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On a numerical method for a homogeneous, nonlinear, nonlocal, elliptic boundary value problem

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ABSTRACT

In this work we develop a numerical method for the equation: $-\alpha\left(\int_0^1 u(t)\mathrm{d}t\right)u''(x) + [u(x)]^{2n+1} = 0, x \in (0, 1), u(0) = a, u(1) = b$. We begin by establishing a priori estimates and the existence and uniqueness of the solution to the nonlinear auxiliary problem via the Schauder fixed point theorem. From this analysis, we then prove the existence and uniqueness to the problem above by defining a continuous compact mapping, utilizing the a priori estimates and the Brouwer fixed point theorem. Next, we analyze a discretization of the above problem and show that a solution to the nonlinear difference problem exists and is unique and that the numerical procedure converges with error $\mathscr{O}(h)$. We conclude with some examples of the numerical process.

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1. Introduction

In [1] the authors analyzed the analytic and numerical solution to the nonlocal elliptic B.V.P.:

$$-\alpha\left(\int_0^1 u(t)\mathrm{d}t\right)u''(x) = f(x), \quad x \in (0, 1), \qquad u(0) = a, \qquad u(1) = b,$$

which was a one-dimensional problem similar to those discussed in [2–5]. The crux of the research in [1] essentially amounted to analyzing conditions on the coefficient and data, which lead to the existence and uniqueness of the solution, the existence and uniqueness of a numerical approximation and the convergence of the numerical approximation to the analytic solution.

This analysis also motivated the consideration of the following equation:

$$-\alpha\left(\int_0^1 u(t)\mathrm{d}t\right)u''(x) + [u(x)]^{2n+1} = 0, \quad x \in (0, 1), \qquad u(0) = a, \qquad u(1) = b, \tag{1.1}$$

which is a nonlinear, nonlocal boundary value problem whose u'' coefficient is dependent upon the integral of the solution over the domain of the solution, analogous to the nonlocal problem in [1]. We ultimately show that the solution of the analytic problem (1.1) exists and is unique, and also demonstrate that a suitable numerical approximation yields similar existence and uniqueness results as well. In addition, we establish that an approximation to (1.1) converges to the analytic

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solution of (1.1) with a satisfactory degree of accuracy, as was discovered for the problem defined in [1], and also supply examples of the numerical process.

It is also important to mention that the numerical methods developed in this paper function as a paradigm for further numerical methods to be developed for various other analytic problems with physical applications. In fact, much research has been done on nonlocal modeling problems with structures very similar to (1.1). Below we discuss some of these problems and their applications. In each example, we also display the modeling equation to emphasize the similarity with (1.1).

A nonlocal problem modeling Ohmic heating with variable thermal conductivity was studied in [6], including an analysis of the asymptotic behavior and the blow-up of solutions. This model has the following form:

$$\begin{aligned} &u_t = (u^3 u_x)_x + \frac{\lambda f(u)}{\left(\int_{-1}^{1} f(u)\mathrm{d}x\right)^2}, \quad -1 < x < 1,\ t > 0, \\ &u(-1,t) = u(1,t) = 0, \quad t > 0 \\ &u(x,0) = u_0(x), \quad -1 < x < 1, \end{aligned}$$

where λ is a positive parameter. This work was motivated by the problem studied in [7,8] that models Ohmic heating, which contains a standard linear diffusion term of the form u_{xx} and is also a nonlocal problem.

In addition, Stańczy [9] studied nonlocal elliptic equations that arise in physical models including systems of particles in thermodynamical equilibrium interacting via gravitational (Coulomb) potential and a similar problem was studied in [10]. The equations that Stańczy studied also arose in fully turbulent behavior of a real flow, thermal runway in Ohmic heating, shear bands in metals deformed under high strain rates and one dimensional fluid flows with rate of strain proportional to a power of stress multiplied by a function of temperature. The model in [9] is defined as follows:

$$-\Delta\varphi = M\frac{f(\varphi)^\alpha}{\left(\int_\Omega f(\varphi)\right)^\beta} \quad \text{in } \Omega \in \mathbb{R}^n,$$

where $\varphi = 0$ on $\partial\Omega$.

Lastly, a nonlocal problem that arises as a local model for the temperature in a thin region, which occurs during linear friction welding, was studied in [11]. A similar model was addressed in [12], which models thermo–viscoelastic flows. The model in [11] has the structure seen below:

$$u_t = u_{xx} + f(u)\left(\int_0^\infty f(u)\mathrm{d}y\right)^{-(1+a)} \quad \text{for } 0 < x < \infty,$$

with $u_x = 0$ on $x = 0$ and $u_x \to -1$ as $x \to \infty$.

2. A priori estimates for the solution of the nonlinear auxiliary equation

We first assume $\alpha := \alpha(q)$ is a continuous positive function defined over $-\infty < q < \infty$ and bounded below by a positive real constant α_0. We define the nonlinear auxiliary problem as

$$-\alpha(q)u''(x) + [u(x)]^{2n+1} = 0, \quad x \in (0,1), \qquad u(0) = a, \qquad u(1) = b, \tag{2.1}$$

where a and b are positive real constants. We then have the following theorem.

Theorem 2.1. *For $u := u(x)$ a solution of (2.1) we have $0 < u \le \max(a, b)$.*

Proof. First assume $u > \max(a, b)$. Then, there must exist a number, say $x_0 \in (0, 1)$, so that $u(x_0)$ is a positive maximum. Therefore,

$$-\alpha(q)u''(x_0) + [u(x_0)]^{2n+1} > 0,$$

which contradicts (2.1). Hence, $u \le \max(a, b)$.

To show $u > 0$ we consider the operator

$$\mathcal{L}(\phi) := -\alpha(q)\phi''(x) + [u(x)]^{2n}\phi(x)$$

and the function

$$\xi(x) := u(x) - \min(a, b)\mathrm{e}^{-\delta x}, \tag{2.2}$$

where δ is a real constant. Now assume $\xi < 0$. Then there must exist a negative minimum of ξ, at say $\check{x}_0$, where $\check{x}_0 \in (0, 1)$. Consequently, at $\check{x}_0$

$$\mathcal{L}(\xi)(\check{x}_0) = -\alpha(q)\xi''(\check{x}_0) + [u(\check{x}_0)]^{2n}\xi(\check{x}_0) \le 0. \tag{2.3}$$

By direct computation of $\mathcal{L}(\xi)(\check{x}_0)$ and (2.1) we obtain

$$-\alpha(q)\xi''(\check{x}_0) + [u(\check{x}_0)]^{2n}\xi(\check{x}_0) \geq [\alpha(q)\delta^2 - [\max(a,b)]^{2n}]\min(a,b)\mathrm{e}^{-\delta\check{x}_0},$$

which is positive when $\delta^2 > [\max(a,b)]^{2n}/\alpha_0$. Hence, we obtain a contradiction from (2.2) and (2.3). Thus, $\xi \geq 0$ implies $u \geq \min(a,b)\mathrm{e}^{-\delta x} > 0$ and therefore $u > 0$. □

3. Existence and uniqueness of the solution to the nonlinear auxiliary problem

We consider the problem of (2.1) and ultimately show the existence of its solution via the Schauder fixed point theorem and conclude with a unicity argument. To begin, let

$$\|f\|_2 := \left(\int_0^1 (f(x))^2 \mathrm{d}x\right)^{1/2}$$

be the norm induced by the inner product

$$\langle f, g\rangle := \int_0^1 f(x)g(x)\mathrm{d}x.$$

We examine the linear problem

$$-\alpha(q)u''(x) + [\phi(x)]^{2n}u(x) = 0, \quad x \in (0,1), \qquad u(0) = a, \qquad u(1) = b, \tag{3.1}$$

which defines the mapping

$$(\mathcal{F}\phi)(x) := u(x;\phi) \tag{3.2}$$

via the existence and uniqueness for the linear elliptic two-point boundary value problem defined in (3.1). By the a priori estimates from Section 2, we can restrict $\{\phi\}$ to the closed convex set of continuous functions over $[0,1]$ satisfying

$$0 \leq \phi(x) \leq \max(a,b). \tag{3.3}$$

We will first show that $\mathcal{F}$ is a compact mapping.

Let $u = v + w$, where $w(x) := a(1-x) + bx$. Therefore, by substituting $u = v + w$ into (3.1) we see that

$$-\alpha(q)v''(x) + \phi^{2n}(x)v(x) = -\phi^{2n}(x)w(x), \qquad v(0) = v(1) = 0. \tag{3.4}$$

Multiplying (3.4) by $v(x)$ and integrating over $0 \leq x \leq 1$ yields

$$\alpha(q)\int_0^1 (v'(x))^2\mathrm{d}x + \int_0^1 \phi^{2n}(x)v^2(x)\mathrm{d}x = -\int_0^1 \phi^{2n}(x)w(x)v(x)\mathrm{d}x. \tag{3.5}$$

Utilizing the Schwarz inequality and the relation $ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$ we see that (3.5) becomes

$$\alpha(q)\int_0^1 (v'(x))^2\mathrm{d}x \leq \frac{1}{2}\int_0^1 \phi^{2n}(x)w^2(x)\mathrm{d}x,$$

which implies

$$\int_0^1 (v'(x))^2\mathrm{d}x \leq \frac{1}{2\alpha_0}\int_0^1 \phi^{2n}(x)w^2(x)\mathrm{d}x \leq \frac{1}{2\alpha_0}[\max(a,b)]^{2n+2}. \tag{3.6}$$

Therefore, $\|v'\|_2^2 \leq C$, where C is defined by the right-hand side of (3.6).

Now, we see that

$$\int_0^1 (u'(x))^2\mathrm{d}x = \int_0^1 (v'(x) + w'(x))^2\mathrm{d}x. \tag{3.7}$$

Expanding the right-hand side of (3.7), using the Schwarz inequality on the middle term $2\int_0^1 v'(x)w'(x)\mathrm{d}x$ and (3.6) we have

$$\int_0^1 (u'(x))^2\mathrm{d}x \leq \frac{1}{\alpha_0}[\max(a,b)]^{2n+2} + 2[\max(a,b)]^2. \tag{3.8}$$

Hence, by again applying the Schwarz inequality and using (3.8) we obtain

$$|u(x_1) - u(x_2)| \leq \int_{x_1}^{x_2} |u'(x)|\mathrm{d}x \leq \sqrt{D(a,b)}|x_1 - x_2|^{1/2}$$

for $x_1, x_2 \in (0,1)$ and $D(a,b)$ defined by the right-hand side of (3.8).

Theorem 3.1. *Under the assumptions of Section 2, the mapping defined by (3.2) is compact.*

Proof. By applying the maximum principle to (3.1) with $\mathcal{L}v := -\alpha v'' + \phi^{2n}v$ and $v = u - \epsilon x(1-x)$ we obtain $u(x;\phi) \leq \max(a,b)$. Acting with $\mathcal{L}$ on the function ξ as defined in (2.2) we also establish $0 < u(x;\phi)$. Hence, the mapping $\mathcal{F}$ carries the closed convex set of continuous functions $\{\phi\}$, defined on $[0,1]$ satisfying $0 < \phi(x) \leq \max(a,b)$ into a compact subset of itself. □

Next we show that

$$|u_1(x;\phi_1) - u_2(x;\phi_2)| \le k\|\phi_1 - \phi_2\|_m,$$

where k is a fixed positive constant, and u_1 and u_2 solve (3.1) respectively for ϕ_1 and ϕ_2 and $\|f\|_m := \max|f(x)|$. Let $z = u_1 - u_2$, as

$$-\alpha(q)u_i''(x) + \phi_i^{2n}(x)u_i(x) = 0, \qquad u_i(0) = a, \qquad u_i(1) = b, \tag{3.9}$$

for $i = 1, 2$. By subtracting (3.9) for $i = 1$ from (3.9) with $i = 2$ we obtain

$$-\alpha(q)z''(x) + \phi_1^{2n}(x)z(x) = u_2(x)[\phi_2^{2n}(x) - \phi_1^{2n}(x)]. \tag{3.10}$$

Since

$$u_2(x)[\phi_2^{2n}(x) - \phi_1^{2n}(x)] = u_2(x)[\phi_2(x) - \phi_1(x)]\sum_{k=0}^{2n-1}[\phi_2(x)]^{2n-k-1}\phi_1^k(x),$$

it follows from (3.3) and (3.10) for $i = 1, 2$ that

$$|u_2(x)[\phi_2^{2n} - \phi_1^{2n}]| \le 2n[\max(a,b)]^{2n}\|\phi_1 - \phi_2\|_m.$$

Now consider the function(s) defined by

$$\eta(x) := [k\|\phi_1 - \phi_2\|_m + \epsilon]x(1-x) \pm z,$$

where $\epsilon > 0$.

Lemma 3.2. *For k sufficiently large, $\eta := \eta(x) > 0$ on $0 < x < 1$.*

Proof. Assume $\eta \le 0$. Then there must exist a negative minimum of η at, say $x_0 \in (0, 1)$, where

$$-\alpha(q)\eta''(x_0) + \phi_1^{2n}(x_0)\eta(x_0) \le 0. \tag{3.11}$$

However,

$$\begin{aligned}-\alpha(q)\eta''(x_0) + \phi_1^{2n}(x_0)\eta(x_0) &= 2k\alpha(q)\|\phi_1 - \phi_2\|_m + 2\alpha(q)\epsilon + \phi_1^{2n}(x_0)[k\|\phi_1 - \phi_2\|_m + \epsilon]x_0(1-x_0)\\ &\quad \pm u_2(x_0)[\phi_2^{2n}(x_0) - \phi_1^{2n}(x_0)].\end{aligned} \tag{3.12}$$

Selecting $k = 2n[\max(a,b)]^{2n}/\alpha_0$ in (3.12) we then have

$$-\alpha(q)\eta''(x_0) + \phi_1^{2n}(x)\eta(x_0) > \frac{2n}{\alpha_0}[\max(a,b)]\|\phi_1 - \phi_2\|_m + 2\alpha_0\epsilon > 0$$

contradicting (3.11). □

Theorem 3.3. *Under the assumptions of Section 2, the mapping defined by (3.2) is continuous.*

Proof. From Lemma 3.2, it follows that $\eta(x) := [k\|\phi_1 - \phi_2\| + \epsilon]x(1-x) \pm z \ge 0$. Hence, since ϵ can be made arbitrarily small and the fact that $\max[x(1-x)] = 1/4$ we readily see that

$$|z| = |u_1 - u_2| \le \frac{n}{2}[\max(a,b)]^{2n}\|\phi_1 - \phi_2\|_m.$$

Thus, the mapping defined by (3.2) is continuous. □

We conclude this section with the following statement.

Theorem 3.4. *Under the assumptions of Section 2, there exists a unique solution to (2.1).*

Proof. Since the mapping $\mathcal{F}$ is a continuous mapping of a closed convex subset of a Banach space into a compact subset of itself, it follows from the Schauder fixed point theorem that a solution $u = u(x)$ of the nonlinear auxiliary problem exists.

Now assume that u_1 and u_2 both solve (2.1), then

$$-\alpha(q)u_i''(x) + [u_i(x)]^{2n+1} = 0, \qquad u_i(0) = a, \qquad u_i(1) = b,$$

for $i = 1, 2$. With $z = u_1 - u_2$ we have

$$-\alpha(q)z''(x) + \left[\sum_{k=0}^{2n}[u_1(x)]^{2n-k}u_2^k(x)\right]z(x) = 0, \qquad z(0) = z(1) = 0. \tag{3.13}$$

Multiplying (3.13) by $z(x)$ and integrating over $(0, 1)$ yields

$$\alpha(q)\int_0^1 (z'(x))^2 dx + \int_0^1 \left[\sum_{k=0}^{2n}[u_1(x)]^{2n-k}u_2^k(x)\right] z^2(x)dx = 0.$$

Notice that $z^2(x) \geq 0$ for all x in $(0, 1)$ and $\sum_{0\leq k\leq 2n}[u_1(x)]^{2n-k}u_2^k(x) > 0$ by Theorem 2.1. Hence, $z(x) \equiv 0$ immediately follows and thus, $u_1 \equiv u_2$. □

4. Existence and uniqueness of the solution to the nonlinear, nonlocal elliptic equation

We first define the mapping

$$T(q) := \int_0^1 u(x; q)dx, \tag{4.1}$$

where u is a solution to

$$-\alpha(q)u''(x) + [u(x)]^{2n+1} = 0, \qquad u(0) = a, \qquad u(1) = b, \quad 0 < x < 1 \tag{4.2}$$

for arbitrary positive constants a and b, where $\alpha(q)$ is continuously differentiable on $0 \leq q \leq \max(a, b)$ and $\alpha'(q) < 0$ with $\alpha(q) \geq \alpha_0 > 0$. From Theorem 2.1, we see that

$$0 < T(q) < \max(a, b)$$

and from the existence and uniqueness results established in Section 3, the mapping $T(q)$ is well defined for $0 \leq q \leq \max(a, b)$.

Lemma 4.1. *If ψ is defined by $\psi(x) := M|\alpha(q_1) - \alpha(q_2)|x(1 - x) \pm z$, for $z := u_1(x) - u_2(x)$ with u_i the solution of (4.2) for q_i, $i = 1, 2$ and for $M \geq \alpha_0^{-2}[\max(a, b)]^{2n+1}$, then $\psi > 0$.*

Proof. The argument is similar to that of Theorem 3.3 and is omitted. □

We can now establish the following result.

Theorem 4.2. *The mapping defined by (4.1) is continuous.*

Proof. Let u_1 and u_2 be solutions to (4.2). Then,

$$-u_i''(x) + \frac{u^{2n+1}(x)}{\alpha(q_i)} = 0, \qquad u_i(0) = a, \qquad u_i(1) = b,$$

for $i = 1, 2$. So, for $z = u_1 - u_2$, it follows that

$$-z''(x) + \frac{1}{\alpha(q_1)}[u_1^{2n+1}(x) - u_2^{2n+1}(x)] = \frac{u_2^{2n+1}(x)}{\alpha(q_1)\alpha(q_2)}[\alpha(q_1) - \alpha(q_2)],$$

which implies

$$-z''(x) + \frac{z(x)}{\alpha(q_1)}\sum_{k=0}^{2n}[u_1(x)]^{2n-k}u_2^k(x) = \frac{u_2^{2n+1}(x)}{\alpha(q_1)\alpha(q_2)}[\alpha(q_1) - \alpha(q_2)]. \tag{4.3}$$

Now define

$$G(x) := \frac{u_2^{2n+1}(x)}{\alpha(q_1)\alpha(q_2)}$$

and note that from Theorem 2.1 we conclude that

$$0 < G(x) < \frac{1}{\alpha_0^2}[\max(a, b)]^{2n+1}.$$

Now, from Lemma 4.1 we conclude by applying the argument of Theorem 3.3 to the operator defined by the left-hand side of (4.3) that $M = [\max(a, b)]^{2n+1}/\alpha_0$. Therefore,

$$\begin{aligned} |u_1(x) - u_2(x)| &< M|\alpha(q_1) - \alpha(q_2)|x(1 - x) \\ &\leq \frac{1}{4\alpha_0^2}[\max(a, b)]^{2n+1}|\alpha(q_1) - \alpha(q_2)|. \end{aligned} \tag{4.4}$$

Hence, from (4.4) we see

$$|T(q_1) - T(q_2)| \le \frac{1}{4\alpha_0^2}[\max(a,b)]^{2n+1}|\alpha(q_1) - \alpha(q_2)|$$

and the proof is complete. □

We can now establish the existence of the solution to (1.1) since the mapping defined by (4.1) is continuous on $0 \le q \le \max(a,b)$ and $0 < T(q) \le \max(a,b)$. It then follows that the graph of $T(q)$ crosses the diagonal of the square $0 \le q, u \le \max(a,b)$, which implies the existence of a fixed point, say $q*$, i.e. $T(q*) = q*$. Thus, there exists a solution to (1.1).

With some additional requirements on $\alpha(q)$, we have the following theorem.

Theorem 4.3. *If $\alpha'(q) < 0$ for $0 < q < \max(a,b)$, then the solution of (1.1) is unique.*

Proof. Assume that both u_1 and u_2 are solutions of (1.1) and let $z = u_1 - u_2$. Then, from the proof of Theorem 4.2 we have for $q_i = \int_0^1 u_i(x)\mathrm{d}x$, $i = 1, 2$

$$-z''(x) + \frac{z(x)}{\alpha(q_1)}\sum_{k=0}^{2n}[u_1(x)]^{2n-k}u_2^k(x) = \frac{u_2^{2n+1}(x)}{\alpha(q_1)\alpha(q_2)}[\alpha(q_1) - \alpha(q_2)]$$

and from the Mean Value Theorem we obtain

$$-z''(x) + \frac{z(x)}{\alpha(q_1)}\sum_{k=0}^{2n}[u_1(x)]^{2n-k}u_2^k(x) = \frac{u_2^{2n+1}(x)\alpha'(\xi)}{\alpha(q_1)\alpha(q_2)}\int_0^1 z(x)\mathrm{d}x, \tag{4.5}$$
$$z(0) = z(1) = 0,$$

where $\xi \in [\min(q_1,q_2), \max(q_1,q_2)]$. Now, α, u_1 and u_2 are all positive, however we have all three cases to consider for $\int_0^1 z(x)\mathrm{d}x$ in (4.5).

Case 1. Assume $\int_0^1 z(x)\mathrm{d}x > 0$. Then, $z > 0$ somewhere in $(0,1)$ so there must exist a positive maximum of z, at say x_0, where $x_0 \in (0,1)$. Therefore, the left-hand side of (4.5) is positive while the right-hand side is negative, which is impossible.

Case 2. Assume $\int_0^1 z(x)\mathrm{d}x < 0$. Then, using a similar argument as in Case 1, we again obtain a contradiction.

Hence, by tricotomy it must be that $\int_0^1 z(x)\mathrm{d}x = 0$ and thus, by the maximum principle for the linear elliptic equation (4.5) with zero boundary conditions we conclude $u_1 \equiv u_2$. □

Lastly, we conclude this section with a summarizing theorem.

Theorem 4.4. *For $\alpha(q) > 0$ continuous and differentiable on $0 \le q \le \max(a,b)$, $\alpha(q) \ge \alpha_0 > 0$ and $\alpha'(q) < 0$, there exists a unique solution of*

$$-\alpha\left(\int_0^1 u(t)\mathrm{d}t\right)u''(x) + [u(x)]^{2n+1} = 0, \qquad u(0) = a, \qquad u(1) = b$$

where a and b are positive constants and $x \in (0,1)$.

Proof. See all of the analysis above. □

5. Discretization of the nonlinear, nonlocal elliptic B.V.P.

In this section we discretize the problem defined by (1.1). We first rewrite the problem as follows:

$$-u''(x) + \frac{1}{\alpha\left(\int_0^1 u(t)\mathrm{d}t\right)}[u(x)]^{2n+1} = 0, \quad x \in (0,1) \qquad u(0) = a, \qquad u(1) = b. \tag{5.1}$$

Next, we note that as u has two continuous derivatives with respect to x and $\alpha(\int_0^1 u(t)\mathrm{d}t)$ is a constant, it follows from Eq. (5.1) that u is infinitely differentiable and its derivatives are bounded.

Now let $x_i = i/N$ for $i = 0, \ldots, N$ and $u_i := u(x_i)$, then (5.1) can be written as

$$-u_i'' + \frac{1}{\alpha\left(\int_0^1 u(t)\mathrm{d}t\right)}u_i^{2n+1} = 0.$$

We discretize $\int_0^1 u(t)\mathrm{d}t$ as

$$Q(\vec{u}) := \sum_{i=0}^{N-1} \frac{u_{i+1}+u_i}{2} h = \int_0^1 u(t)\mathrm{d}t + \mathcal{O}(h^2), \tag{5.2}$$

where $\vec{u} := (a, u_1, \ldots, u_i, \ldots, u_{n-1}, b)$. Now, we take into account the fact that

$$u_i'' = \Delta_h^2 u_i + \mathcal{O}(h^2),$$
$$\text{with } \Delta_h^2 u_i = (u_{i-1} - 2u_i + u_{i+1})/h^2,$$

see [13]. Thus, we can write (1.1) in the following way

$$-\Delta_h^2 u_i + \mathcal{O}(h^2) + \frac{1}{\alpha(Q(\vec{u}))} u_i^{2n+1} + \frac{1}{\alpha\left(\int_0^1 u\mathrm{d}x\right)} u_i^{2n+1} - \frac{1}{\alpha(Q(\vec{u}))} u_i^{2n+1} = 0$$

where $h = 1/N$. Thus, by utilizing the Mean Value Theorem we obtain

$$-\Delta_h^2 u_i + \frac{1}{\alpha(Q(\vec{u}))} u_i^{2n+1} = \frac{u_i^{2n+1}\alpha'(\xi)}{\alpha\left(\int_0^1 u\mathrm{d}x\right)\alpha(Q(\vec{u}))}\left[Q(\vec{u}) - \int_0^1 u\mathrm{d}t\right] + \mathcal{O}(h^2)$$

where $0 < \alpha_0 \le \alpha$ and $|u_i| \le \max(a, b)$ with α restricted such that $|\alpha'(\xi)| \le M$. Hence, it follows that

$$-\Delta_h^2 u_i + \frac{1}{\alpha(Q(\vec{u}))} u_i^{2n+1} = \mathcal{O}(h^2). \tag{5.3}$$

Remark. By replacing u_i by w_i in the left-hand side of (5.3) and setting the right-hand side of (5.3) to zero leads to the nonlinear, nonlocal finite-difference boundary value problem:

$$-\Delta_h^2 w_i + \frac{w_i^{2n+1}}{\alpha(Q(\vec{w}))} = 0, \quad i = 1, \ldots, N-1 \tag{5.4}$$

where $w_0 = a$ and $w_N = b$.

6. The existence and uniqueness of the solution to the nonlinear difference equation

We begin by showing the existence and uniqueness of the solution $\vec{w} := (w_0, w_1, \ldots, w_N)$ of the auxiliary problem:

$$-\alpha(q)\Delta_h^2 w_i + w_i^{2n+1} = 0, \qquad w_0 = a, \qquad w_N = b, \quad i = 1, \ldots, N-1 \tag{6.1}$$

where $\Delta_h^2 = (w_{i-1} - 2w_i + w_{i+1})/h^2$, for $h = 1/N$ and N a positive integer. We then have the following.

Theorem 6.1. *For w_i a solution to (6.1), we have $0 < w_i \le \max(a, b)$, for $a, b > 0$.*

Proof. First, suppose $w_i > \max(a, b)$ for some interior i, say i_0, then $\vec{w}$ must have a positive maximum, therefore $-\alpha(q)\Delta_h^2 w_{i_0} + w_{i_0}^{2n+1} > 0$, which is a contradiction of Eq. (6.1). Thus, it follows $w_i \le \max(a, b)$ for $i = 0, \ldots, N$.

Now assume $w_i < 0$ for some i. Then, $\vec{w}$ must have a negative minimum, at say i_1, where $-\alpha(q)\Delta_h^2 w_{i_1} + w_{i_1}^{2n+1} < 0$. This is a contradiction of (6.1) and it follows that $0 \le w_i$. Hence, we see that $0 \le w_i \le \max(a, b)$ for $i = 0, \ldots, N$.

To show that $w_i > 0$, $i = 1, \ldots, N-1$ we consider the vector

$$\varsigma_i := w_i - \min(a, b)\mathrm{e}^{-\delta x_i}, \quad i = 0, \ldots, N,$$

where $x_i = ih$. Now, observe that

$$\begin{aligned}\Delta_h^2 \mathrm{e}^{-\delta h_i} &= 2\mathrm{e}^{-\delta x_i}\left[\sum_{k=1}^{\infty} \frac{(\delta h)^{2k}}{(2k)!}\right]\frac{1}{h^2} \\ &= \mathrm{e}^{-\delta x_i}\left[\delta^2 + 2\sum_{k=2}^{\infty} \frac{\delta^{2k} h^{2k-2}}{(2k)!}\right].\end{aligned}$$

Therefore, substituting ς_i into the left-hand side of (5.1) yields

$$\begin{aligned}-\alpha(q)\Delta_h^2[w_i - \min(a, b)\mathrm{e}^{-\delta x_i}] + w_i^{2n}[w_i - \min(a, b)\mathrm{e}^{-\delta x_i}] &= \alpha(q)\min(a, b)\mathrm{e}^{-\delta x_i}\left[\delta^2 + 2\sum_{k=2}^{\infty} \frac{\delta^{2k} h^{2k-2}}{(2k)!}\right] \\ &\quad - \min(a, b) w_i^{2n}\mathrm{e}^{-\delta x_i} \\ &> [\alpha_0\delta^2 - [\max(a, b)]^{2n}]\min(a, b)\mathrm{e}^{-\delta x_i}.\end{aligned}$$

By selecting δ such that $\delta^2 > [\max(a,b)]^{2n}/\alpha_0 > 0$ we conclude via the argument for $w_i \geq 0$ that $\varsigma_i \geq 0$, from which it follows that $w_i \geq \min(a,b)e^{-\delta x_i} > 0$, $i = 1, \ldots, N-1$. Hence, we have $0 < w_i \leq \max(a,b)$. □

Now consider the following linear auxiliary difference problem:

$$-\alpha(q)\Delta_h^2 w_i + \phi_i^{2n} w_i = 0, \qquad w_0 = a, \qquad w_N = b, \tag{6.2}$$

with $a, b > 0$ where $\vec{\phi} := (a, \phi_1, \ldots, \phi_{N-1}, b)$ and $0 \leq \phi_i \leq \max(a,b)$, $i = 1, \ldots, N-1$. It is also important to note that the solution $\vec{w}(\vec{\phi})$ is well defined since by the maximum principle $0 \leq w_i(\vec{\phi}) \leq \max(a,b)$, $\forall i$ and for $a = b = 0$ and $w_i \equiv 0$, which implies the unique solvability of (6.2).

Next, we define the mapping

$$\vec{W} := \mathcal{F}(\vec{\phi}). \tag{6.3}$$

We show that $\mathcal{F}$ is a continuous mapping of $\phi \in [0, \max(a,b)]$. Define $\vec{z} := \vec{w}(\vec{\phi}_1) - \vec{w}(\vec{\phi}_2)$, where $\vec{w}(\vec{\phi}_1)$ is the solution of (6.2) with ϕ_i replaced by $\phi_{1,i}$ and $\vec{w}(\vec{\phi}_2)$ is the solution of (6.2) with ϕ_i replaced by $\phi_{2,i}$. Thus, by subtracting the difference equation for $\vec{w}(\vec{\phi}_2)$ from the difference equation for $\vec{w}(\vec{\phi}_1)$ we obtain

$$-\alpha(q)\Delta_h^2 z_i + \phi_{1,i}^{2n} z_i = w_i(\phi_2)[\phi_{2,i}^{2n} - \phi_{1,i}^{2n}.].$$

Defining the vector $\vec{\sigma}$ as

$$\sigma_i := Cx_i(1-x_i) \pm z_i, \quad i = 0, \ldots, N \tag{6.4}$$

we see that $\Delta_h^2 x_i(1-x_i) = -2$. Using the vector norm $\|\vec{\phi}\| := \max_{0\leq i\leq N} |\phi_i|$, we observe that

$$\begin{aligned} -\alpha(q)\Delta_h^2\sigma_i + \phi_1^{2n}\sigma_i &= 2C\alpha(q) + C\phi_1^{2n}(x_i(1-x_i)) \pm w_i(\phi_2)[\phi_{2,i}^{2n} - \phi_{1,i}^{2n}] \\ &\geq 2\alpha_0 C - 2n[\max(a,b)]^{2n}\|\vec{\phi}_1 - \vec{\phi}_2\| = 0 \end{aligned}$$

by selecting $C = n[\max(a,b)]^{2n}\|\vec{\phi}_1 - \vec{\phi}_2\|/\alpha_0$. This contradicts the assumption that $\sigma < 0$, which would imply a negative minimum at some i between 0 and N. Hence, $\sigma_i \geq 0$ for $i = 0, \ldots, N$, which implies $Cx_i(1-x_i) \geq |z_i|$ and

$$\|\vec{w}(\vec{\phi}_1) - \vec{w}(\vec{\phi}_2)\| \leq C\|\vec{\phi}_1 - \vec{\phi}_2\|.$$

Thus, the mapping defined by (6.3) is continuous.

Since $\vec{W}$ defines a continuous mapping of a closed and bounded convex set of Euclidean $N+1$ space into itself, it has at least one fixed point via the Brouwer fixed point theorem. Therefore, there exists a solution of (6.1). We conclude this section by proving that the solution of (6.1) is unique.

Let $\vec{w}$ and $\vec{v}$ be two solutions of (6.1) and set $\vec{z} := \vec{w} - \vec{v}$. Subtracting (6.1) for v_i from (6.1) for w_i, we see that

$$-\alpha(q)\Delta_h^2 z_i + \left[\sum_{k=0}^{2n} w_i^{2n-k} v_i^k\right] z_i = 0, \quad i = 1, \ldots, N-1$$

with $z_0 = z_N = 0$. Hence, it follows from the maximum principle that $z_i \equiv 0$ since all of $\alpha(q)$, v_i and w_i are positive. These results are summarized in the following statement.

Theorem 6.2. *There exists a unique solution to the nonlinear auxiliary equation defined by*

$$-\alpha(q)\Delta_x^2 w_i + w_i^{2n+1} = 0, \qquad w_0 = a > 0, \qquad w_N = b > 0, \tag{6.5}$$

where $\alpha(q) > \alpha_0 > 0$ is continuous on $0 \leq q < \infty$.

Proof. See the entire argument above. □

From Theorem 6.2 we see that the mapping $T(q) = Q(\vec{w})$, where $Q(\vec{w})$ is defined by (5.2) and $\vec{w}$ is the solution of (6.5), which is well defined. Also, from the estimate obtained in the proof of Theorem 6.1 it follows that $0 < T(q) \leq \max(a,b)$.

Now let $\vec{w}(q_1)$ and $\vec{w}(q_2)$ denote the solutions of (6.5) with q replaced by q_1 and q_2 respectively. Setting $\vec{z} := \vec{w}(q_1) - \vec{w}(q_2)$ and subtracting one difference equation from the other, we obtain

$$-\Delta_h^2 z_i + \frac{w_{1,i}^{2n+1} - w_{2,i}^{2n+1}}{\alpha(q_1)} = \frac{w_{2,i}^{2n+1}}{\alpha(q_2)} - \frac{w_{2,i}^{2n+1}}{\alpha(q_1)}, \quad i = 1, \ldots, N-1,$$

where $w_{j,i}$ denotes the ith component of $\vec{w}(q_j)$, $j = 1, 2$. This reduces to $z_0 = z_n = 0$ and

$$-\Delta_h^2 z_i + \frac{1}{\alpha(q_1)}\left[\sum_{k=0}^{2n} w_{1,i}^{2n-k} w_{2,i}^k\right] z_i = \frac{w_{2,i}^{2n+1}(\alpha(q_1) - \alpha(q_2))}{\alpha(q_1)\alpha(q_2)}, \quad i = 1, \ldots, N-1. \tag{6.6}$$

Now, recall the vector $\vec{\sigma}$ as defined in (6.4) with C a positive constant to be selected below. As $\Delta_h^2 x_i(1-x_i) = -2$, we obtain

$$-\Delta_h^2\sigma_i + \frac{1}{\alpha(q_1)}\left[\sum_{k=0}^{2n} w_{1,i}^{2n-k} w_{2,i}^k\right]\sigma_i = 2C + C\frac{1}{\alpha(q_1)}\left[\sum_{k=0}^{2n} w_{1,i}^{2n-k} w_{2,i}^k\right]x_i(1-x_i) \pm \frac{w_{2,i}^{2n+1}(\alpha(q_1)-\alpha(q_2))}{\alpha(q_1)\alpha(q_2)}$$
$$> 2C - \frac{[\max(a,b)]^{2n+1}}{\alpha_0^2}|\alpha(q_1)-\alpha(q_2)|.$$

Since Theorem 6.1 implies that $0 < w_{1,i}$ and $w_{2,i} < \max(a,b)$ we see that

$$C\frac{1}{\alpha(q_1)}\left[\sum_{k=0}^{2n} w_{1,i}^{2n-k} w_{2,i}^k\right]x_i(1-x_i) > 0$$

and $\alpha(q) \geq \alpha_0 > 0$. Choosing

$$C = \frac{[\max(a,b)]^{2n+1}}{\alpha_0^2}|\alpha(q_1)-\alpha(q_2)|$$

it follows that

$$-\Delta_h^2\sigma_i + \frac{1}{\alpha(q_1)}\left[\sum_{k=0}^{2n+1} w_{1,i}^{2n-k} w_{2,i}^k\right]\sigma_i > 0, \quad i = 1,\ldots,N-1. \tag{6.7}$$

Next, we assert that $\sigma_i \geq 0$ for $i = 1,\ldots,N-1$. Otherwise, then as $\sigma_0 = \sigma_N = 0$ and some $\sigma_{i_0} < 0$, we have a negative minimum at i_0 where σ_{i_0-1} or σ_{i_0+1} is greater than σ_{i_0}. As $\Delta_h^2\sigma_{i_0} > 0$ and $\sigma_{i_0} < 0$, we see that at i_0

$$-\Delta_h^2\sigma_{i_0} + \frac{1}{\alpha(q_1)}\left[\sum_{k=0}^{2n+1} w_{1,i_0}^{2n-k} w_{2,i_0}^k\right]\sigma_{i_0} < 0,$$

which contradicts (6.7). Hence, $\sigma_i \geq 0$ for $i = 1,\ldots,N-1$ and $\pm z_i \leq |z_i| \leq Cx_i(1-x_i)$.

Since $x(1-x) \leq 1/4$ for $0 \leq x \leq 1$, we have

$$|w_{1,i} - w_{2,i}| \leq \frac{[\max(a,b)]^{2n+1}}{\alpha_0^2}|\alpha(q_1)-\alpha(q_2)|,$$

which implies that

$$T(q_1) - T(q_2) = Q(\vec{w}(q_1) - \vec{w}(q_2)) = \mathcal{O}(|\alpha(q_1)-\alpha(q_2)|).$$

Thus, the continuity of $T(q)$ follows from the continuity of $\alpha(q)$ and we see that $T(q)$ is a continuous function on $0 \leq q \leq \max(a,b)$ whose graph must cross the diagonal of the square formed from the Cartesian product of $0 \leq q \leq \max(a,b)$ with itself. Hence, the mapping $T(q)$ has at least one fixed point and the problem defined by (5.4) has a solution.

For the uniqueness, we assume that w_1 and w_2 are two solutions of (5.4). Setting $\vec{z} := \vec{w}_1 - \vec{w}_2$ and substituting the difference equation for $\vec{w}_i$, we obtain $z_0 = z_N = 0$ and

$$-\Delta_h^2 z_i + \frac{W_{1,i}^{2n+1} - w_{2,i}^{2n+1}}{\alpha(Q(\vec{w}_1))} = \frac{w_{2,i}^{2n+1}}{\alpha(Q(\vec{w}_2))} - \frac{w_{2,i}^{2n+1}}{\alpha(Q(\vec{w}_1))}, \quad i = 1,\ldots,N-1.$$

Assuming that α is differentiable and $\alpha' < 0$, we see that $z_0 = z_N = 0$ and

$$-\Delta_h^2 z_i + \frac{1}{\alpha(Q(\vec{w}_1))}\left[\sum_{k=0}^{2n} w_{1,i}^{2n-k} w_{2,i}^k\right]z_i = \frac{w_{2,i}^{2n+1}Q(\vec{z})\alpha'(\xi)}{\alpha(Q(\vec{w}_1))\alpha(Q(\vec{w}_2))}, \quad i = 1,\ldots,N-1. \tag{6.8}$$

The remainder of the argument follows that of Section 4, where $Q(\vec{z}) > 0$ implies the existence of an interior positive maximum, which implies that the left-hand side of (6.8) is positive while the right-hand side is negative, which is impossible. $Q(\vec{z}) < 0$ implies a negative minimum, which again produces a contradiction in (6.8). Finally, $Q(\vec{z}) = 0$ implies $\vec{z} = 0$ via the maximum principle, which the first two cases imply via the absence of a positive maximum and a positive minimum. We summarize the above in the following statement.

Theorem 6.3. *If $\alpha(q) \geq 0$ is continuous on $0 \leq q \leq \max(a,b)$, where a and b are positive constants, and if α is differentiable on $0 \leq q \leq \max(a,b)$ with $\alpha'(q) < 0$, then there exists a unique solution $\vec{w}$ to the problem (5.4).*

7. Convergence of the solution of the difference equation to the solution of the nonlinear, nonlocal B.V.P.

Next, we analyze the error $z_i := u_i - w_i$ noting that $z_0 = z_N = 0$. By subtracting (5.4) from (5.3) we immediately see that

$$-\Delta_h^2 z_i + \frac{1}{\alpha(Q(\vec{u}))} u_i^{2n+1} - \frac{1}{\alpha(Q(\vec{w}))} w_i^{2n+1} = \mathcal{O}(h^2),$$

which implies that

$$-\Delta_h^2 z_i + \frac{1}{\alpha(Q(\vec{u}))}[u_i^{2n+1} - w_i^{2n+1}] = w_i^{2n+1}\left[\frac{1}{\alpha(Q(\vec{w}))} - \frac{1}{\alpha(Q(\vec{u}))}\right] + \mathcal{O}(h^2).$$

It then follows that

$$-\Delta_h^2 z_i + \frac{1}{\alpha(Q(\vec{u}))}\left[\sum_{k=0}^{2n} u_i^{2n-k} w_i^k\right] z_i = \frac{w_i^{2n+1}\alpha'(\xi)Q(\vec{z})}{\alpha(Q(\vec{u}))\alpha(Q(\vec{w}))} + \mathcal{O}(h^2), \tag{7.1}$$

where $\xi \in (\min(Q(\vec{u}), Q(\vec{w})), \max(Q(\vec{u}), Q(\vec{w})))$.

We now partition the analysis into three parts, which are based on the tricotomy of $Q(\vec{z})$. For ease of notation, we define

$$A_i = \frac{1}{\alpha(Q(\vec{u}))}\left[\sum_{k=0}^{2n} u_i^{2n-k} w_i^k\right] \quad \text{and} \quad B_i = \frac{w_i^{2n+1}\alpha'(\xi)}{\alpha(Q(\vec{u}))\alpha(Q(\vec{w}))}$$

so that the relationship in (7.1) becomes:

$$-\Delta_h^2 z_i + A_i z_i = B_i Q(\vec{z}) + \mathcal{O}(h^2).$$

It is also important to recall that A_i is positive and bounded and B_i is negative and bounded.

Case 1. $Q(\vec{z}) > 0$.

As $B_i Q(\vec{z}) < 0$ we have $-\Delta_h^2 z + A_i z < Ch^2$, for some positive constant C. Now consider a test function analogous to the ones we have previously utilized: $\varsigma_i := Kx_i(1 - x_i) - z_i$. Assume that $\varsigma_i < 0$ for some i. Then there is an interior negative minimum, at say i_0. Hence, $-\Delta_h^2 \varsigma_{i_0} + A_i \varsigma_{i_0} < 0$. However, direct computation shows that

$$-\Delta_h^2 \varsigma_{i_0} + A_i \varsigma_{i_0} = 2K + A_i K x_i(1 - x_i) + \Delta_h^2 z_i - A_i z > 2K - Ch^2$$

where $2K - Ch^2$ can be made positive by selecting $K = Ch^2$.

Thus, $\varsigma_i \geq 0$ and we see that $z_i \leq Ch^2/4$, $\forall z_i$. Since $x(1 - x) \leq 1/4$ on $[0, 1]$

$$Q(\vec{z}) = \sum z_i^+ h + \sum z_i^- h > 0,$$

where z_i^+ and z_i^- are respectively the positive and negative z_i-terms. Hence, it follows that

$$\sum z_i^+ h > -\sum z_i^- h = \sum |z_i^-| h > 0.$$

An upper bound for $\sum z_i^+ h$ is achieved by regarding all but one of the $N - 1$ z_i-values that are positive and bounded above by $Ch^2/4$. Thus, we have

$$\frac{1}{4}Ch^2 > \frac{1}{4}Ch^3(N - 2) > \sum z_i^+ h > \sum |z_i^-| h > \max_i(|z_i^-|)h.$$

Therefore, we have shown that

$$\frac{1}{4}Ch^2 > \max(z_i^+) > \min(z_i^-) > -\frac{1}{4}Ch$$

implying that $|z_i| < Ch/4$.

Case 2. $Q(\vec{z}) < 0$.

Here, we see that $B_i Q(\vec{z}) > 0$ and we have $-\Delta_h z_i + A_i z_i > -Ch^2$, for some positive constant C. Considering the test function $\tau_i := Kx_i(1 - x_i) + z_i$ the assumption $\tau_i < 0$ implies that there is an interior i_0 at which τ_i has a negative minimum and $-\Delta_h^2 \tau_{i0} + A_i \tau_{i0} < 0$. However, direct computation shows that

$$-\Delta_h^2 \tau_i + A_i \tau_i = 2K + A_i K x_i(1 - x_i) - \Delta_h z_i + A_i z_i > 2K - Ch^2 > 0,$$

provided we select $K = Ch^2$. Thus $\tau \geq 0$, for all i and $Ch^2 x_i(1 - x_i) \geq -z_i$ or $Ch^2/4 \geq -z_i$. Similarly to Case 1 we have

$$Q(\vec{z}) = \sum z_i^+ h + \sum z_i^- h < 0$$

and

$$(\max_i z_i^+)h < \sum z_i^+ h < -\sum z_i^- h = \sum |z_i^-| h < \frac{1}{4}Ch^3(N - 2) < \sum z_i^+ h < \frac{1}{4}Ch^2.$$

Hence,

$$-\frac{1}{4}Ch^2 \le z_i \le \max_i(z_i^+) < \frac{1}{4}Ch.$$

Thus, $|z_i| \le Ch/4$.

Case 3. $Q(\vec{z}) = 0$.
As $Q(\vec{z}) = 0$, (7.1) becomes

$$-\Delta_h^2 z_i + A_i z_i = \mathcal{O}(h^2)$$

and the application to the vector $\tau_i := Kx_i(1 - x_i) \pm z_i$ leads directly to the estimate $z_i \le Ch^2/4$ for some positive constant C.

We summarize this analysis with the following statement.

Theorem 7.1. *Under the assumptions for the existence and uniqueness of the solution u of the nonlinear, nonlocal boundary value problem (1.1) and the existence and uniqueness of the difference approximation $\vec{w}$ as defined by (6.1), there exists a constant C such that $|u(x) - w_{i(x)}| < Ch$ for each $x \in (0, 1)$, which remains in successive refinements of a grid, say $x_i = ih$ for $h = 2^{-m}$ with $m \to \infty$, where $i(x)$ is the index of x_i within the grid for each m.*

Proof. See above analysis. □

8. Numerical experiments

In this section we present the results of two numerical examples that were based on an interval-halving scheme. These results are displayed in Tables 1 and 2. The numerical scheme searched for a change of sign in $q_k = Q(\vec{w}(q_k))$ proceeded by a interval-halving procedure until $|Q(\vec{w}(q_k)) - q|$ diminished below a pre-set precision. In each table below, N represents the number of iterations and ALPHA corresponds to the numerical computation of the value $\alpha(\int_0^1 u\mathrm{d}x)$. The J-VALUE symbolizes the number of iterations required for $|Q(\vec{w}(q_k)) - q|$ to diminish below the pre-set precision 10^{-8}, MAX signifies the maximum error $|u(x_i) - w(x_i)|$ and ERROR represents the error $|Q(\vec{w}(q_k)) - q|$.

We first experiment with the equation

$$-\frac{1}{q}u'' + \frac{3}{4(2\sqrt{2} - 2)}u^5 = 0, \qquad u(0) = 1, \qquad u(1) = \sqrt{2}/2, \tag{8.1}$$

which has the solution $u = 1/\sqrt{1 + x}$, with $q = \int_0^1 (1/\sqrt{1 + x})\mathrm{d}x = 2\sqrt{2} - 2$ and therefore $\alpha(q) = 1/q = 1/(2\sqrt{2} - 2) \approx 1.20711$. We then obtain the values displayed in Table 1:

Table 1
Numerical experiment for Eq. (8.1).

N	10	100	1000	10,000
ALPHA	1.2066	1.2071	1.2071	1.2071
J-VALUE	22	18	23	15
MAX	5.0401−009	5.4479e−008	5.4651e−011	5.1181e−014
ERROR	7.6495e−009	5.1275e−009	6.8017e−010	1.0801e−009

We see that $|u(x_i) - w(x_i)|$ became sufficiently small and $|Q(\vec{w}(q_k)) - q|$ essentially remained at the same order of magnitude.

For our second example we modify Eq. (8.1) by replacing $\alpha = 1/q$ with the non-decreasing function $\alpha = q$, resulting in

$$-qu'' + \frac{3(2\sqrt{2} - 2)}{4}u^5 = 0, \qquad u(0) = 1, \qquad u(1) = \sqrt{2}/2, \tag{8.2}$$

which has the solution $u = 1/\sqrt{1 + x}$, with $\alpha(q) = q = \int_0^1 (1/\sqrt{1 + x})\mathrm{d}x = 2\sqrt{2} - 2 \approx .828427$. The results are given in Table 2:

Table 2
Numerical experiment for Eq. (8.2).

N	10	100	1000	10,000
ALPHA	0.8288	0.8284	0.8284	0.8284
J-VALUE	18	19	23	15
MAX	6.4974e−005	7.1316e−008	7.1578e−011	7.8826e−015
ERROR	9.5892e−009	4.1351e−009	1.1226e−009	1.0406e−009

In conclusion, we have seen from the above numerical experiments that the condition $\alpha'(q) < 0$ is sufficient for unicity and convergence; however it was not necessary, which leaves open the analysis of the problem under the assumption $\alpha'(q) \geq 0$.

Dedication

This paper is dedicated to the memory of the first author's first Ph.D. student, Richard E. Ewing, who was an excellent human being and mathematician.

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