

On the Higher-Order Sheffer Orthogonal Polynomial Sequences (Preface)

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In 1939, I.M. Sheffer published seminal results regarding the characterizations of polynomials via general degree-lowering operators and showed that every polynomial sequence can be classified as belonging to exactly one *Type*. A large portion of his work was dedicated to developing a wealth of aesthetic results regarding the most basic type set, entitled *B-Type 0* (or equivalently *A-Type 0*), which included the development of several interesting characterizing theorems. In particular, one of Sheffer's most important results was his classification of which *B-Type 0* sets were also orthogonal, which are now known to be the very well-studied and applicable Laguerre, Hermite, Charlier, Meixner, Meixner-Pollaczek and Krawtchouk polynomials, which are often called the *Sheffer Sequences*. As it turned out, Sheffer proved that every *B-Type 0* set $\{P_n(x)\}_{n=0}^{\infty}$ can be characterized by the generating function:

$$A(t)e^{xH(t)} = \sum_{n=0}^{\infty} P_n(x)t^n,$$

where $A(t)$ and $H(t)$ are formal power series in t , with certain restrictions. Furthermore, Sheffer also briefly described how this generating function can also be extended to the case of arbitrary *B-Type k* as follows:

$$A(t)\exp[xH_1(t) + \cdots + x^{k+1}H_{k+1}(t)] = \sum_{n=0}^{\infty} P_n(x)t^n,$$

with $H_i(t) = h_{i,i}t^i + h_{i,i+1}t^{i+1} + \cdots$, $h_{1,1} \neq 0$, $i = 1, 2, \dots, k+1$.

Thus far, a very large amount of research has been completed regarding the theory and applications of the *B-Type 0* sets. Therefore, it is natural to attempt to determine whether or not orthogonal sets can be extracted from the higher-order classes, i.e. $k \geq 1$ in the generating relation directly above. In fact, no results have been published to date that *specifically* analyze the higher-order Sheffer classes. With this in mind, we have constructed the novel results of this monograph (Chapter 3), wherein we present a preliminary analysis of a special case of the *B-Type 1* class. We conduct this analysis for the following reasons. One, most importantly, our method functions as a template, which can be applied to other characterization problems as well. In order to effectively apply this method, computer algebra was found to be essential and Mathematica® was determined to be the most efficient platform for performing each of our manipulations. Therefore, our second motivation is the fact that the novel analysis of Chapter 3 lends itself as a paradigm regarding how computer algebra packages, like Mathematica®, can play an important role in developing rigorous mathematics. Lastly, we intend for this work to eventually lead to a complete characterization of the general *B-Type k* orthogonal sets and foster future research on other types of similar characterization problems as well.

We certainly wish to emphasize that Mathematica® was utilized only for *managing* many of the algebraic manipulations involved in establishing the original results within. The relationships achieved with the aid of Mathematica® were used to rigorously construct the novel theorems of this work, which were proven via algebraic techniques and rudimentary linear algebra, without the usage of computer algebra.

Now, it is well known that symbolic computations are becoming utilized more frequently in mathematical research and are also becoming increasingly more accepted. Several journals include various results that are based on computer algebra and some may even include, essentially, diminutive fragments of code, pseudo-code or computer algebra outputs. In Chapter 3 of this work, a wealth of the Mathematica® inputs and their respective outputs are displayed in a reader-friendly format and written in a distinctive font class that is similar to the Mathematica® notebook. This amount of displayed code is certainly not typical in any of the current peer reviewed mathematical science journals and is a luxury we are afforded in this monograph. Such uniquely detailed code displays are intended to increase the reader's understanding of our usage of Mathematica®, assist in verifications and facilitate further experimentation.

It is also worth mentioning here that upon initially implementing the method of Chapter 3, it was also evident that the preliminary results were void of the level of elegance that mathematicians strive to achieve. Therefore, an approach was sought that would simultaneously give insights into existence / non-existence the *B-Type 1* orthogonal polynomial sequences and also yield a tractable problem that would ultimately admit elegant results. These goals were accomplished using simplifying assumptions that reduced the problem to a manageable format that was as similar in structure as possible to the *B-Type 0* class that Sheffer analyzed.

To enhance the novel results of this monograph, in Chapter 1 we additionally include an overview of the research that motivated their establishment. We begin by addressing Sheffer's derivation of the *B-Type 0* generating function defined above, as well as the characterizations of the *B-Type 0* orthogonal sets. Since, in 1934, J. Meixner initially studied this generating function and determined which sets were also orthogonal using a different approach than Sheffer, we also cover the central details of Meixner's method and results. We then briefly allude to W.A. Al-Salams's extension of Meixner's analysis.

Additionally, we discuss that there were actually three classifications that Sheffer developed; *A-Type*, *B-Type* (our focus in Chapter 3) and *C-Type*. The discrepancies between these types will also be addressed. We then present a summary of the σ -*Type* classification developed by E. D. Rainville, which is an extension of the Sheffer *A-Type* classification. All together, the Sheffer Sequences, the notion of *Type* and the relevant background material is elucidated in order to facilitate the transition into the latter material.

To enhance this monograph even further, in Chapter 2 we discuss several of the many applications that classical orthogonal polynomials satisfy, which include; first and second-order differential equations, quantum mechanics, difference equations and numerical integration. We first develop each of our applications in a general context and then show the specific roles played particular *A-Type 0* orthogonal sets. Through covering each of these applications, we also develop additional fundamental terminologies, definitions, lemmas and theorems etc. that are very important in the field of orthogonal polynomials and special functions in a broad context.

In essence, Chapters 1 and 2 are intended to be used as (1.) a concise, but informative, reference for developing new results related to the *A-Type 0* orthogonal sets and classical orthogonal polynomials in general and (2.) provide material for advanced undergraduate courses, or graduate courses, in pure and applied analysis.

For the benefit of the reader, each chapter is self-contained. In addition, with respect to space constraints, this entire monograph has been written with as much detail, rigor and supplementation via informative concrete examples as feasible.

This work represents the culmination of approximately three years of research on the Sheffer Sequences and related structures, which was conducted at Penn State Erie, The Behrend College.

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