

Research Statement

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My research area is algebraic topology. My specific focus is stable homotopy theory, where a key problem is to compute the stable homotopy groups of spheres $\pi_k S$ for $0 \leq k \in \mathbb{Z}$. The group $\pi_k S$ is the k th stable stem and is defined to be

$$\pi_k S := \pi_{n+k} S^n \quad \text{for any integer } n > k + 1$$

where “stable” refers to the group being independent of n for n sufficiently large. In particular, a stable stem comprises all possible continuous maps between two spheres, up to deformation. Such maps are fundamental as they give “gluing data” for building CW complexes. Despite their geometrically simple definition, the stable stems have a surprisingly rich and unpredictable algebraic structure. Perhaps even more surprising is the major role number theory has played in uncovering this structure in recent years. Elliptic curves and modular forms, for example, have been mainstays in the field since the 1990s, and together with more classical results they can account for at least the first 45 stable stems. My work leverages this fruitful connection with number theory to study the p -torsion of the stable stems at odd primes p .

Most recently, I have been exploring the implications of a known correspondence between the stable stems and modular forms. For every prime p , there is a distinguished infinite family of elements in the p -torsion of the stable stems known as the divided beta family. Behrens has characterized each family for $p \geq 5$ in terms of modular forms satisfying certain congruences modulo p . However, the proof does not construct the modular form tied to each beta family element. In [11] I show that the *Victor Miller basis*, a computational tool in the theory of modular forms, is ideal for making for such constructions. My goals for this work going forward are to study the divided beta family via its corresponding modular forms, and to identify number theoretic properties that distinguish the particular modular forms arising in this way.

In my 2013 PhD thesis [13] and in a subsequent paper [12], I use spectral sequences to study algebraic properties of a geometric object $Q(2)$. Using elliptic curves and certain morphisms between them, M. Behrens constructed $Q(2)$ as a tool to approximate the 3-torsion of the stable stems. My spectral sequence computations unveil portions of this approximation. As an application, I also give computational evidence that $Q(2)$ detects the divided beta family at the prime 3. In future work I hope to prove that this detection indeed occurs.

I shall freely use the language of spectra and their associated homology theories in what follows. Spectra are the stable homotopy theory analogs of topological spaces, and they yield homology theories in much the same way that Eilenberg-Mac Lane spaces yield the ordinary cohomology functors. A good reference for this is Part III of [2].

I Modular forms and the divided beta family

As the notation suggests, the stable stems are precisely the homotopy groups of the sphere spectrum S . Homotopy theorists prefer to approach these groups “one prime at a time;” that is, to fix a prime p and study $\pi_* S \otimes \mathbb{Z}_{(p)}$. An important computational tool is the Adams-Novikov spectral sequence (ANSS) [15], which takes the form

$$E_2^{s,t} = \text{Ext}_{\Gamma}^{s,t}(BP_*, BP_*) \Rightarrow \pi_{t-s} S \otimes \mathbb{Z}_{(p)}$$

where $BP_* = \mathbb{Z}_{(p)}[v_1, v_2, \dots]$ and $\Gamma = BP_*[t_1, t_2, \dots]$ are algebras related to the theory of formal group laws [14]. The E_2 -term is prohibitively difficult to compute fully. However, for each $n \geq 1$, $\text{Ext}_{\Gamma}^{n,*}(BP_*, BP_*)$ contains an infinite family of cohomology classes with representatives expressible in terms of p, v_1, \dots, v_n . Each such family is “periodic” in the sense that its elements are related multiplicatively. Known as the Greek letter families, they bring some semblance of order to an otherwise chaotic algebraic object.

The first of the Greek letter families is the alpha family $\{\alpha_{i/j}\} \subset \text{Ext}_{\Gamma}^{1,*}(BP_*, BP_*)$ where i and j are certain positive integers that dictate how a given element is represented in terms of p and v_1 . The alpha family admits a global number-theoretic description, as follows: $\alpha_{i/j}$ has order p^j , and p^j is equal to the p -factor of the denominator of B_t/t where $t = (p-1)i$ and B_t is the t -th Bernoulli number.

Next is the beta family $\{\beta_{i/j,k}\} \subset \text{Ext}_{\Gamma}^{2,*}(BP_*, BP_*)$ with i, j, k dictating how a given element is represented in terms of p, v_1 , and v_2 . Using analogs of the spectrum $Q(2)$, Behrens [6] has proven that for all $p \geq 5$, there exists a 1-1 correspondence $\beta_{i/j,k} \leftrightarrow f_{i/j,k}$, where $f_{i/j,k}$ is a modular form over \mathbb{Z} of weight $t = (p^2 - 1)i$ with Fourier coefficients that satisfy certain congruence conditions modulo p . These Fourier coefficients are like Bernoulli numbers in that they encode important number theoretic information. Behrens’ proof is not constructive, however, so a natural question arises: What *is* the modular form $f_{i/j,k}$?

The Fourier expansion of a modular form f over \mathbb{Z} has the form $f(q) = \sum_{n=0}^{\infty} a_n q^n$ where the Fourier coefficients a_n are integers. If $M_t(\mathbb{Z})$ denotes the set of modular forms of weight t over \mathbb{Z} , then $M_t(\mathbb{Z})$ is a free \mathbb{Z} -module whose rank d is computable in terms

of t . Each $M_t(\mathbb{Z})$ has a Victor Miller basis [10], which is an ordered basis $\{\varphi_0, \dots, \varphi_{d-1}\}$ where $\varphi_i(q) = q^i + O(q^d)$. It is therefore a basis in reduced row-echelon form with respect to the powers of q in each Fourier expansion.

Among the conditions that $f_{i/j,k}$ must satisfy in Behrens' theorem is a constraint on the lowest power of q that can appear in its Fourier expansion. This makes the Victor Miller basis the ideal tool for constructing $f_{i/j,k}$. I have proven a theorem that describes how this can be done for beta family elements with $j = k = 1$.

Theorem A (Larson). *Let $1 \leq i \in \mathbb{Z}$.*

- (a) *If $p = 5$, $f_{i/1,1}$ is an integer multiple of $\varphi_{d-1} = \Delta^{d-1} \in M_{12(d-1)}(\mathbb{Z})$.*
- (b) *If $p > 5$ and $p \equiv 11 \pmod{12}$, $f_{i/1,1}$ is a \mathbb{Z} -linear combination of the last $\frac{p+13}{12}$ Victor Miller basis elements for $M_{i(p^2-1)}(\mathbb{Z})$.*
- (c) *If $p > 5$ and $p \equiv 1, 5$, or $7 \pmod{12}$, $f_{i/1,1}$ is a \mathbb{Z} -linear combination of the last $\left\lceil \frac{p-1}{12} \right\rceil$ Victor Miller basis elements for $M_{i(p^2-1)}(\mathbb{Z})$.*

In future work, I hope to obtain a precise description of all modular forms $f_{i/j,k}$ in terms of the Victor Miller basis, and to study their number-theoretic properties.

2 $Q(2)$ and the $K(2)$ -local sphere

The p -torsion of the stable stems can be recovered as the homotopy groups of the p -local sphere $S_{(p)}$, a spectrum with the property that $\pi_* S_{(p)} = \pi_* S \otimes \mathbb{Z}_{(p)}$. The construction of $S_{(p)}$ is a special case of constructing the Bousfield E -localization $L_E X$ of a spectrum X with respect to the homology theory E . Specifically, $S_{(p)}$ is equivalent to $L_{M\mathbb{Z}_{(p)}} S$ where $M\mathbb{Z}_{(p)}$ is the Moore spectrum for $\mathbb{Z}_{(p)}$. For an arbitrary X , $L_E X$ is a spectrum that, roughly speaking, is “the part of X that E can see.” The homotopy groups of $L_E X$ will generally be more accessible than those of X .

Homotopy theorists have learned that localizing $S_{(p)}$ further, with respect to more exotic homology theories, can yield useful approximations of $\pi_* S_{(p)}$. Chief among these exotic homology theories are the Morava K -theories $K(n)$ for $n \geq 0$. In fact, the chromatic convergence theorem of Hopkins and Ravenel implies that the spectra $L_{K(n)} S_{(p)}$ —the $K(n)$ -local spheres—offer better and better approximations to $S_{(p)}$ as n gets large. Said differently, $\pi_* S_{(p)}$ is governed by $\pi_* L_{K(n)} S_{(p)}$.

The $K(0)$ -local sphere sees the non-torsion in $\pi_* S_{(p)}$, while the $K(1)$ -local sphere is closely tied to the alpha family and image of the J -homomorphism [1]; both are completely understood at all primes. By contrast, very little is known about $\pi_* L_{K(n)} S_{(p)}$ for

$n \geq 3$. Sitting in the middle of these two extremes is the $K(2)$ -local sphere $L_{K(2)}S_{(p)}$, whose homotopy groups have been a primary target of computational efforts over the last 15 years. For the remainder of this section I shall focus on the $K(2)$ -local sphere at the prime 3.

Behrens [5], in an effort to reinterpret and extend previous work of Shimomura [16] and Goerss-Henn-Mahowald-Rezk [8], studies $L_{K(2)}S_{(3)}$ by constructing a ring spectrum $Q(2)$ and observing that there is a fiber sequence

$$DQ(2) \rightarrow L_{K(2)}S_{(3)} \xrightarrow{\eta} Q(2)$$

where $DQ(2)$ is the $K(2)$ -local Spanier-Whitehead dual of $Q(2)$ and η is the $K(2)$ -localization of its unit map. This fiber sequence links the homotopy groups of $Q(2)$ with those of $L_{K(2)}S_{(3)}$ in a long exact sequence, making $\pi_*Q(2)$ an algebraic object of significant interest.

By definition, $Q(2)$ is the homotopy limit of a semi-cosimplicial diagram of spectra. More precisely,

$$Q(2) = \operatorname{holim}(TMF \rightrightarrows TMF \vee TMF_0(2) \rightrightarrows TMF_0(2)) \quad (1)$$

where TMF and $TMF_0(2)$ are variants of the spectrum of topological modular forms [9]. The arrows “ \rightrightarrows ” and “ \Rightarrow ” are alternating sums of maps induced by degree 2 isogenies of elliptic curves (hence the “2” in the notation for $Q(2)$) that satisfy the usual cosimplicial identities. By virtue of its construction, $Q(2)$ has an Adams-Novikov style spectral sequence converging to its homotopy groups that is stitched together from similar spectral sequences for TMF and $TMF_0(2)$. The input is the cohomology of a certain elliptic curve Hopf algebroid. This cohomology, hereafter denoted $\operatorname{Ext}^{*,*}$, has been completely computed by Hopkins and Miller (see, e.g., [3]).

In [13] I produce a complete computation of the Adams-Novikov E_2 -term for $Q(2)$ in terms of $\operatorname{Ext}^{*,*}$ up to a small ambiguity in two submodules U^1 and U^2 , for which explicit generators could not be found. Fortunately, this ambiguity is nearly solved, and the answer will appear in a forthcoming version of the paper [12].

Theorem B (Larson). *Let $E_2^{s,t}Q(2)$ denote the Adams-Novikov E_2 -term for $Q(2)$.*

(a) $E_2^{0,0}Q(2) = \mathbb{Z}_{(3)}$ and $E_2^{0,t}Q(2) = 0$ for $t > 0$.

$$(b) E_2^{1,t}Q(2) = \begin{cases} \bigoplus_{n \in \mathbb{N}} \mathbb{Z}_{(3)}, & t = 0, \\ \mathbb{Z}/(3) \oplus \mathbb{Z}/(3), & t = 4, \\ \mathbb{Z}/(3^{\nu_3(3m)}), & t = 4m, m \geq 2, \\ U^{1,t}, & t = 4m + 2, m \geq 1, m \equiv 13 \pmod{27}, \\ \mathbb{Z}/(3^{\nu_3(6m+3)}), & t = 4m + 2, m \geq 1, m \not\equiv 13 \pmod{27}, \\ 0, & \text{otherwise.} \end{cases}$$

$$(c) E_2^{2,t}Q(2) = \text{Ext}^{2,t} \oplus \text{Ext}^{1,t} \oplus M, \text{ where}$$

$$M = \begin{cases} \bigoplus_{n \in \mathbb{N}} \mathbb{Z}/3^{\nu_3(6m+3)}, & t = 4m + 2, m \leq -1, \\ U^{2,t} \oplus \left(\bigoplus_{n \in \mathbb{N}} \mathbb{Z}/3^{\nu_3(6m+3)} \right), & t = 4m + 2, m \geq 1, m \equiv 13 \pmod{27}, \\ \bigoplus_{n \in \mathbb{N}} \mathbb{Z}/3^{\nu_3(6m+3)}, & t = 4m + 2, m \geq 1, m \not\equiv 13 \pmod{27}, \\ 0, & \text{otherwise.} \end{cases}$$

$$(d) E_2^{s,t}Q(2) = \text{Ext}^{s,t} \oplus \text{Ext}^{s-1,t} \text{ for } s \geq 3.$$

My proof of Theorem B adapts to the 3-local setting the methods used by Behrens [4] to compute the rational homotopy of $Q(2)$. As a side-effect, I produce evidence ([13], Section 7, Examples 1-3) that $E_2^{*,*}Q(2)$ detects the divided beta family for $p = 3$. If true, this would in turn support a conjecture of Behrens ([5], Section 1) that $\pi_*Q(2)$ detects the divided beta family on the level of homotopy.

Because $Q(2)$ is defined in terms of a semi-cosimplicial diagram (Eq. (1)), it has a Bousfield-Kan spectral sequence [7] converging to its homotopy groups. This spectral sequence takes as input the semi-cosimplicial abelian group obtained from applying the homotopy functor π_* to the aforementioned diagram. In [12] I compute the E_2 -term of this spectral sequence using methods similar to those used in [13], thereby producing a second possible entryway to $\pi_*Q(2)$. I will not state the results of this computation here, but instead give a useful consequence.

Theorem C (Larson). *In the Bousfield-Kan spectral sequence ${}_{BK}E_r^{*,*}Q(2)$ for $Q(2)$,*

(a) *The only possibly nontrivial d_2 -differentials have the form*

$$d_2 : {}_{BK}E_2^{0,t}Q(2) \rightarrow {}_{BK}E_2^{2,t-1}Q(2),$$

(b) *All elements in ${}_{BK}E_2^{1,*}Q(2)$ are permanent cycles,*

(c) *The spectral sequence collapses at E_3 , i.e., ${}_{BK}E_3^{s,t}Q(2) = {}_{BK}E_\infty^{s,t}Q(2)$.*

My long-term goal is to use Theorems B and C to completely compute $\pi_*Q(2)$.

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