Combinatorial Proofs of Various q-Pell Identities via Tilings

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Joint work with Karen Briggs and James Sellers

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Pell Numbers

Recursive definition:

$$p_0 = 1$$
$$p_1 = 2$$
$$p_n = 2p_{n-1} + p_{n-2} \qquad \text{for } n \geq 2$$

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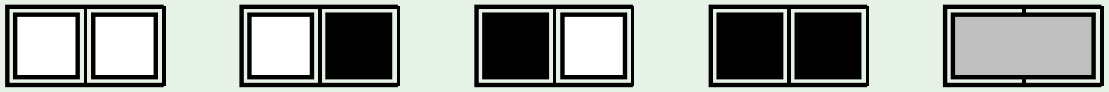
Example: $p_2 = 5$



Pell Identities

Position of last square

$$p_{2n} = 1 + 2\sum_{k=1}^{n} p_{2k-1} \qquad\qquad p_{2n+1} = 2\sum_{k=0}^{n} p_{2k}$$

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Number of dominoes

$$p_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} 2^{n-2k} \qquad p_{n-2} + p_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-k} \binom{n-k}{k} 2^{n-2k}$$

q-Pell Numbers

Recursive definition: (Santos and Sills, 2002)

$$P_0(q) = 1$$
$$P_1(q) = 1 + q$$
$$P_n(q) = (1 + q^n)P_{n-1}(q) + q^{n-1}P_{n-2}(q) \qquad \text{for } n \geq 2$$

Note that

$$P_n(1) = p_n$$

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Combinatorial definition (Idea):

Use the above recursion to introduce weighted Pell tilings so that $P_n(q)$ can be defined as the sum of the weights of all Pell tilings of length n.

q-Pell Tilings

Weight of tile t

$$w(t) = \begin{cases} q^i & \text{if } t \text{ is a gray domino at position } i \\ q^i & \text{if } t \text{ is a black square at position } i \\ 1 & \text{if } t \text{ is a white square at position } i \end{cases}$$

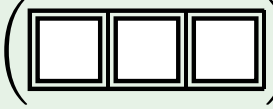
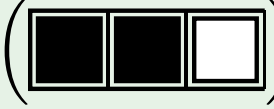
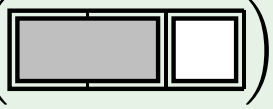
Weight of tiling T

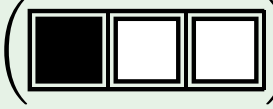
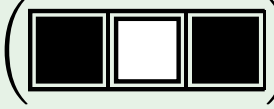
$$w(T) = \prod_{t \in T} w(t)$$

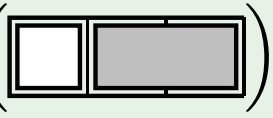
Combinatorial definition:

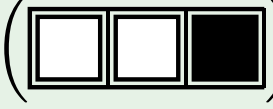
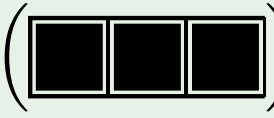
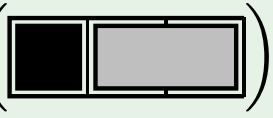
$$P_n(q) = \sum_{T \in \mathcal{T}_n} w(T)$$

Example: $P_3(q) = 1 + 2q + 2q^2 + 3q^3 + 2q^4 + q^5 + q^6$

$w\Big($  $\Big) = 1$ $\quad w\Big($  $\Big) = q^3$ $\quad w\Big($  $\Big) = q$

$w\Big($  $\Big) = q$ $\quad w\Big($  $\Big) = q^4$ $\quad w\Big($ $\Big) = q^4$

$w\Big($ $\Big) = q^2$ $\quad w\Big($ $\Big) = q^5$ $\quad w\Big($  $\Big) = q^2$

$w\Big($  $\Big) = q^3$ $\quad w\Big($  $\Big) = q^6$ $\quad w\Big($  $\Big) = q^3$

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Theorem: For all $n \geq 0$,

$$P_{2n}(q) = q^{n^2} + \sum_{k=1}^{n} q^{n^2-k^2}(1+q^{2k})P_{2k-1}(q).$$

Proof.

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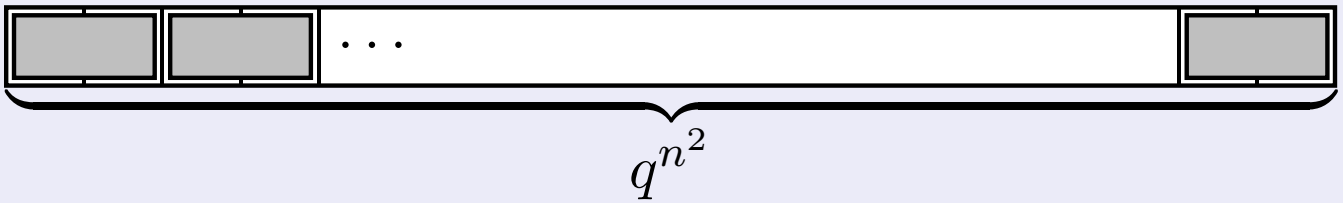

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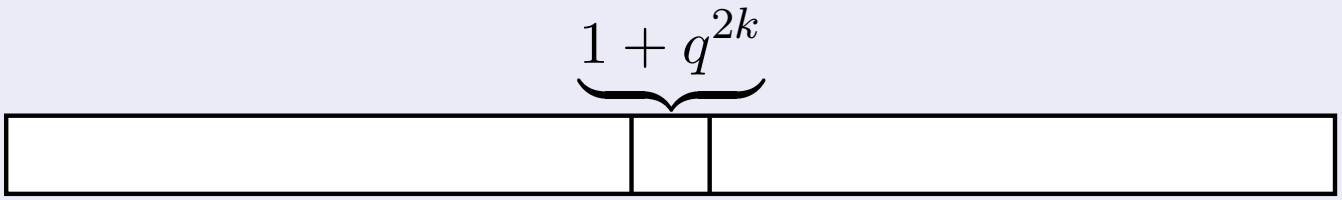

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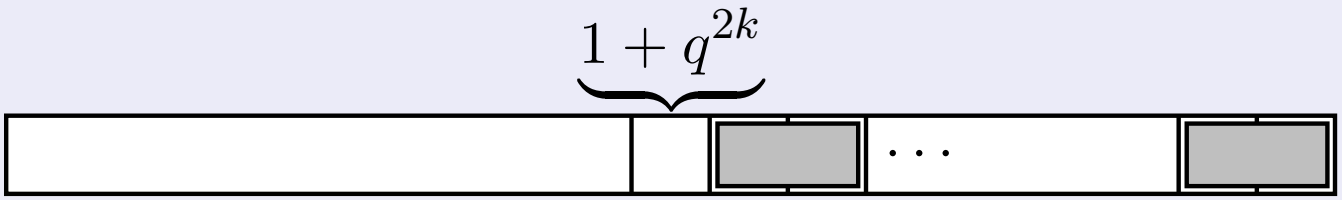

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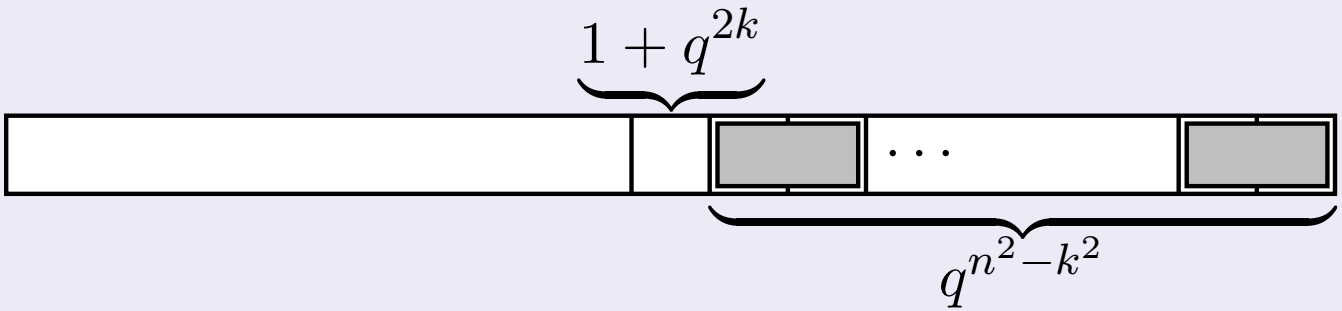

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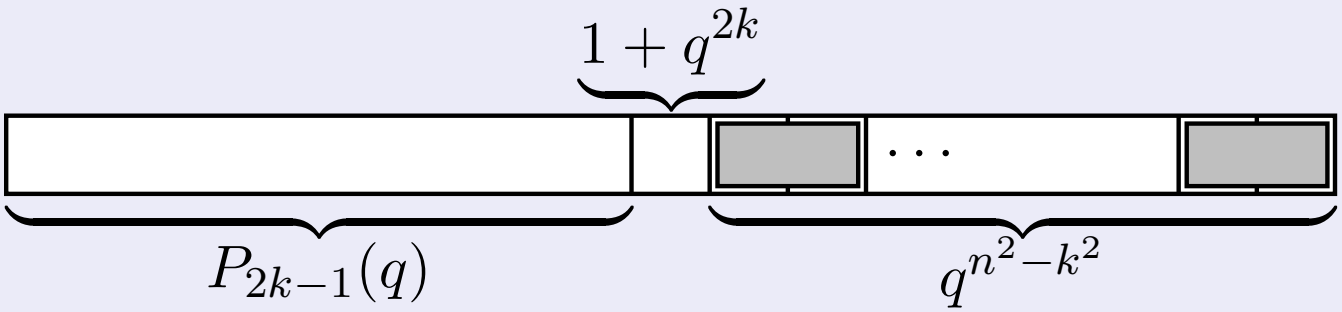

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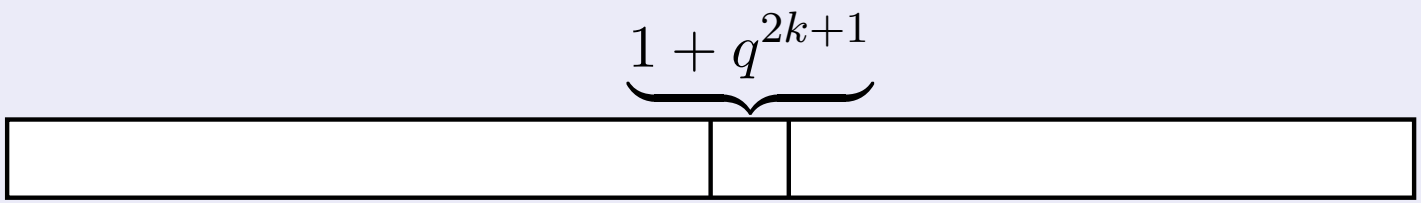

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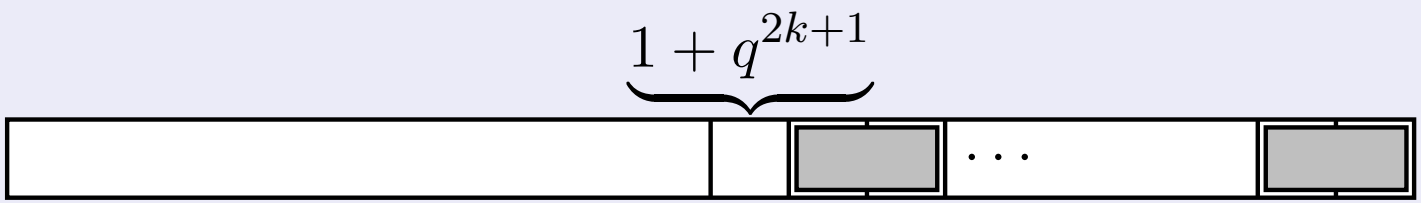

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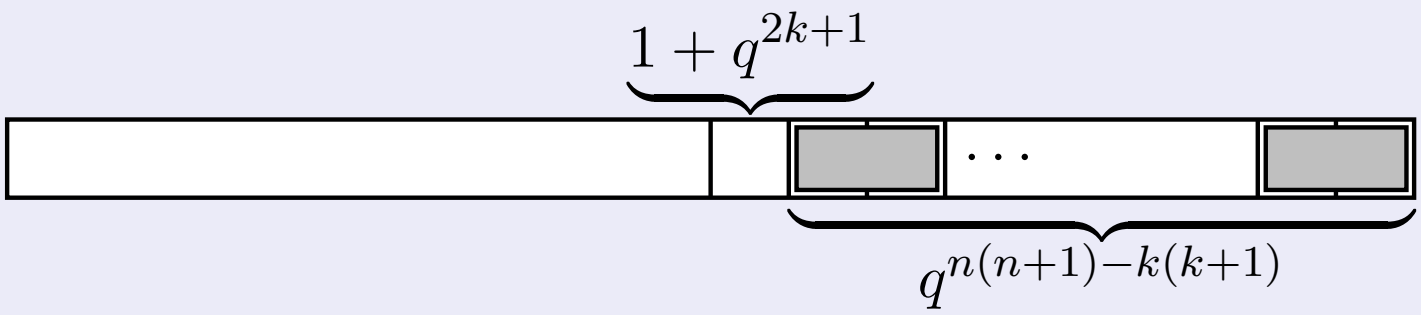

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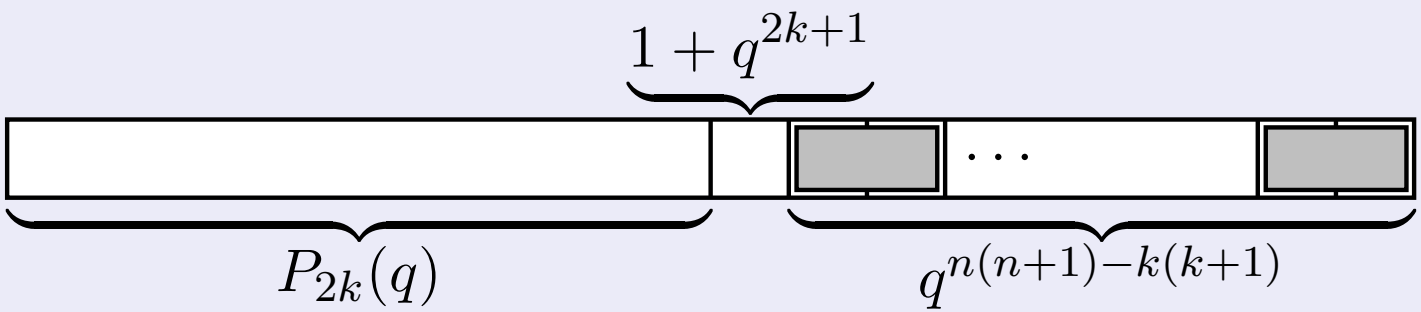


$\square$

m–shifted q–Pell numbers

Recursive definition:

$$P_0^{(m)}(q) = 1$$

$$P_1^{(m)}(q) = 1 + q^{m+1}$$

$$P_n^{(m)}(q) = (1 + q^{m+n})P_{n-1}^{(m)}(q) + q^{m+n-1}P_{n-2}^{(m)}(q) \qquad \text{for } n \geq 2$$

Note that

$$P_n^{(0)}(q) = P_n(q) \quad \text{and} \quad P_n^{(m)}(1) = p_n.$$

Combinatorial definition:

$P_n^{(m)}(q)$ is the sum of all weighted Pell Tilings that cover positions $m+1$ through $m+n$.

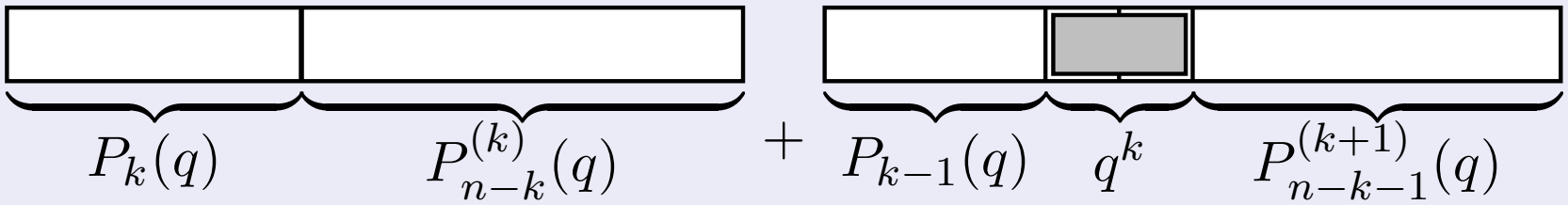
q-Analogue of $p_{2n} = p_n^2 + p_{n-1}^2$

Theorem: For all $n \geq 2$ and $1 \leq k \leq n-1$,

$$P_n(q) = P_k(q)P_{n-k}^{(k)}(q) + q^k P_{k-1}(q)P_{n-k-1}^{(k+1)}(q).$$

Proof.

No domino at position k | Domino at position k



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Consider $P_{2n}(q)$ and let $k = n$:

$$P_{2n}(q) = P_n(q)P_n^{(n)}(q) + q^n P_{n-1}(q)P_{n-1}^{(n+1)}(q)$$

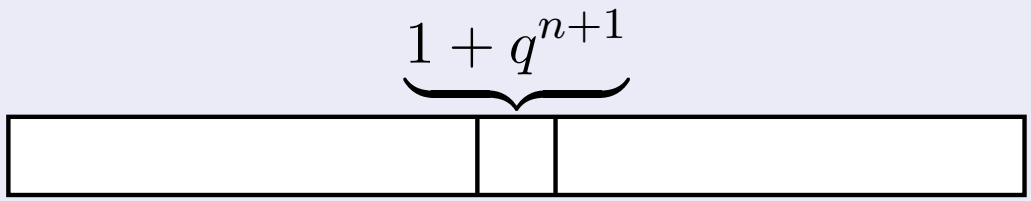
q-Analogue of $p_{2n+1} = 2(p_n^2 + 2\sum_{k=1}^{n} p_{n-k}^2)$

Theorem: For all $n \geq 0$,

$$P_{2n+1} = (1+q^{n+1})\left(P_n P_n^{(n+1)} + \sum_{k=1}^{n} q^{kn}(1+q^k)P_{n-k}P_{n-k}^{(n+k+1)}\right)$$

Proof. Locate square closest to the center of the board.

CASE I: Square in position $n+1$



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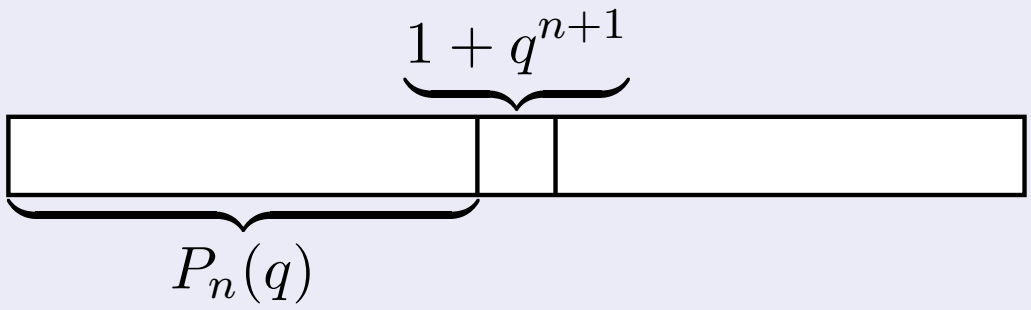

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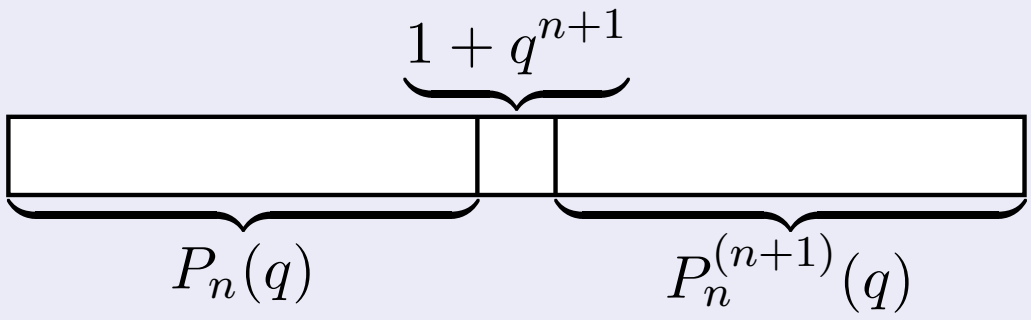

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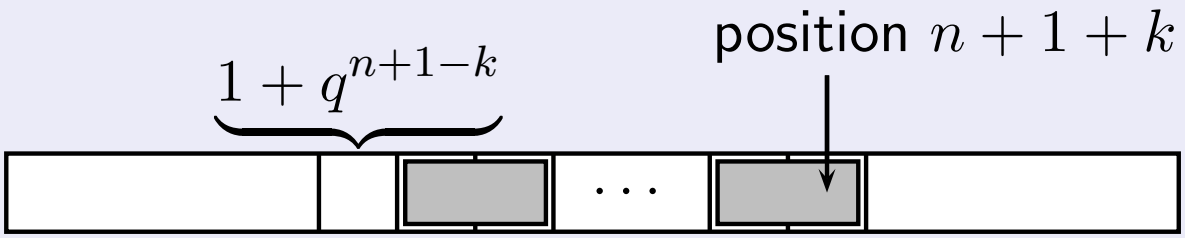

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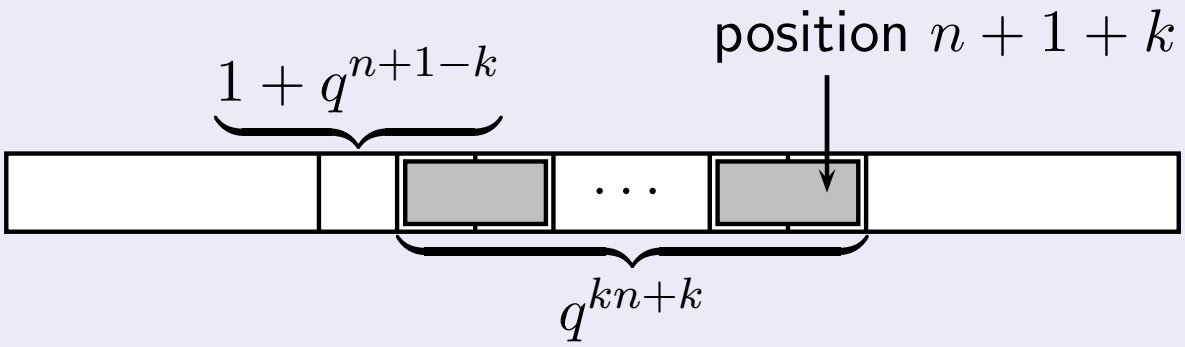

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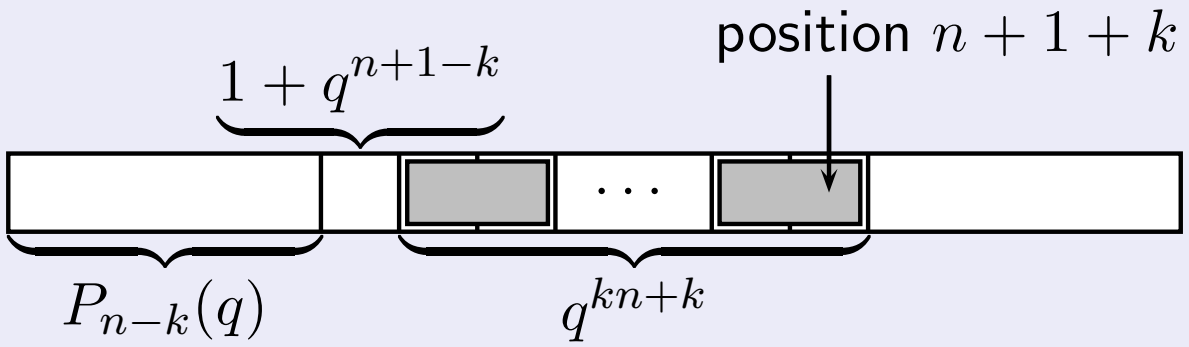

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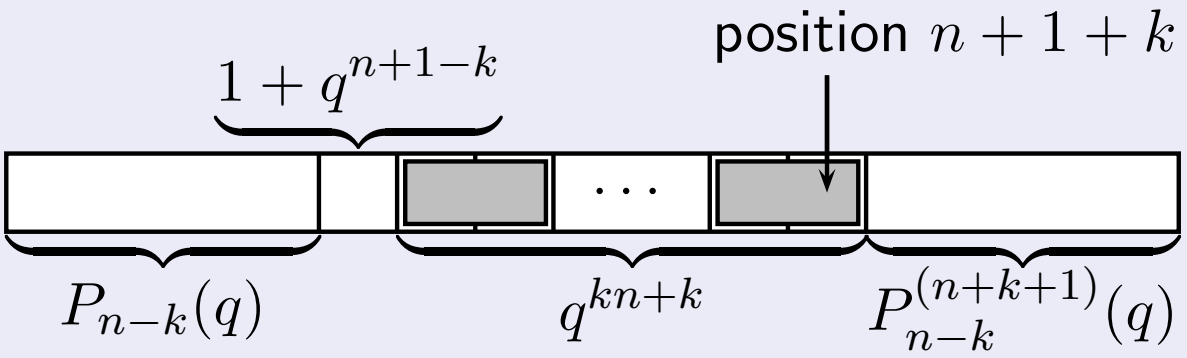

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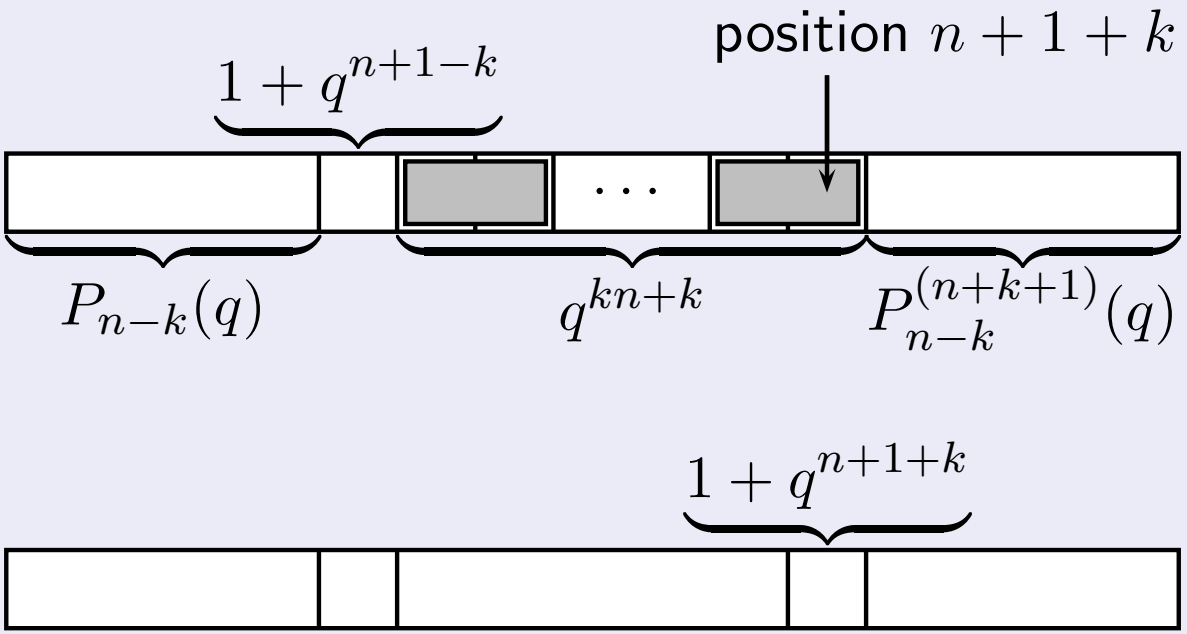

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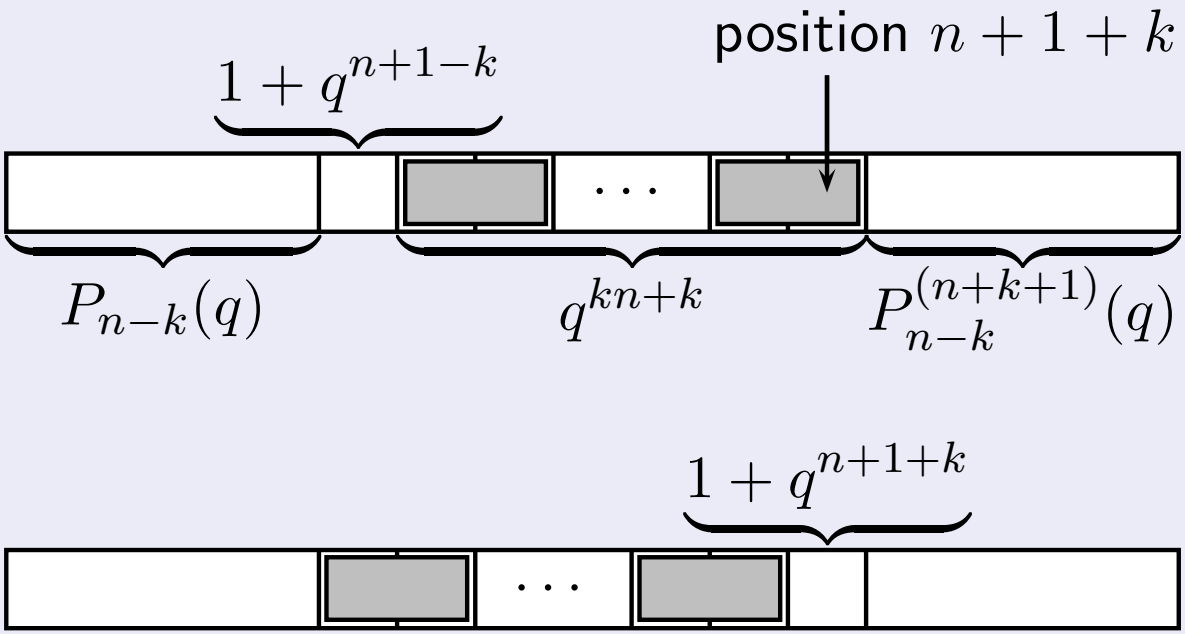

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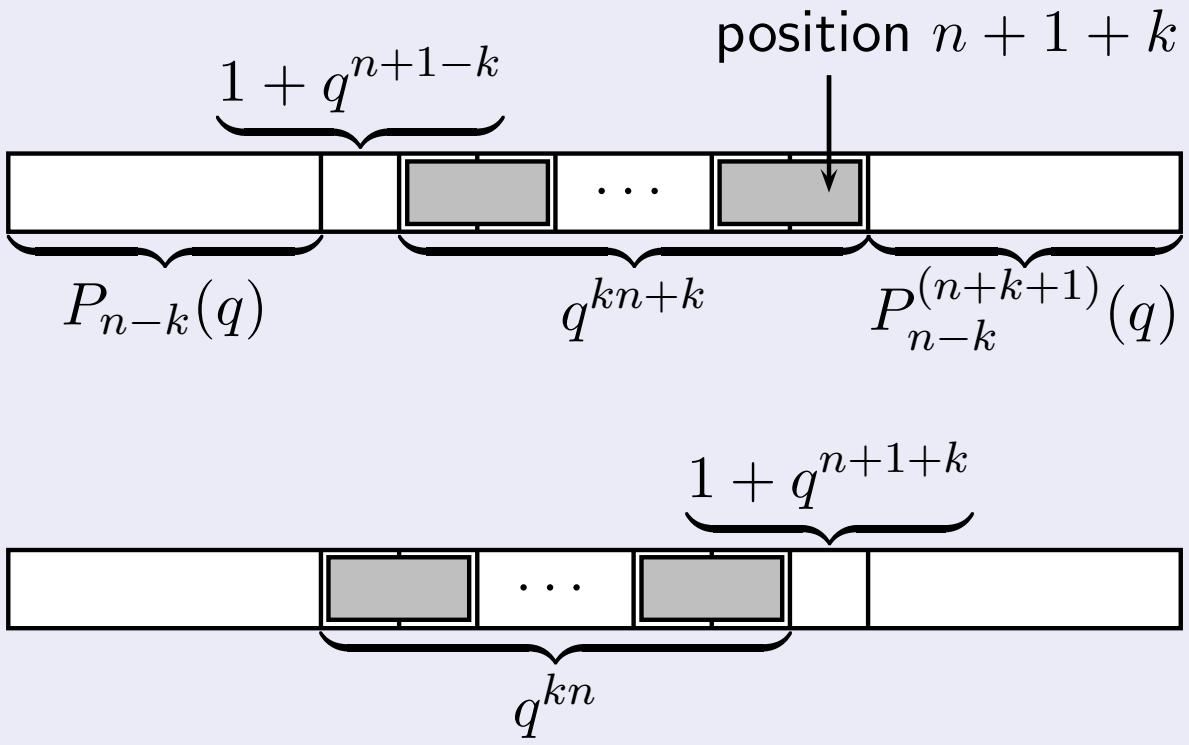

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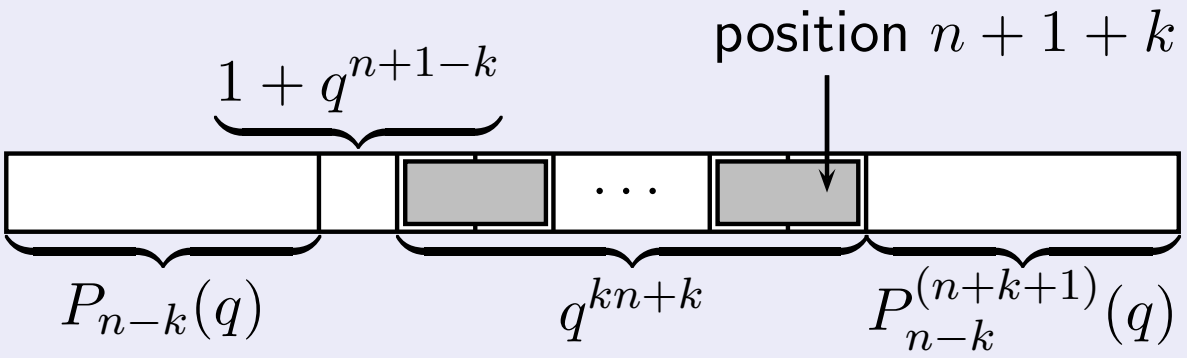

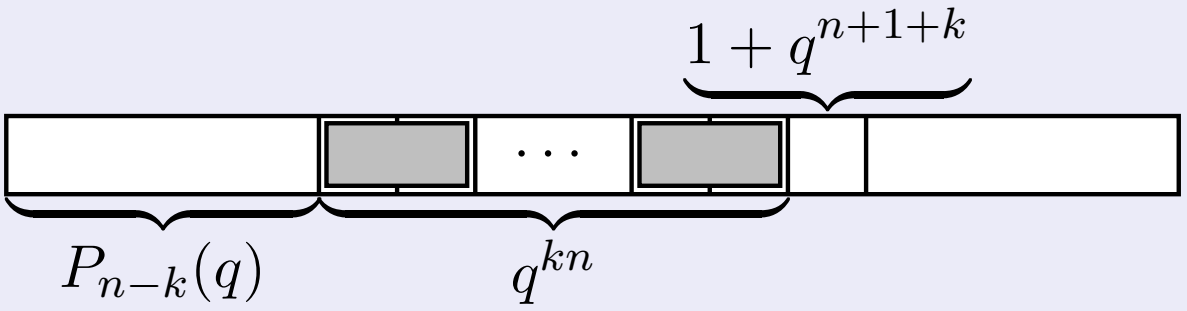

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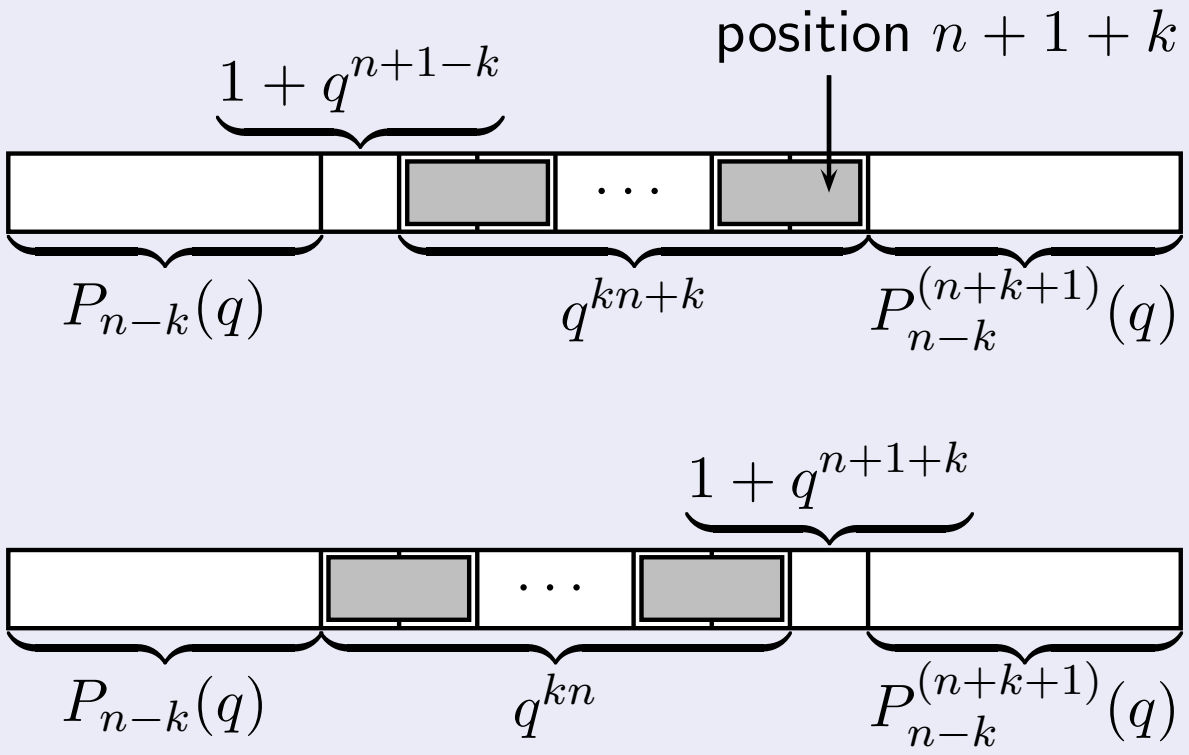

$\square$

q-Multinomial Coefficients and Inversions

Recall the usual q–analogue of the multinomial coefficients:

$$\begin{bmatrix} n_1 + n_2 + \cdots + n_r \\ n_1, n_2, \ldots, n_r \end{bmatrix}_q = \frac{[n_1 + n_2 + \cdots + n_r]_q!}{[n_1]_q![n_2]_q! \cdots [n_r]_q!}$$

where $[n]_q = 1 + q + q^2 + \cdots + q^{n-1} = \frac{1-q^n}{1-q}$ and $[n]_q! = [1]_q[2]_q \cdots [n]_q$

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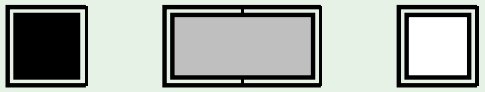
Theorem: (MacMahon)

$$\sum_{\sigma \in \mathcal{R}(1^{n_1} 2^{n_2} \cdots r^{n_r})} q^{inv(\sigma)} = \begin{bmatrix} n_1 + n_2 + \cdots + n_r \\ n_1, n_2, \ldots, n_r \end{bmatrix}_q$$

What does this have to with tilings?

$T \in \mathcal{T}_{i,j,k}$



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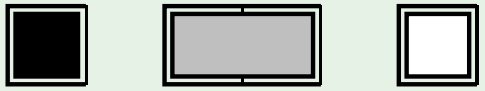

$$\updownarrow \quad \updownarrow \quad \updownarrow$$

$$1 \quad 2 \quad 3$$

$$T_\sigma \in \mathcal{T}_{i,j,k} \quad \uparrow \quad \sigma \in \mathcal{R}(1^i 2^j 3^k)$$

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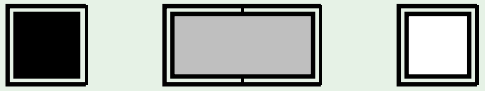

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T_{min}, the tiling of minimum weight, corresponds to the rearrangement

$$\sigma_{min} = \underbrace{111...11}_{j}\underbrace{222...22}_{k}\underbrace{333...33}_{l}$$

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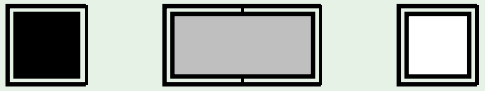

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and has weight

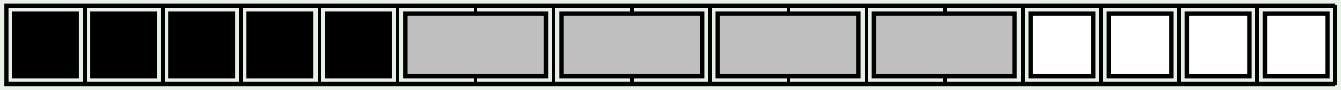
$$\sum_{r=1}^{j} r + \sum_{s=1}^{k}(j+2s-1) = \binom{j+1}{2} + jk + k^2.$$

Question:

How does the weight of $T \in \mathcal{T}_{i,j,k}$ differ from the weight of T_{min}?

Example

$T_{min} =$



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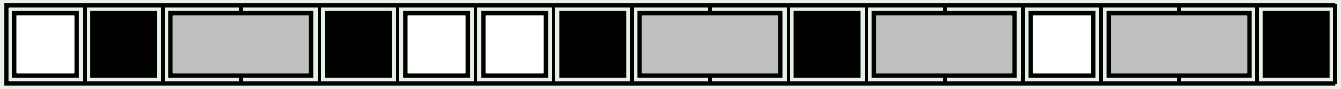

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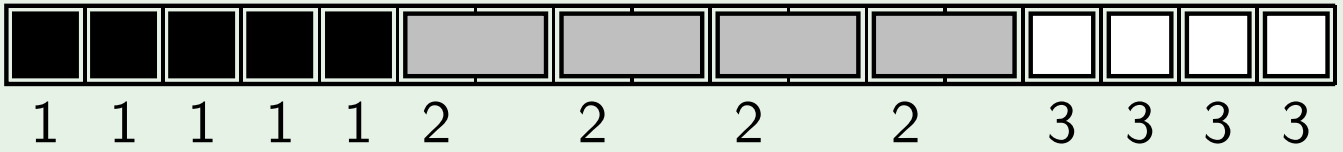

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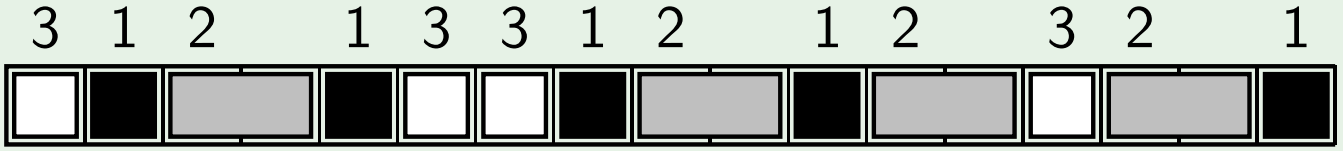

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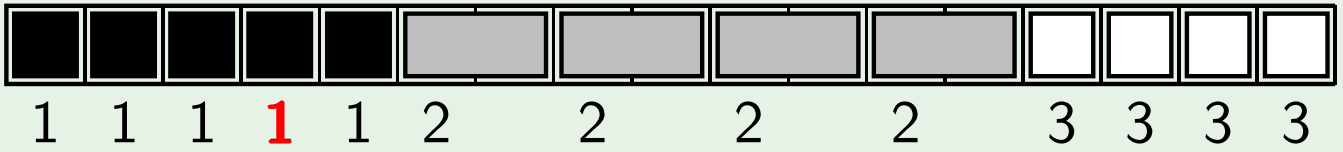

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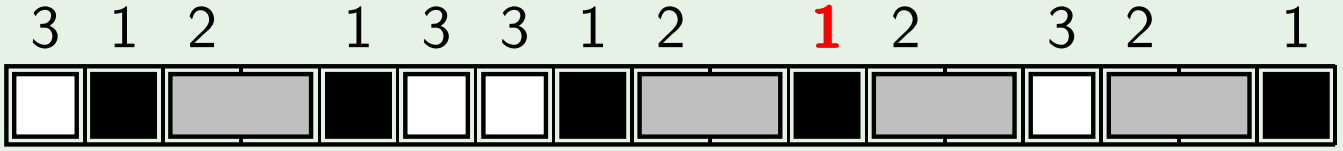

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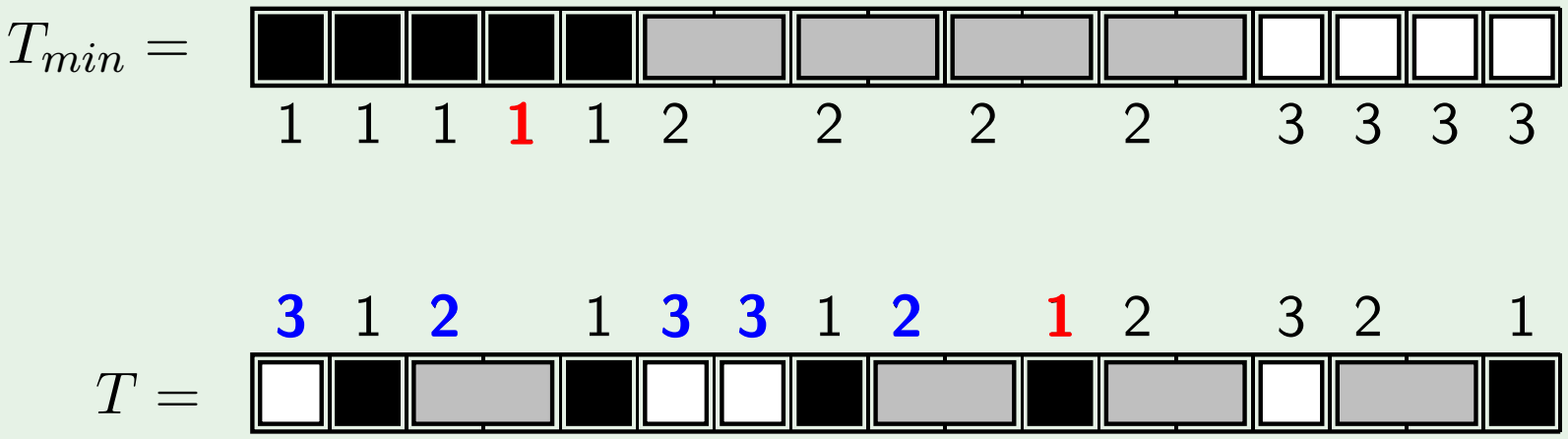

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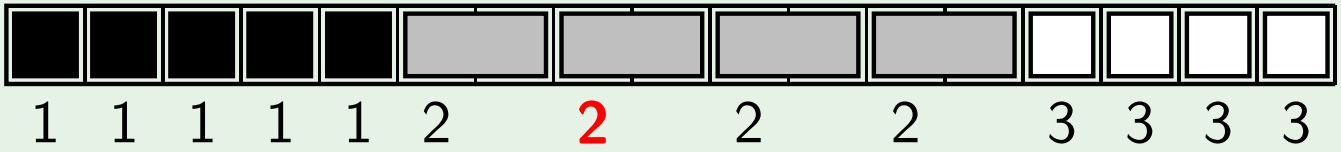

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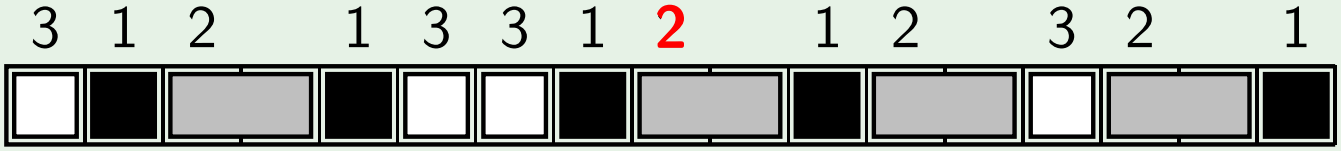

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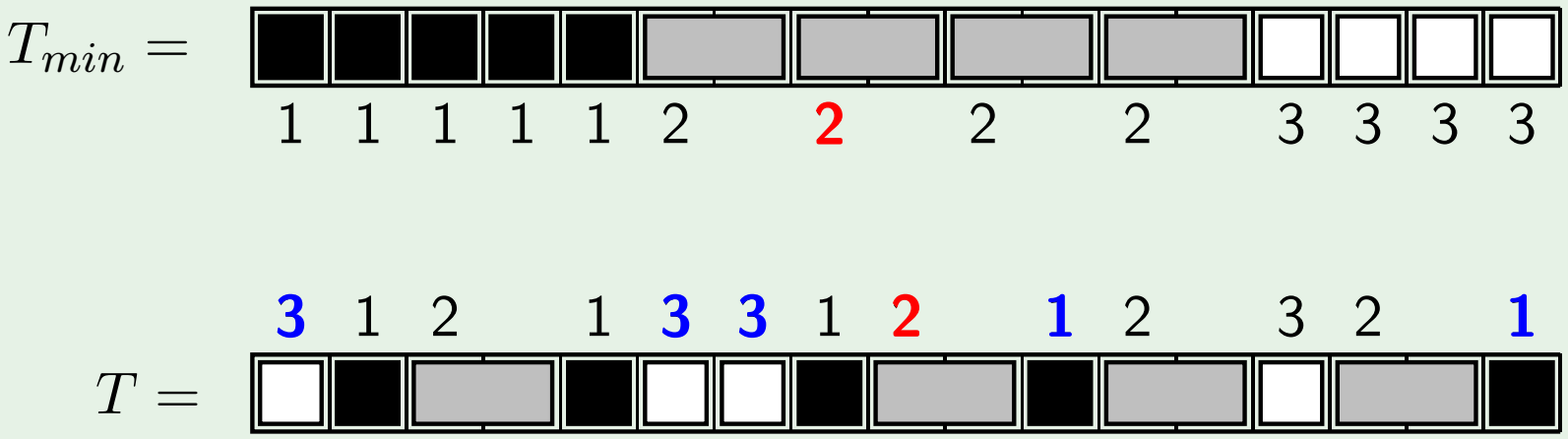

Recap

Total change in weight of black squares:

$$|\{i < j \mid \sigma_i = 3,\ \sigma_j = 1\}| + 2|\{i < j \mid \sigma_i = 2,\ \sigma_j = 1\}|.$$

Total change in weight of gray dominoes:

$$|\{i < j \mid \sigma_i = 3,\ \sigma_j = 2\}| - |\{i < j \mid \sigma_i = 2,\ \sigma_j = 1\}|.$$

Total change in weight:

$$inv(\sigma_T)$$

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Lemma: For all $\sigma \in \mathcal{R}(1^j 2^k 3^l)$,

$$\begin{aligned} w(T_\sigma) &= q^{inv(\sigma)} w(T_{min}) \\ &= q^{inv(\sigma) + \binom{j+1}{2} + jk + k^2} \end{aligned}$$

$$
\begin{aligned}
\sum_{T \in \mathcal{T}_{j,k,l}} w(T) &= \sum_{\sigma \in \mathcal{R}(1^j 2^k 3^l)} w(T_\sigma) \\
&= \sum_{\sigma \in \mathcal{R}(1^j 2^k 3^l)} q^{\binom{j+1}{2} + jk + k^2 + inv(\sigma)} \\
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Theorem

The generating function for q–Pell tilings with exactly j black squares, k gray dominoes and l white squares is given by

$$
q^{\binom{j+1}{2} + jk + k^2} \begin{bmatrix} j+k+l \\ j,k,l \end{bmatrix}_q .
$$

Theorem

The generating function for m–shifted q–Pell tilings with exactly j black squares, k gray dominoes and l white squares is given by

$$q^{m(j+k)+\binom{j+1}{2}+jk+k^2}\begin{bmatrix} j+k+l \\ j,k,l \end{bmatrix}_q.$$

Proof.

Starting with a tiling of length n, move each tile m positions to the right.

The weight of each black square and gray domino increases by a factor of q^m.

Thus the $j+k$ black squares and gray dominoes increase the weight of the tiling by a factor of $q^{m(k+j)}$, as claimed.

$\square$

q-Analogue of $p_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} 2^{n-2k}$

Theorem: For all $n \geq 0$,

$$P_n(q) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} q^{k^2} \begin{bmatrix} n-k \\ k \end{bmatrix}_q \prod_{i=1}^{n-2k} (1 + q^{k+i}).$$

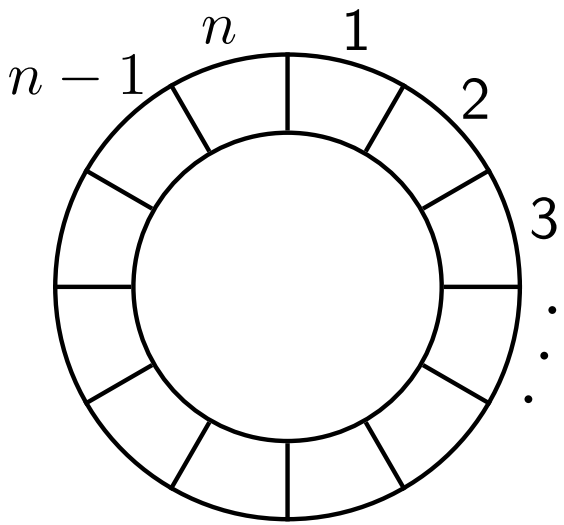
Proof.

Count tilings of length n according to k, the number of gray dominoes, and j, the number of black squares.

$$\begin{aligned} P_n(q) &= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{j=0}^{n-2k} q^{\binom{j+1}{2}+jk+k^2} \begin{bmatrix} n-k \\ j, k, n-j-2k \end{bmatrix}_q \\ &= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} q^{k^2} \begin{bmatrix} n-k \\ k \end{bmatrix}_q \sum_{j=0}^{n-2k} q^{\binom{j+1}{2}+jk} \begin{bmatrix} n-2k \\ j \end{bmatrix}_q \end{aligned}$$

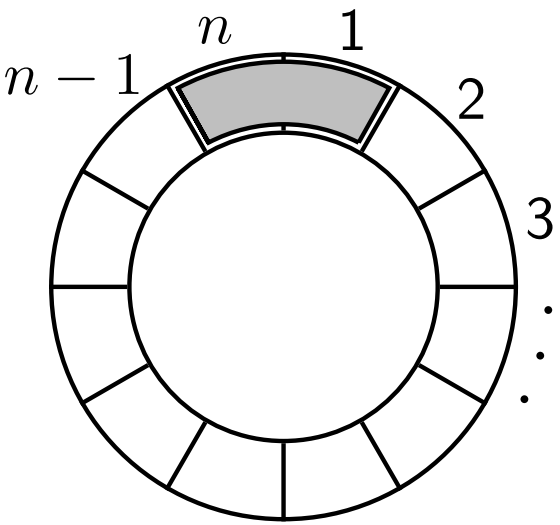
$\square$

q–Pell Bracelets

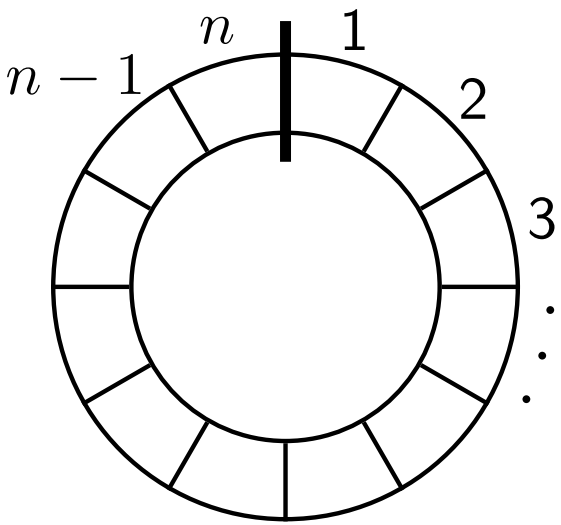


q–Pell Bracelets

Domino at position n



No domino at position n



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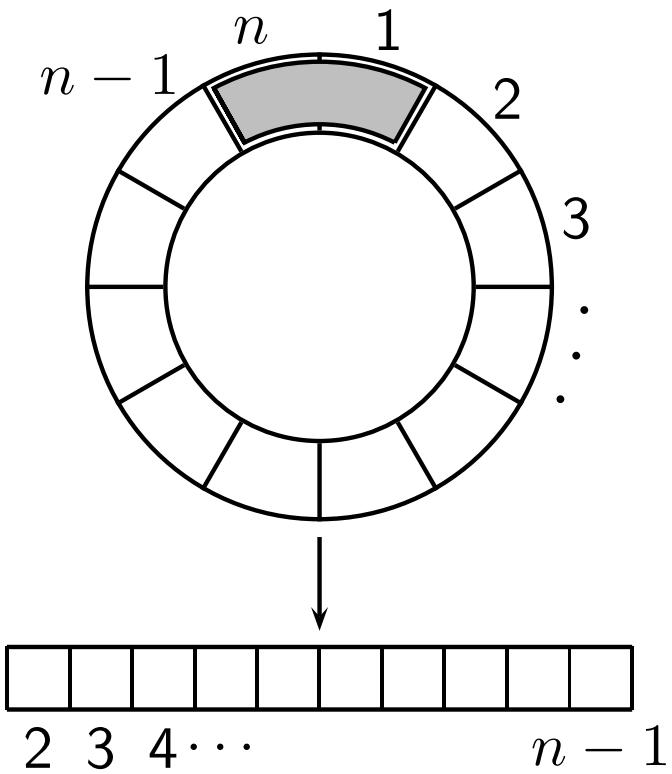

$q^n P^{(1)}_{n-2}(q)$

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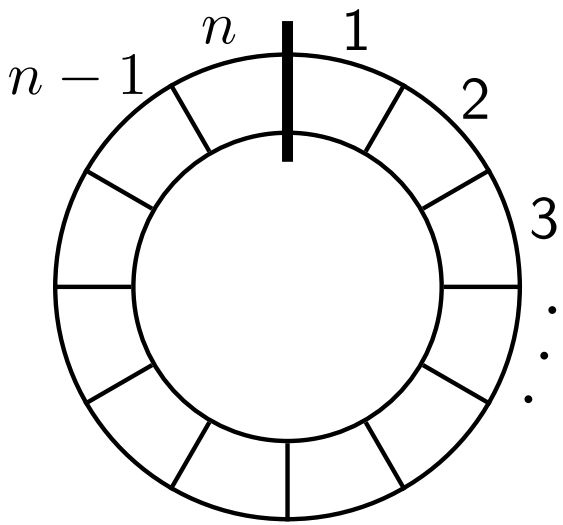

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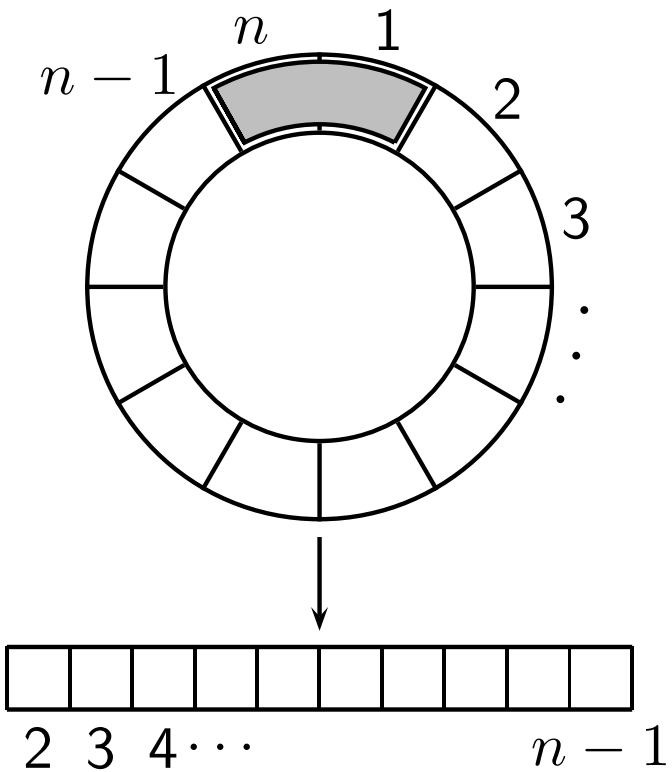

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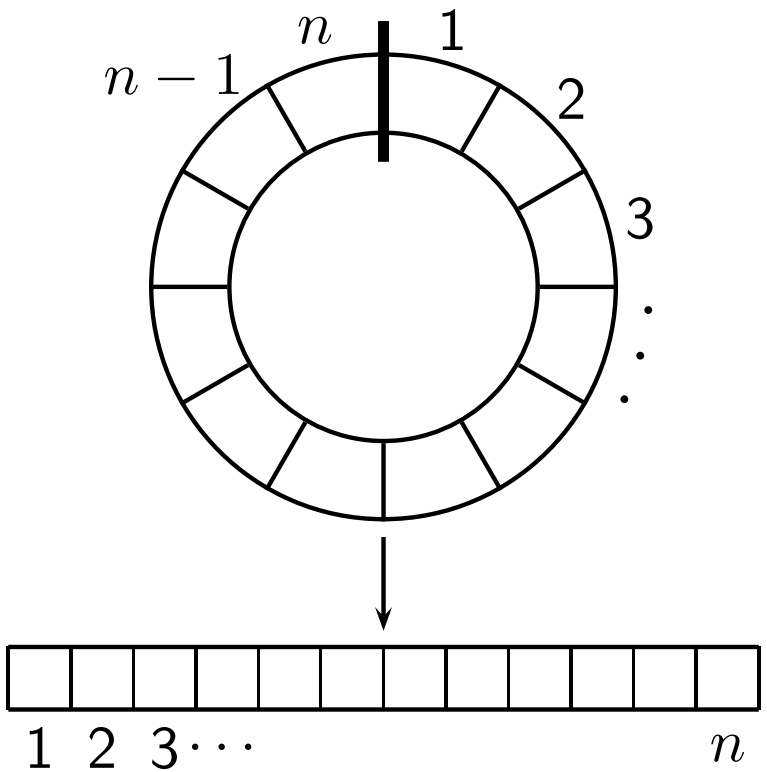

$P_n(q)$

Theorem

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$$q^{\binom{j+1}{2}+jk+k^2}\frac{[j+2k+l]_q}{[j+k+l]_q}\begin{bmatrix} j+k+l \\ j,k,l \end{bmatrix}_q$$

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$$q^{\binom{j+1}{2}+jk+k^2}\begin{bmatrix} j+k+l \\ j,k,l \end{bmatrix}_q + q^{\binom{j+1}{2}+jk+k^2+j+k+l}\begin{bmatrix} j+k-1+l \\ j,k-1,l \end{bmatrix}_q$$

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$$q^{\binom{j+1}{2}+jk+k^2}\frac{[j+2k+l]_q}{[j+k+l]_q}\begin{bmatrix} j+k+l \\ j,k,l \end{bmatrix}_q$$

$$\begin{aligned}
q^{\binom{j+1}{2}+jk+k^2}\begin{bmatrix} j+k+l \\ j,k,l \end{bmatrix}_q + q^{\binom{j+1}{2}+jk+k^2+j+k+l}\begin{bmatrix} j+k-1+l \\ j,k-1,l \end{bmatrix}_q \\
&= q^{\binom{j+1}{2}+jk+k^2}\begin{bmatrix} j+k+l \\ j,k,l \end{bmatrix}_q\left(1+\frac{q^{j+k+l}[k]_q}{[j+k+l]_q}\right) \\
&= q^{\binom{j+1}{2}+jk+k^2}\begin{bmatrix} j+k+l \\ j,k,l \end{bmatrix}_q\left(\frac{[j+k+l]_q+q^{n-k}[k]_q}{[j+k+l]_q}\right) \\
&= q^{\binom{j+1}{2}+jk+k^2}\frac{[j+2k+l]_q}{[j+k+l]_q}\begin{bmatrix} j+k+l \\ j,k,l \end{bmatrix}_q.
\end{aligned}$$

q-Analogue of $p_n + p_{n-2} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-k}\binom{n-k}{k}2^{n-2k}$

Theorem: For all $n \geq 2$,

$$P_n(q) + q^n P_{n-2}^{(1)}(q) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} q^{k^2} \frac{[n]_q}{[n-k]_q} \begin{bmatrix} n-k \\ k \end{bmatrix}_q \prod_{i=1}^{n-2k} (1 + q^{k+i}).$$

Proof.

Count bracelets of length n according to k, the number of gray dominoes, and j, the number of black squares.

$$\begin{aligned} P_n(q) + q^n P_{n-2}^{(1)}(q) &= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{j=0}^{n-2k} q^{\binom{j+1}{2}+jk+k^2} \frac{[n]_q}{[n-k]_q} \begin{bmatrix} n-k \\ j, k, n-j-2k \end{bmatrix}_q \\ &= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} q^{k^2} \frac{[n]_q}{[n-k]_q} \begin{bmatrix} n-k \\ k \end{bmatrix}_q \sum_{j=0}^{n-2k} q^{\binom{j+1}{2}+jk} \begin{bmatrix} n-2k \\ j \end{bmatrix}_q \end{aligned}$$

$\square$

Generalizations

Suppose we have a colors of squares and b colors of dominoes.

$$w(t) = \begin{cases} q^{ij} & \text{if } t \text{ is a } j\text{–colored domino at position } i \\ q^{i(j-1)} & \text{if } t \text{ is a } j\text{–colored square at position } i. \end{cases}$$

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$$\begin{aligned} P_n(a,b;q) = {} & (1 + q^n + \cdots + q^{(a-1)n}) P_{n-1}(a,b;q) \\ & + (q^{n-1} + q^{2(n-1)} + \cdots + q^{b(n-1)}) P_{n-2}(a,b;q) \end{aligned}$$

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$$\begin{aligned} P_n(a,b;q) &= (1 + q^n + \cdots + q^{(a-1)n}) P_{n-1}(a,b;q) \\ &\quad + (q^{n-1} + q^{2(n-1)} + \cdots + q^{b(n-1)}) P_{n-2}(a,b;q) \\ &= \frac{1-q^{an}}{1-q^n} P_{n-1}(a,b;q) + q^{n-1} \frac{1-q^{b(n-1)}}{1-q^{n-1}} P_{n-2}(a,b;q) \end{aligned}$$

with initial conditions $P_0(a,b;q) = 1$ and $P_1(a,b;q) = \frac{1-q^a}{1-q}$.

Position of Last Square

Theorem: For all $n \geq 0$,

$$P_{2n}(a,b;q) = q^{n^2} \prod_{k=1}^{n} \frac{1-q^{(2k-1)b}}{1-q^{2k-1}} + \sum_{k=0}^{n} q^{n^2-k^2} \frac{1-q^{2ka}}{1-q^{2k}} \prod_{j=k+1}^{n} \frac{1-q^{(2j-1)b}}{1-q^{2j-1}} P_{2k-1}(a,b;q).$$

Theorem: For all $n \geq 0$,

$$P_{2n+1}(a,b;q) = \sum_{k=0}^{n} q^{n(n+1)-k(k+1)} \frac{1-q^{a(2k+1)}}{1-q^{2k+1}} \prod_{j=k+1}^{n} \frac{1-q^{2jb}}{1-q^{2j}} P_{2k}(a,b;q).$$

Square Closest to the Center

Theorem: For all $n \geq 2$ and $1 \leq i \leq n-1$,

$$P_n(a,b;q) = P_i(a,b;q)P_{n-i}^{(i)}(a,b;q) + q^i \frac{1-q^{bi}}{1-q^i} P_{i-1}(a,b;q)P_{n-i-1}^{(i+1)}(a,b;q).$$

Theorem: For all $n \geq 0$,

$$P_{2n+1}(a,b;q) = \frac{1-q^{a(n+1)}}{1-q^{n+1}} P_n(a,b;q)P_n^{(n+1)}(a,b,q) + \sum_{i=1}^{n} A_{n,i}(q)P_{n-i}(a,b;q)P_{n-i}^{(n+i+1)}(a,b;q)$$

where $A_{n,i}(q) = q^{in+i}\frac{1-q^{a(n+1-i)}}{1-q^{n+1-i}} \prod_{j=1}^{i} \frac{1-q^{b(n-i+2j)}}{1-q^{n-i+2j}} + q^{in}\frac{1-q^{a(n+1+i)}}{1-q^{n+1+i}} \prod_{j=1}^{i} \frac{1-q^{b(n-i+2j-1)}}{1-q^{n-i+2j-1}}.$