Classical q-Series Identities via Tilings

David P. Little

November 13, 2007

www.math.psu.edu/dlittle

Pell Tilings

Definition

A Pell tiling is a covering of an infinitely long board:

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using three different types of tiles:

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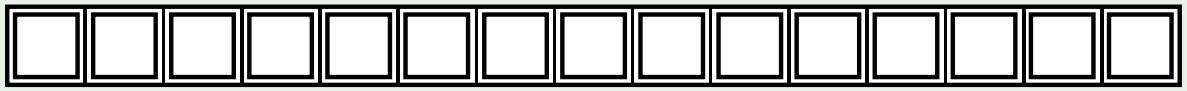

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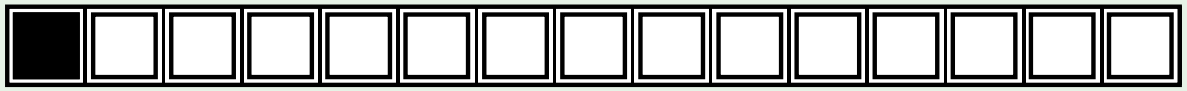

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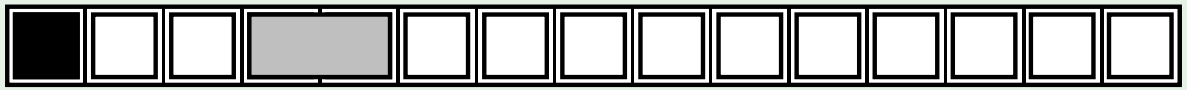

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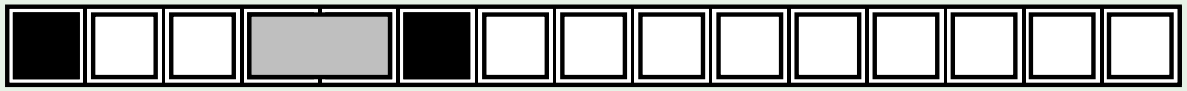

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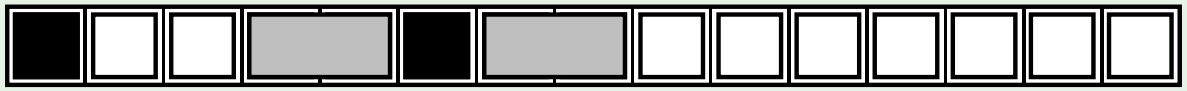

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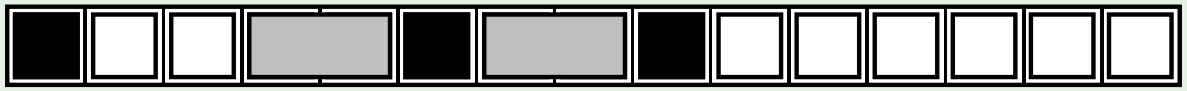

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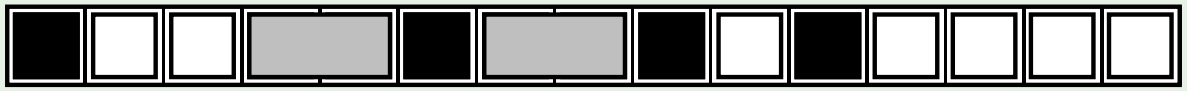

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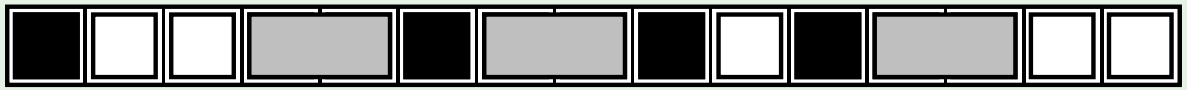

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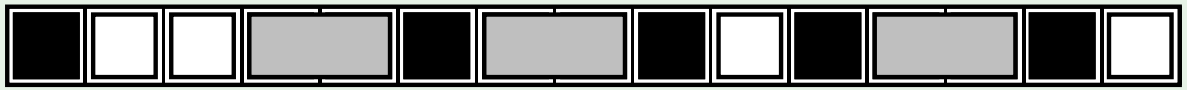

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Weighted Pell Tilings

The weight of tile t:

$$w(t) = \begin{cases} q^i & \text{if } t \text{ is a black square in position } i \\ zq^i & \text{if } t \text{ is a gray domino in position } i \\ 1 & \text{if } t \text{ is a white square in position } i \end{cases}$$

The weight of tiling T:

$$w(T) = \prod_{t \in T} w(t)$$

Generating Function:

$$P(z; q) = \sum_{T \in \mathcal{T}} w(T)$$

From Last Time...

By constructing tilings in two different ways, we obtained:

Theorem (Lebesgue)

$$\sum_{n\geq 0} \frac{(-z;q)_n}{(q;q)_n} q^{\binom{n+1}{2}} = \prod_{n\geq 1} (1+q^n)(1+zq^{2n-1})$$

where $(z;q)_n = (1-z)(1-zq)\cdots(1-zq^{n-1})$.

Theorem

The generating function for Pell tilings where at least $m \geq 0$ white squares appear before the first domino, if any, is given by

$$P(zq^m;q).$$

A Theorem of Euler

Theorem

$$\prod_{n\geq 1}(1+q^n) = \prod_{n\geq 1}\frac{1}{1-q^{2n-1}}$$

Proof.

Let $z=-1$ in the product side of Lebesgue: $\prod_{n\geq 1}(1+q^n)(1-q^{2n-1})$

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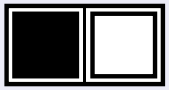
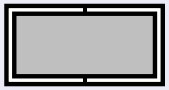
Find first occurrence of:  or 

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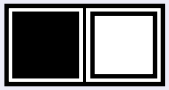
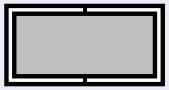
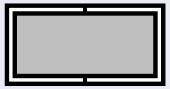
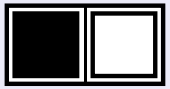
Find first occurrence of:  or 

and replace with:  or  (respectively)

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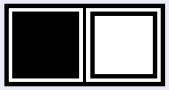
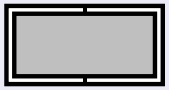
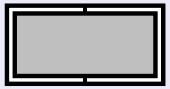
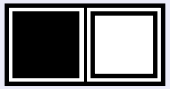
Find first occurrence of:  or 

and replace with:  or  (respectively)

Only remaining tiling consists of all white squares (weight 1).

□

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Theorem

$$\sum_{n\geq 0} q^{\binom{n+1}{2}} = \prod_{n\geq 1}(1+q^n)(1-q^{2n})$$

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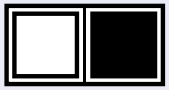
Find first occurrence of:  or

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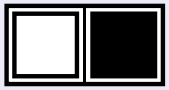
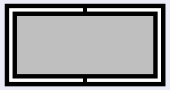
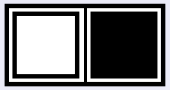
Find first occurrence of:  or

and replace with:  or  (respectively)

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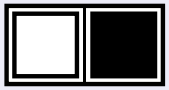
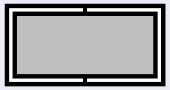
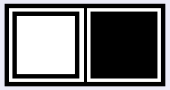
Find first occurrence of:  or

and replace with:  or  (respectively)

Remaining tilings consist of black squares in positions 1 through n (weight $q^{\binom{n+1}{2}}$) for some $n \geq 0$ and white squares everywhere else.

$\square$

A Generalization

Theorem For all $m \geq 0$,

$$\sum_{n\geq 0} q^{\binom{n+1}{2}} \begin{bmatrix} n+m-1 \\ m-1 \end{bmatrix} = \prod_{n\geq 1} (1+q^n)(1-q^{2n+m-1})$$

Proof.

Let $z = zq^m$ in the product side of Lebesgue: $\prod_{n\geq 1} (1+q^n)(1+zq^{2n+m-1})$ (at least m white squares appear before first domino.) Let $z = -1$.

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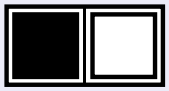
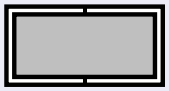
Find first occurrence of:  or  after mth white square.

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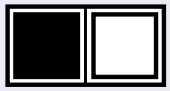
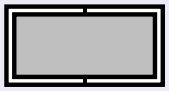
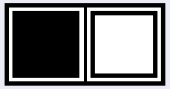
Find first occurrence of:  or  after mth white square.

and replace with: or  (respectively)

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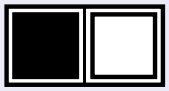
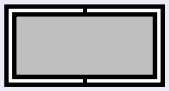
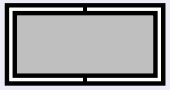
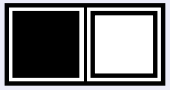
Find first occurrence of:  or  after mth white square.

and replace with:  or  (respectively)

Remaining tilings consist of n black squares in positions 1 through $n+m-1$ for some $n \geq 0$ and white squares everywhere else.

□

Fibonacci Tilings

The weight of tile t, denoted $w(t)$, is defined as follows:

$$w(t) = \begin{cases} 0 & \text{if } t \text{ is a black square at position } i \\ zq^i & \text{if } t \text{ is a gray domino at position } i \\ 1 & \text{if } t \text{ is a white square at position } i. \end{cases}$$

The weight of a tiling T is defined as

$$w(T) = \prod_{t \in T} w(t)$$

Generating Function:

$$F(z; q) = \sum_{T \in \mathcal{T}} w(T).$$

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The generating function for Fibonacci tilings where at least $m \geq 0$ white squares appear before the first domino, if any, is given by

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Definition

A domino is said to be even (odd) if its position is even (resp. odd).

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Define a slide operation on dominoes that:

- can be performed on a domino that is followed by a white square,
- preserves number of even/odd dominoes, and
- increases the weight of the tiling by a factor of q^2.

Domino Parity and Sliding

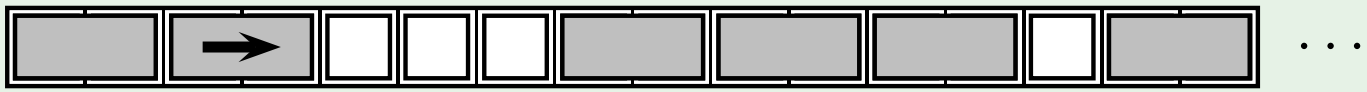
Definition

A domino is said to be even (odd) if its position is even (resp. odd).

Define a slide operation on dominoes that:

- can be performed on a domino that is followed by a white square,
- preserves number of even/odd dominoes, and
- increases the weight of the tiling by a factor of q^2.

Example



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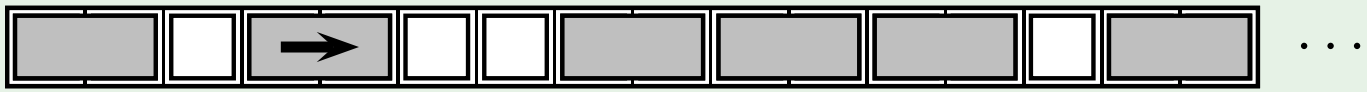

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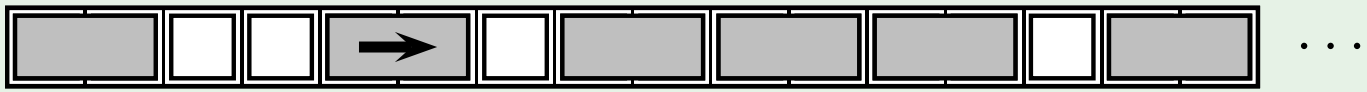

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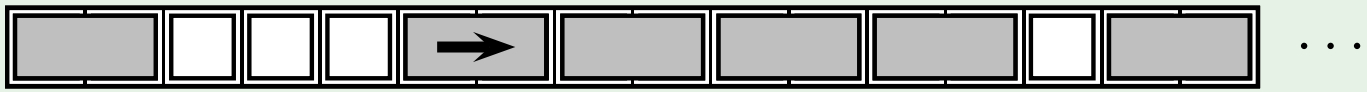

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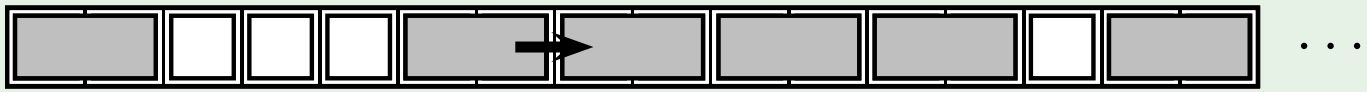

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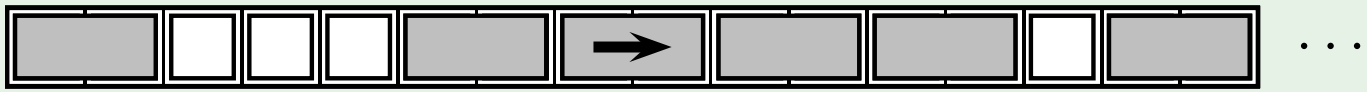

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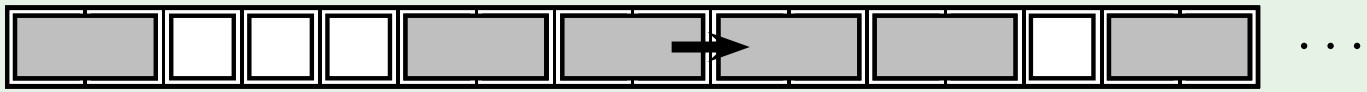

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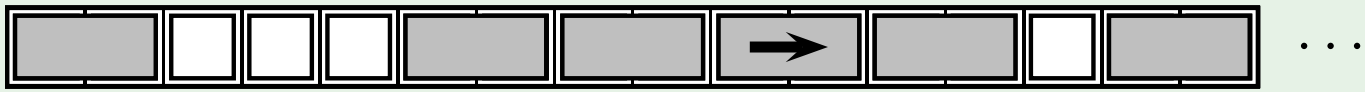

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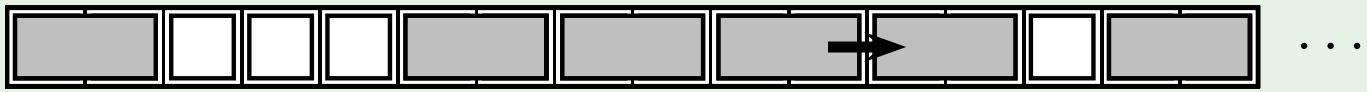

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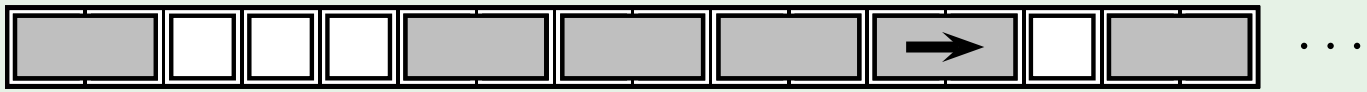

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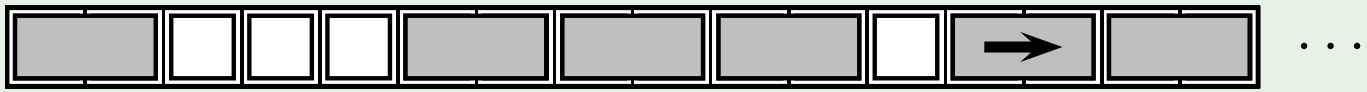

A Theorem of Sylvester

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$$\sum_{n\geq 0} \frac{z^n q^{n^2}}{(q^2;q^2)_n} = \prod_{n\geq 1}(1+zq^{2n-1})$$

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RHS: Generating function for Fibonacci tilings that use only white squares and odd dominoes.

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RHS: Generating function for Fibonacci tilings that use only white squares and odd dominoes.

LHS: We can construct the same objects in the following manner:

1. Place n odd dominoes in positions $1, 3, 5, \ldots, 2n-1$.
2. Arbitrarily slide odd dominoes.

$$\frac{1}{(1-q^2)(1-q^4)\cdots(1-q^{2n})} = \frac{1}{(q^2;q^2)_n}$$

□

Theorem

$$\sum_{n\geq 0} \frac{z^n q^{n^2}}{(q;q)_n} = (-zq^2;q^2)_\infty \sum_{n\geq 0} \frac{z^n q^{n^2}}{(q^2;q^2)_n(-zq^2;q^2)_n}$$

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Proof.

Count all Fibonacci tilings according to number of odd dominoes.

1. Place n odd dominoes in positions $1, 3, 5, \ldots, 2n-1$.
2. Arbitrarily place even dominoes in positions $2n+2, 2n+4, 2n+6, \ldots$

$$z^n q^{n^2} \prod_{j\geq n+1} (1+zq^{2j}) = (-zq^2;q^2)_\infty \frac{z^n q^{n^2}}{(-zq^2;q^2)_n}$$

3. Arbitrarily slide odd dominoes.

$\square$

Rogers Identities, Part I

$$\sum_{n\geq 0} \frac{z^n q^{n^2}}{(q;q)_n} = (-zq^2;q^2)_\infty \sum_{n\geq 0} \frac{z^n q^{n^2}}{(q^2;q^2)_n(-zq^2;q^2)_n}$$

Setting $z = 1$ yields

$$\sum_{n\geq 0} \frac{q^{n^2}}{(q;q)_n} = (-q^2;q^2)_\infty \sum_{n\geq 0} \frac{q^{n^2}}{(q^4;q^4)_n}$$

Setting $z = q$ yields

$$\begin{aligned}\sum_{n\geq 0} \frac{q^{n^2+n}}{(q;q)_n} &= (-q^3;q^2)_\infty \sum_{n\geq 0} \frac{q^{n^2+n}}{(q^2;q^2)_n(-q^3;q^2)_n} \\ &= (-q;q^2)_\infty \sum_{n\geq 0} \frac{q^{n^2+n}}{(q^2;q^2)_n(-q;q^2)_{n+1}}\end{aligned}$$

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Proof.

LHS: $F(zq;q)$ is generating function for weighted Fibonacci tilings that have a white square in position one.

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Proof.

LHS: $F(zq;q)$ is generating function for weighted Fibonacci tilings that have a white square in position one.

RHS: Count same collection of tilings based on number of odd dominoes.

1. Place n odd dominoes in positions $1, 3, 5, \ldots, 2n-1$.
2. Arbitrarily place even dominoes in positions $2n+2, 2n+4, 2n+6, \ldots$

$$z^n q^{n^2} \prod_{j\geq n+1} (1+zq^{2j}) = (-zq^2;q^2)_\infty \frac{z^n q^{n^2}}{(-zq^2;q^2)_n}$$

3. Arbitrarily slide every odd domino at least once.

$\square$

Rogers Identities, Part II

$$\sum_{n\geq 0} \frac{z^n q^{n^2+n}}{(q;q)_n} = (-zq^2;q^2)_\infty \sum_{n\geq 0} \frac{z^n q^{n^2+2n}}{(q^2;q^2)_n(-zq^2;q^2)_n}$$

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Setting $z = 1/q$ yields

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Signed Even Weighted Fibonacci Tilings

Weight tiles in the following manner:

$$w(t) = \begin{cases} -zq^{2i} & \text{if } t \text{ is an even domino in position } 2i \\ zq^{2i} & \text{if } t \text{ is an odd domino in position } 2i-1 \\ 1 & \text{if } t \text{ is a white square covering position } i. \end{cases}$$

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How does this affect sliding?

- Increasing position of domino by two increases its weight by q^2.

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How does this affect sliding?

- Increasing position of domino by two increases its weight by q^2.
- Increasing position of odd domino by one does not change its weight.
- Increasing position of even domino by one increases its weight by q^2.

Thus, slide operation still increases the weight of a tiling by q^2

Theorem

$$\sum_{n\geq 0} \frac{z^{2n} q^{4n^2+2n}}{(q^4;q^4)_n} = (zq^2;q^2)_\infty \sum_{n\geq 0} \frac{z^n q^{n^2+n}}{(q^2;q^2)_n (zq^2;q^2)_n}$$

Proof.

Step I: RHS counts Fibonacci tilings according to new weights.

1. Place n odd dominoes in positions $1, 3, 5, \ldots, 2n-1$.
2. Arbitrarily place even dominoes in positions $2n+2, 2n+4, 2n+6, \ldots$

$$z^n q^{2+4+\cdots+2n} \prod_{j\geq n+1} (1 - zq^{2j}) = (zq^2;q^2)_\infty \frac{z^n q^{n^2+n}}{(zq^2;q^2)_n}$$

3. Arbitrarily slide odd dominoes

Theorem

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Proof.

Step II: Cancel out terms via involution

Find first occurrence of:

Theorem

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and replace with:

and vice versa.

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Proof.

Step II: Cancel out terms via involution

Find first occurrence of:

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Remaining tilings cannot have any even dominoes nor an odd number of consecutive odd dominoes.

Theorem

$$\sum_{n\geq 0} \frac{z^{2n} q^{4n^2+2n}}{(q^4;q^4)_n} = (zq^2;q^2)_\infty \sum_{n\geq 0} \frac{z^n q^{n^2+n}}{(q^2;q^2)_n (zq^2;q^2)_n}$$

Proof.

Step III: Count fixed points of involution

Fixed points consist of sequences of consecutive odd dominoes that contain an even number of dominoes.

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Proof.

Step III: Count fixed points of involution

Fixed points consist of sequences of consecutive odd dominoes that contain an even number of dominoes.

1. Place $2n$ odd dominoes in positions $1, 3, 5, \ldots, 4n-1$.

$$z^{2n} q^{2+4+6+\cdots+4n} = z^{2n} q^{4n^2+2n}$$

2. Arbitrarily slide odd dominoes in pairs

$$\frac{1}{(1-q^4)(1-q^8)\cdots(1-q^{4n})} = \frac{1}{(q^4;q^4)_n}$$

□

Rogers Identities, Part III

$$\sum_{n\geq 0} \frac{z^{2n} q^{4n^2+2n}}{(q^4;q^4)_n} = (zq^2;q^2)_\infty \sum_{n\geq 0} \frac{z^n q^{n^2+n}}{(q^2;q^2)_n (zq^2;q^2)_n}$$

Setting $z = 1/q$ yields

$$\sum_{n\geq 0} \frac{q^{4n^2}}{(q^4;q^4)_n} = (q;q^2)_\infty \sum_{n\geq 0} \frac{q^{n^2}}{(q^2;q^2)_n (q;q^2)_n}$$

Setting $z = q$ yields

$$\begin{aligned}\sum_{n\geq 0} \frac{q^{4n^2+4n}}{(q^4;q^4)_n} &= (q^3;q^2)_\infty \sum_{n\geq 0} \frac{q^{n^2+2n}}{(q^2;q^2)_n (q^3;q^2)_n} \\ &= (q;q^2)_\infty \sum_{n\geq 0} \frac{q^{n^2+2n}}{(q^2;q^2)_n (q;q^2)_{n+1}}\end{aligned}$$

Rogers Identities, Part IV

$$\sum_{n\geq 0} \frac{z^n q^{2n^2+n}}{(q^2;q^2)_n} = (zq^2;q^2)_\infty \sum_{n\geq 0} \frac{z^n q^{(3n^2+n)/2}}{(q;q)_n (zq^2;q^2)_n}$$

Setting $z = 1/q$ yields

$$\sum_{n\geq 0} \frac{q^{2n^2}}{(q^2;q^2)_n} = (q;q^2)_\infty \sum_{n\geq 0} \frac{q^{(3n^2-n)/2}}{(q;q)_n (q;q^2)_n}$$

Setting $z = q$ yields

$$\sum_{n\geq 0} \frac{q^{2n^2+2n}}{(q^2;q^2)_n} = (q^3;q^2)_\infty \sum_{n\geq 0} \frac{q^{(3n^2+3n)/2}}{(q;q)_n (q^3;q^2)_n}$$

$$= (q;q^2)_\infty \sum_{n\geq 0} \frac{q^{(3n^2+3n)/2}}{(q;q)_n (q;q^2)_{n+1}}$$

Rogers Identities, Recap

$$\sum_{n\geq 0} \frac{z^n q^{n^2}}{(q;q)_n} = (-zq^2;q^2)_\infty \sum_{n\geq 0} \frac{z^n q^{n^2}}{(q^2;q^2)_n(-zq^2;q^2)_n}$$

$$\sum_{n\geq 0} \frac{z^n q^{n^2+n}}{(q;q)_n} = (-zq^2;q^2)_\infty \sum_{n\geq 0} \frac{z^n q^{n^2+2n}}{(q^2;q^2)_n(-zq^2;q^2)_n}$$

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Future Work

Theorem (Watson's q-analog of Whipple's theorem)

$$
{}_8\phi_7\left(\begin{matrix} z, q\sqrt{z}, -q\sqrt{z}, a, b, c, d, q^{-N} \\ \sqrt{z}, -\sqrt{z}, zq/a, zq/b, zq/c, zq/d, zq^{N+1} \end{matrix}; q, \frac{z^2 q^{N+2}}{abcd}\right)
$$
$$
= \frac{(zq)_N (zq/cd)_N}{(zq/c)_N (zq/d)_N} {}_4\phi_3\left(\begin{matrix} zq/ab, c, d, q^{-N} \\ zq/a, zq/b, cdq^{-N}/z \end{matrix}; q, q\right)
$$

Letting $a, b, c, d, N \to \infty$ yields

Theorem (Rogers and Ramanujan)

$$
(zq;q)_\infty \sum_{n\geq 0} \frac{z^n q^{n^2}}{(q;q)_n} = 1 + \sum_{n\geq 1} \frac{(-1)^n (zq;q)_{n-1} (1 - zq^{2n}) z^{2n} q^{(5n^2-n)/2}}{(q;q)_n}
$$

Theorem (Rogers-Fine)

$$(1-t)\sum_{n\geq 0}\frac{(a;q)_n t^n}{(b;q)_n}=\sum_{n\geq 0}\frac{(a;q)_n(atq/b)_n b^n t^n q^{n^2-n}(1-atq^{2n})}{(b;q)_n(tq;q)_n}$$

Theorem (Jacobi Triple Product)

$$\sum_{n=-\infty}^{\infty} z^n q^{n^2}=\prod_{n\geq 1}(1-q^{2n})(1+zq^{2n-1})(1+z^{-1}q^{2n-1})$$

Theorem (Gauss)

$$\sum_{n=-\infty}^{\infty}(-1)^n q^{n^2}=\prod_{n\geq 1}\frac{(1-q^n)}{(1+q^n)}$$

Theorem (Cauchy)

$$1+\sum_{n\geq 1}\frac{(z;q)_n t^n}{(q;q)_n}=\prod_{n\geq 0}\frac{(1-ztq^n)}{(1-tq^n)}$$

Theorem (Sylvester)

$$\sum_{n\geq 0}\frac{(-1)^n z^n q^{(3n^2+n)/2}(1-zq^{2n+1})(zq;q)_n}{(q;q)_n}=\prod_{n\geq 1}(1-zq^n)$$