_n} = \prod_{n=0}^{\infty} \frac{1+aq^n}{1-zq^n}$$

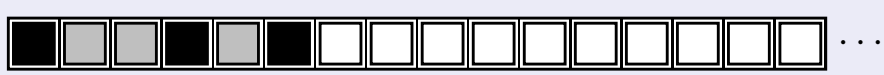
$$\begin{aligned} & 1 + \frac{z+a}{1-q} + \frac{(z+a)(z+aq)}{(1-q)(1-q^2)} + \frac{(z+a)(z+aq)(z+aq^2)}{(1-q)(1-q^2)(1-q^3)} + \cdots \\ = \quad & \frac{1+a}{1-z} \cdot \frac{1+aq}{1-zq} \cdot \frac{1+aq^2}{1-zq^2} \cdot \frac{1+aq^3}{1-zq^3} \cdots \end{aligned}$$

THEOREM (Cauchy)

$$\sum_{n=0}^{\infty} \frac{(-a/z;q)_n z^n}{(q;q)_n} = \prod_{n=0}^{\infty} \frac{1+aq^n}{1-zq^n}$$

PROOF. PART I: Interpret infinite series

STEP 1: Place n black or gray squares in positions $1, 2, 3, \ldots, n$.



A $\blacksquare$ in position i accounts for a weight of z.

A $\square$ in position i accounts for a weight of aq^{n-i}.

This process accounts for a weight of

$$\prod_{i=1}^{n} (z + aq^{n-i}) = (-a/z;q)_n z^n$$

Theorem (Cauchy)

$$\sum_{n=0}^{\infty} \frac{(-a/z;q)_n z^n}{(q;q)_n} = \prod_{n=0}^{\infty} \frac{1+aq^n}{1-zq^n}$$

Proof. PART I: Interpret infinite series

STEP 2: Insert white squares to the left of each black/gray square

Inserting j white squares increases the weight by a factor of q^{3j}

Accounting for all values of j: $\displaystyle\sum_{j=0}^{\infty} (q^3)^j = \frac{1}{1-q^3}$

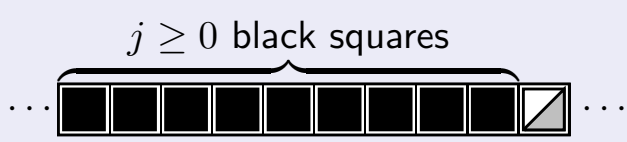
Accounting for all positions: $\displaystyle\prod_{i=1}^{n} \frac{1}{1-q^i} = \frac{1}{(q;q)_n}$

Theorem (Cauchy)

$$\sum_{n=0}^{\infty} \frac{(-a/z;q)_n a^n}{(q;q)_n} = \prod_{n=0}^{\infty} \frac{1+aq^n}{1-zq^n}$$

Proof. PART II: Interpret infinite product

Each tiling can be broken up into segments:



The weight of the nth segment for $n \geq 0$ is given by

$$(1+aq^n)\sum_{j=0}^{\infty} (zq^n)^j = \frac{1+aq^n}{1-zq^n}$$

Multiplying over $n \geq 0$ completes the construction.

Specializations

$z = q,\ a = 0$

No gray squares, black squares weighted by q^j if it has $j-1$ white squares to its left:

$$\sum_{n=0}^{\infty} \frac{q^n}{(q;q)_n} = \prod_{n=1}^{\infty} \frac{1}{1-q^n}$$

Generating function for partitions.

$z = 0,\ a = q$

No black squares, gray squares weighted by q^j if it is in position j:

$$\sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}}{(q;q)_n} = \prod_{n=1}^{\infty} (1+q^n)$$

Generating function for partitions into distinct parts.

Specializations

$z = q,\ a = -q^{N+1}$

$$\sum_{n=0}^{\infty} \begin{bmatrix} N+n-1 \\ n \end{bmatrix} q^n = \frac{1}{(1-q)(1-q^2)\cdots(1-q^N)}$$

Generating function for partitions using the numbers 1 through N.

where

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}$$

is a q-analog of $\binom{n}{k}$.

$$\lim_{q \to 1} \begin{bmatrix} n \\ k \end{bmatrix} = \binom{n}{k}$$

Other Identities

Heine

$$(cq;q)_\infty \sum_{n=0}^{\infty} \frac{(-c/a;q)_n(-q/b;q)_n a^n b^n}{(q;q)_n(cq;q)_n} = \prod_{n=0}^{\infty} \frac{(1+bcq^n)(1+aq^{n+1})}{1-abq^n}$$

Lebesgue:

$$\sum_{n=0}^{\infty} \frac{(-z;q)_n}{(q;q)_n} q^{\binom{n+1}{2}} = \prod_{n=1}^{\infty}(1+q^n)(1+zq^{2n-1})$$

Cauchy:

$$(zq;q)_\infty \sum_{n=0}^{\infty} \frac{z^n q^{n^2}}{(q;q)_n(zq;q)_n} = 1$$

Sylvester:

$$\sum_{n=0}^{\infty} \frac{z^n q^{n^2}}{(q^2;q^2)_n} = \prod_{n=1}^{\infty}(1+zq^{2n-1})$$

Rogers:

$$\sum_{n=0}^{\infty} \frac{z^n q^{n^2}}{(q;q)_n} = (-zq^2;q^2)_\infty \sum_{n=0}^{\infty} \frac{z^n q^{n^2}}{(q^2;q^2)_n(-zq^2;q^2)_n}$$

and many many many more....

References

- Free Books:
  - "generatingfunctionology" by H. Wilf
  - "A=B" by H. Wilf, D. Zeilberger, M. Petkovsek
- More Texts:
  - "Basic Hypergeometric Series" by G. Gasper & M. Rahman
  - "The Theory of Partitions" by G. E. Andrews
  - "Special Functions" by G. E. Andrews, R. Askey, R. Roy
- Papers:
  - L. J. Slater, Further Identities of the Rogers-Ramanujan Type, Proc. London Math. Soc. (2) 54 (1952), 147-167
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