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An epidemiology model suggested by yellow fever

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In this work, we construct and analyze a nonlinear reaction–diffusion epidemiology model consisting of two integral-differential equations and an ordinary differential equation, which is suggested by various insect borne diseases, for example, Yellow Fever. We begin by defining a nonlinear auxiliary problem and establishing the existence and uniqueness of its solution via a priori estimates and a fixed point argument, from which we prove the existence and uniqueness of the classical solution to the analytic problem. Next, we develop a finite-difference method to approximate our model and perform some numerical experiments. We conclude with a brief discussion of some subsequent extensions. Copyright © 2011 John Wiley & Sons, Ltd.

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1. Introduction

In recent years, various reaction–diffusion systems have been constructed for epidemiological models, some of which have lead to nonlocal reaction–diffusion models. For example, in order to study the fecally–orally transmitted diseases in the Mediterranean regions of Europe, Capasso and Maddalena [1] constructed and studied the reaction–diffusion model:

$$\begin{aligned}\frac{\partial}{\partial t}u_1(t,x) &= \mathrm{d}\Delta u_1 - a_{11}u_1(t,x) + a_{12}u_2(t,x),\\ \frac{\partial}{\partial t}u_2(t,x) &= -a_{22}u_2(t,x) + g(u_1(t,x)),\end{aligned} \tag{1}$$

with $u_1(t,x)$ representing the spatial densities of infectious agents and $u_2(t,x)$ representing the infective human population at a point x in the habitat $\Omega \subseteq \mathbb{R}^n$ and time $t \geq 0$.

The model in (1) was also further considered in [2] as a two-component reaction–diffusion system modeling a class of spatially structured epidemic systems, which describes the spatial spread of infectious diseases interceded by environmental pollution (additionally refer to [3,4]). This model has the following form:

$$\begin{aligned}\frac{\partial}{\partial t}u_1(x,t) &= \mathrm{d}\Delta u_1(x,t) - a_{11}u_1(x,t) + \int_\Omega k(x,x')u_2(x',t)dx'\\ \frac{\partial}{\partial t}u_2(x,t) &= -a_{22}u_2(x,t) + g(u_1(x,t)).\end{aligned} \tag{2}$$

Moreover, in [2], the authors focused on the stabilization and provided conditions for the exponential decay of the epidemic in the entire habitat. In addition, the stabilization of (2) was also analyzed in [5] (see references therein).

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Also, in regards to (1), Thieme and Zhao [6] subsequently considered the latent period of a virus incorporating disturbed delay, which resulted in the nonlocal model seen below:

$$\begin{aligned}\frac{\partial}{\partial t}u_1(t,x) &= \mathrm{d}\Delta u_1 - a_{11}u_1(t,x) + a_{12}u_2(t,x),\\ \frac{\partial}{\partial t}u_2(t,x) &= -a_{22}u_2(t,x) + \int_0^\infty g(u_1(t-s,x))P(\mathrm{d}s),\end{aligned} \tag{3}$$

for habitat Ω, where $x \in \Omega \subseteq \mathbb{R}^n$, $t \geq 0$ and P is a probability measure on $\mathbb{R}_+$, which describes the distribution of the latent period. We also mention that in [7], Wu and Liu further generalized (1) and (3):

$$\begin{aligned}\frac{\partial}{\partial t}u_1(t,x) &= \mathrm{d}\Delta u_1 - a_{11}u_1(t,x) + \int_\Omega K(x-y)u_2(t,y)\mathrm{d}y,\\ \frac{\partial}{\partial t}u_2(t,x) &= -a_{22}u_2(t,x) + \int_0^\infty g(u_1(t-s,x))P(\mathrm{d}s).\end{aligned} \tag{4}$$

The system (4) consists of an additional nonlocal term appearing in the first equation, which aids in modeling indirect transmission diseases, like typhoid fever and malaria, for example.

In this paper, we construct and analyze an epidemiology model suggested by yellow fever and other insect borne diseases, which is also essentially a nonlocal model:

$$u_t = u_{xx} - \alpha\left(\int_a^b w(x,t)\mathrm{d}x\right)u + \gamma w, \quad -\infty < x < \infty, \quad t \leq T, \tag{5}$$

$$v_t = v_{xx} + \alpha\left(\int_a^b w(x,t)\mathrm{d}x\right)u - \beta v, \quad -\infty < x < \infty, \quad t \leq T, \tag{6}$$

$$w_t = \beta v - \gamma w, \quad -\infty < x < \infty, \quad t \leq T, \tag{7}$$

$$\begin{aligned}u(x,0) &= f(x), \quad -\infty < x < \infty,\\ v(x,0) &= 0, \quad -\infty < x < \infty,\\ w(x,0) &= 0, \quad -\infty < x < \infty,\end{aligned} \tag{8}$$

with β and γ arbitrary positive constants. The unknown u represents the general population density, the unknown v represents the infected population density and the unknown w represents the population density of those incapacitated with the disease. The non-negative function $\alpha(q)$, $0 \leq q < \infty$ denotes the rate of change of infection, which is decreasing with q. The argument $\int_a^b w(x,t)\mathrm{d}x$, representing the number of the population incapacitated with the disease, when substituted into α brings about a reduction in the rate of infection via actions taken at large, and at the individual level, to reduce the risk of infections.

Now, it is important to mention how the specific nature of yellow fever and similar insect borne diseases influenced the structure of our model, which is somewhat in contrast to the models discussed in (1)–(4). First of all, there is no diffusion term in (7) because the people that were infected with yellow fever became incapacitated—for insights into the nature and history of yellow fever see [8]. In addition, the reaction–diffusion terms of our model consist of the coefficient $\alpha\left(\int_a^b w(x,t)\mathrm{d}x\right)$, which acts as the probability of a susceptible being bitten by a mosquito carrying the yellow fever disease. Therefore, $\alpha \equiv$ const is equivalent to people taking no preventive measures in regards to contracting the disease. For other physical models consisting of a nonlocal term of the form $\alpha\left(\int_a^b w(x,t)\mathrm{d}x\right)$ consider [9] and [10].

This article is divided into seven sections. In Section 2, we introduce an auxiliary problem in which the argument $\int_a^b w(x,t)\mathrm{d}x$ is replaced by an arbitrary continuous function $q(t)$, $0 \leq t \leq T$. Then, the auxiliary system is consequently reduced to an equivalent integral equation and a priori estimates are derived. The existence and uniqueness of the solution $w(x,t;q)$ of the auxiliary problem are proven in Section 3 via a Picard iteration procedure using the estimates obtained in Section 2. This existence and uniqueness result allows us to define a nonlinear mapping

$$(\mathcal{F}q)(t) := \int_a^b w(x,t;q)\mathrm{d}x, \quad 0 \leq t \leq T$$

and to demonstrate the existence of a solution via the Schauder fixed point theorem in Section 4. Uniqueness of the fixed point is established utilizing the a priori estimates in Section 2 and an iteration procedure similar to that for the initial-value problem for a first-order differential equation (ODE). In Section 5, we formulate a finite-difference method for approximating the solution of the problem defined by (5)–(8). Some numerical experiments implementing the procedure outlined in Section 5, which compare various $\alpha(q)$-values against $\alpha \equiv$ const, $0 \leq q < \infty$, are addressed in Section 6. We finish with discussions regarding our conclusions and future research in Section 7.

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2. The auxiliary problem

In this section, we consider the auxiliary system:

$$u_t = u_{xx} - \alpha(q(t))\,u + \gamma w, \quad -\infty < x < \infty, \quad t \le T, \tag{9}$$

$$v_t = v_{xx} + \alpha(q(t))\,u - \beta v, \quad -\infty < x < \infty, \quad t \le T, \tag{10}$$

$$w_t = \beta v - \gamma w, \quad -\infty < x < \infty, \quad t \le T, \tag{11}$$

which is subject to the initial conditions (8).

We shall assume that $f(x)$ is a non-negative, real-valued function, which is continuous and bounded. We also assume that $q(t)$ is continuous and bounded for $0 \le t \le T$ and that $\alpha(q)$ is a positive, real-valued function defined on $0 \le q < \infty$, which is continuous and bounded. In addition, we shall assume that $w(x,t;q)$ is bounded and has continuous first partial derivatives with respect to x and t.

Now, we first derive an integral equation for w via the solution representations for $u(x,t;w(x,t;q))$ and $v(x,t;w(x,t;q))$. From the analysis below, it will be clear that the existence and uniqueness of the solution to the problem defined by (9)–(11) and (8) is equivalent to the existence and uniqueness of the integral equation for w.

Rewriting (9) as

$$u_t + \alpha(q(t))\,u = u_{xx} + \gamma w$$

and using the integrating factor $\exp\left[\int_0^t \alpha(q(\eta))\mathrm{d}\eta\right]$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\exp\left(\int_0^t \alpha(q(\eta))\mathrm{d}\eta\right)u\right] = (u_{xx} + \gamma w)\exp\left(\int_0^t \alpha(q(\eta))\mathrm{d}\eta\right). \tag{12}$$

Then, assigning

$$A(x,t) := \exp\left(\int_0^t \alpha(q(\tau))\mathrm{d}\tau\right)u,$$

we have

$$A_{xx}(x,t) = \exp\left(\int_0^t \alpha(q(\tau))\mathrm{d}\tau\right)u_{xx}$$

$$A_t(x,t) = \exp\left(\int_0^t \alpha(q(\tau))\mathrm{d}\tau\right)(u_t + \alpha(q(t))\,u).$$

Therefore, (12) becomes

$$A_t(x,t) = A_{xx}(x,t) + \exp\left(\int_0^t \alpha(q(\tau))\mathrm{d}\tau\right)\gamma w,$$

which has the solution (see [11])

$$A(x,t) = \int_{-\infty}^{\infty} K(x-\xi,t)f(\xi)\mathrm{d}\xi + \gamma\int_0^t\int_{-\infty}^{\infty} K(x-\xi,t-\tau)w(\xi,\tau)\exp\left(\int_0^\tau \alpha(q(\eta))\mathrm{d}\eta\right)\mathrm{d}\xi\mathrm{d}\tau, \tag{13}$$

where

$$K(x,t) := \frac{1}{2\sqrt{\pi t}}\exp\left(\frac{-x^2}{4t}\right), \quad t > 0, \quad -\infty < x < \infty.$$

Hence,

$$u(x,t;w) = \exp\left(-\int_0^t \alpha(q(\tau))\mathrm{d}\tau\right)A(x,t) \tag{14}$$

with $A(x,t)$ as in (13).

Next, for (10), we consider $B(x,t) := ve^{\beta t}$ and obtain

$$B_t(x,t) = B_{xx}(x,t) + \alpha(q(t))ue^{\beta t},$$

which has the solution

$$B(x,t) = \int_0^t\int_{-\infty}^{\infty} K(x-\xi,t-\tau)\alpha(q(\tau))u(\xi,\tau)e^{\beta\tau}\mathrm{d}\xi\mathrm{d}\tau.$$

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Therefore,

$$v(x,t;w) = e^{-\beta t}B(x,t). \tag{15}$$

Lastly, we solve (11) via the integrating factor $e^{\gamma t}$, which yields the integral equation

$$w(x,t) = \beta \int_0^t v(x,\tau;w)e^{-\gamma(t-\tau)}\mathrm{d}\tau. \tag{16}$$

Next, we derive several a priori estimates for w, which utilize our relationships (14), (15), and (16) and the following lemma (refer to [11]).

Lemma 2.1
For f, a bounded and continuous function on $(-\infty,\infty)$, $0 \le t$, which is uniformly Hölder continuous with exponent μ, $0 < \mu < 1$, with respect to x, the potential

$$z(x,t) = \int_0^t \int_{-\infty}^{\infty} K(x-\xi, t-\tau)f(\xi,\tau)\mathrm{d}\xi\mathrm{d}\tau$$

possesses the following properties:

1.) z, z_x, z_{xx}, and z_t are continuous;

2.) $z_t = z_{xx} + f(x,\ t),\ -\infty < x < \infty,\ 0 < t;$

3.) $|z(x,\ t)| \le t\,\|f\|_t,\ -\infty < x < \infty,\ 0 \le t,$ where

$$\|f\|_t = \sup\{|f(x,\ \tau)|:\quad -\infty < x < \infty,\quad 0 \le \tau \le t\};$$

4.) $|z_x(x,\ t)| \le 2\pi^{-1/2}\,\|f\|_t\, t^{1/2},\ -\infty < x < \infty,\ 0 \le t;$

5.) $|z_{xx}(x,\ t)| \le C\,|f|_\mu\, t^{\mu/2},\ -\infty < x < \infty,\ 0 \le t,$ where C is a positive constant and

$$|f|_\mu = \sup\left[|f(x+\delta,\ t) - f(x,\ t)|\,\delta^{-\mu}\right];$$

6.) $|z_t(x,\ t)| \le \|f\|_t + C\,|f|_\mu\, t^{\mu/2},\ -\infty < x < \infty,\ 0 \le t.$

First, from Property 3 in Lemma 2.1, we see that

$$|u(x,t;w)| \le \|f\| + \gamma t\,\|w\|_t, \tag{17}$$

where

$$\|f\| := \sup_{-\infty<x<\infty} |f|$$

and

$$|v(x,t;w)| \le |B(x,t)| \le t\,\|\alpha\|\,\|u\|_t. \tag{18}$$

Then, because from (16), we have

$$|w(x,t;q)| \le \beta \int_0^t |v(x,\tau;w)|\,\mathrm{d}\tau$$

and utilizing (17), we see that

$$|w(x,t;q)| \le \beta\|\alpha\|\|f\|t^2 + \beta\gamma\,\|\alpha\|\,t^2 \int_0^t \|w\|_\tau\,\mathrm{d}\tau,\quad \forall t. \tag{19}$$

Replacing t with $0 < \eta < t$ in (19), we obtain

$$\begin{aligned}|w(x,\eta;q)| &\le \beta\|\alpha\|\|f\|\eta^2 + \beta\gamma\,\|\alpha\|\,\eta^2 \int_0^\eta \|w\|_\tau\,\mathrm{d}\tau\\ &\le \beta\|\alpha\|\|f\|t^2 + \beta\gamma\,\|\alpha\|\,t^2 \int_0^t \|w\|_\tau\,\mathrm{d}\tau,\end{aligned}$$

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from which it follows that

$$\|w\|_t \leq \beta\|\alpha\|\|f\|t^2 + \beta\gamma\,\|\alpha\|\,t^2\int_0^t \|w\|_\tau\,\mathrm{d}\tau. \tag{20}$$

Then, we define

$$y(t) := \int_0^t \|w\|_\tau\,\mathrm{d}\tau.$$

From (20), we attain the differential inequality

$$y'(t) - \beta\gamma\,\|\alpha\|\,t^2 y(t) \leq \frac{1}{\gamma}\beta\gamma\,\|f\|\,\|\alpha\|\,t^2. \tag{21}$$

Integrating both sides of (21) leads to

$$y(t)\exp\left(-\frac{1}{3}\beta\gamma\,\|\alpha\|\,t^3\right) \leq \frac{1}{\gamma}\,\|f\|\left[1-\exp\left(-\frac{1}{3}\beta\gamma\,\|\alpha\|\,t^3\right)\right],$$

which yields the Gronwall Inequality

$$y(t) \leq \frac{1}{\gamma}\,\|f\|\left[\exp\left(\frac{1}{3}\beta\gamma\,\|\alpha\|\,t^3\right)-1\right]. \tag{22}$$

Then, substituting (22) into (20), we obtain

$$\|w(x,t)\|_t \leq \beta\,\|\alpha\|\,\|f\|\,t^2\exp\left(\frac{1}{3}\beta\gamma\,\|\alpha\|\,t^3\right), \tag{23}$$

which implies the uniqueness of the solution.

Now, after differentiating (16) with respect to x and taking the absolute value of both sides of the result yields

$$|w_x(x,t;q)| \leq \beta\int_0^t |v_x(x,t;w)|\,\mathrm{d}\tau. \tag{24}$$

Then, differentiating (15) with respect to x and utilizing Part 4 of Lemma 2.1, we arrive at

$$|v_x(x,t;w)| \leq 2\pi^{-1/2}\,\|\alpha\|\,\|u\|_t\,t^{1/2},$$

so that (24) becomes

$$|w_x(x,t;q)| \leq 2\pi^{-1/2}\beta\,\|\alpha\|\,\|u\|_t\,t^{3/2}.$$

Therefore, substituting (17) into the above result and utilizing (22), we obtain

$$\begin{aligned}
|w_x(x,t;q)| &\leq 2\pi^{-1/2}\beta\,\|\alpha\|\left[\frac{2}{3}\,\|f\|\right]t^{3/2} + 2\pi^{-1/2}\beta\gamma\,\|\alpha\|\,t^{3/2}\int_0^t \|w\|_\tau\,d\tau \\
&\leq 2\pi^{-1/2}\beta\,\|\alpha\|\,\|f\|\,t^{3/2} + 2\pi^{-1/2}\beta\gamma\,\|\alpha\|\,t^{3/2}\int_0^t \|w\|_\tau\,d\tau \\
&\leq 2\pi^{-1/2}\beta\,\|\alpha\|\,\|f\|\,t^{3/2} + 2\pi^{-1/2}\beta\,\|\alpha\|\,\|f\|\,t^{3/2}\left[\exp\left(\frac{1}{3}\beta\gamma\,\|\alpha\|\,t^3\right)-1\right] \\
&= 2\pi^{-1/2}\beta\,\|\alpha\|\,\|f\|\,t^{3/2}\exp\left(\frac{1}{3}\beta\gamma\,\|\alpha\|\,t^3\right),
\end{aligned}$$

which implies that the a priori solution $w(x,t;q)$ is uniformly Lipschitz continuous, thus justifying the equivalence of the problem defined by (9)–(11) and (8), and the integral equation (16).

3. A Picard iteration procedure

In this section, we implement a Picard iteration procedure for the integral equation (16) via the representations (14) and (15) in order to establish the existence and uniqueness of the solution to (16). We begin by defining our preliminary approximation based on our initial data as seen below:

$$w_0(x,t) = \exp\left[-\int_0^t \alpha(q(\tau))\mathrm{d}\tau\right]\int_{-\infty}^{\infty} K(x-\xi,t)f(\xi)\mathrm{d}\xi.$$

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We then assign the following values:

$$u_1(x,t) = w_0 + \gamma \int_0^t \int_{-\infty}^{\infty} K(x-\xi, t-\tau) w_0(\xi,\tau) \exp\left[-\int_{-\tau}^{t} \alpha(q(\tau))\mathrm{d}\tau\right] \mathrm{d}\xi \mathrm{d}\tau$$
$$v_1(x,t) = \int_0^t \int_{-\infty}^{\infty} K(x-\xi, t-\tau)\alpha(q(\tau)) u_1(\xi,\tau) \exp\left[-\beta(t-\tau)\right] \mathrm{d}\xi \mathrm{d}\tau$$
$$w_1(x,t) = \beta \int_0^t v_1(x,\tau) \exp\left[-\gamma(t-\tau)\right] \mathrm{d}\tau,$$

which generalizes to

$$u_{n+1}(x,t) = w_0 + \gamma \int_0^t \int_{-\infty}^{\infty} K(x-\xi, t-\tau) w_n(\xi,\tau) \exp\left[-\int_{-\tau}^{t} \alpha(q(\tau))\mathrm{d}\tau\right] \mathrm{d}\xi \mathrm{d}\tau$$
$$v_{n+1}(x,t) = \int_0^t \int_{-\infty}^{\infty} K(x-\xi, t-\tau)\alpha(q(\tau)) u_{n+1}(\xi,\tau) \exp\left[-\beta(t-\tau)\right] \mathrm{d}\xi \mathrm{d}\tau$$
$$w_{n+1}(x,t) = \beta \int_0^t v_{n+1}(x,\tau) \exp\left[-\gamma(t-\tau)\right] \mathrm{d}\tau.$$

It then immediately follows that

$$|w_{n+1}(x,t) - w_n(x,t)| \le \beta \int_0^t |v_{n+1}(x,\tau) - v_n(x,\tau)| \, \mathrm{d}\tau \tag{25}$$
$$|v_{n+1}(x,t) - v_n(x,t)| \le \int_0^t \int_{-\infty}^{\infty} K(x-\xi, t-\tau)\alpha(q(\tau)) \, |u_{n+1}(\xi,\tau) - u_n(\xi,\tau)| \, \mathrm{d}\xi \mathrm{d}\tau \tag{26}$$
$$|u_{n+1}(x,t) - u_n(x,t)| \le \gamma \int_0^t \int_{-\infty}^{\infty} K(x-\xi, t-\tau) \, |w_n(\xi,\tau) - w_{n-1}(\xi,\tau)| \, \mathrm{d}\xi \mathrm{d}\tau. \tag{27}$$

Then from (25)–(27) and Lemma 2.1, we have

$$|u_{n+1} - u_n| \le \gamma t \, \|w_n - w_{n-1}\|_t$$

and therefore,

$$|v_{n+1} - v_n| \le \beta\gamma \, \|\alpha\| \, t^2 \, \|w_n - w_{n-1}\|_t,$$

which leads to

$$\|w_{n+1} - w_n\|_t \le \beta\gamma \, \|\alpha\| \int_0^t \tau^2 \, \|w_n - w_{n-1}\|_\tau \, \mathrm{d}\tau.$$

Next, we consider (25) with $n = 0$, use the fact that $v_0 = v(x,0) = 0$ and utilize the estimate in (17) yielding

$$\begin{aligned} |w_1 - w_0| &\le \beta \int_0^t |v_1| \, \mathrm{d}\tau \\ &\le \beta \int_0^t \tau \, \|\alpha\| \, \|u_1\|_\tau \, \mathrm{d}\tau \\ &\le \beta \int_0^t \tau \, \|\alpha\| \, [\|f\| + \gamma\tau \, \|w_0\|_\tau] \, \mathrm{d}\tau \\ &\le \frac{1}{2}\beta \, \|\alpha\| \, \|f\| \, t^2 + \frac{1}{3}\beta\gamma \, \|\alpha\| \, \|f\| \, t^3, \end{aligned}$$

which implies $\|w_1 - w_0\|_t \le C_1$ where $C_1 := \beta \, \|\alpha\| \, \|f\| \, T^2 + \beta\gamma \, \|\alpha\| \, \|f\|_T \, T^3$, for $0 < t < T$.

We then see that

$$\|w_2 - w_1\|_t \le \beta\gamma \, \|\alpha\| \int_0^t \tau^2 \, \|w_1 - w_0\|_\tau \, \mathrm{d}\tau \le \beta\gamma \, \|\alpha\| \, C_1 T^2 t$$

and similarly

$$\|w_3 - w_2\|_t \le \beta^2\gamma^2 \, \|\alpha\|^2 \, C_1 T^2 t^2/2$$

and inductively

$$\|w_{n+1} - w_n\|_t \le \beta^n\gamma^n \, \|\alpha\|^n \, C_1 T^2 t^n/n!, \quad n = 1, 2, \ldots$$

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Next, we observe that

$$S = \|w_0\|_T + \sum_{k=0}^{\infty} \|w_{k+1} - w_k\|_T$$
$$\leq \|w_0\|_T + C_1 T^2 \sum_{k=0}^{\infty} \frac{\beta^k \gamma^k \|\alpha\|^k T^k}{k!} < \infty. \tag{28}$$

Consequently, (28) implies the absolute and uniform convergence of the telescoping series above to obtain the existence of

$$w(x,t;q) = \lim_{n\to\infty} w_n = w_0 + \lim_{n\to\infty} \sum_{k=0}^{n-1} (w_{k+1} - w_k)$$
$$= w_0 + \sum_{k=0}^{\infty} (w_{k+1} - w_k).$$

By recalling the a priori estimate (23), we see that the solution $w(x,t;q)$ of the auxiliary problem is unique.

4. Existence and uniqueness of a classical solution to the original system

We first define the mapping

$$(\mathcal{F}q)(t) := \int_a^b w(x,t;q)\mathrm{d}x. \tag{29}$$

Clearly, the map is well-defined and a fixed point of the map is a solution to (9)–(11) and (8). Then, recalling the a priori estimate (23) for w, we have the bound:

$$\|(\mathcal{F}q)(t)\|_t \leq \beta \|\alpha\| \|f\| t^2 \exp\left(\frac{1}{3}\beta\gamma \|\alpha\| t^3\right) [b-a] := C_2(t). \tag{30}$$

Let $\mathcal{K} = \{q(t) \in C([0,T]) : \|q\|_t \leq C_2(t)\}$. For $q_1(t)$ and $q_2(t)$ belonging to $\mathcal{K}$, we see that

$$\|\lambda q_1 + (1-\lambda) q_2\|_t \leq \lambda \|q_1\|_t + (1-\lambda) \|q_2\|_t \leq C_2(t)$$

via the triangle inequality for $0 \leq \lambda \leq 1$. Hence, $\mathcal{K}$ is convex. From the differential equation (11) and the estimates (17), (18), and (23), it follows that $|w_t(x,t;q)| \leq C_3(t)$ and therefore, we see that $\left|(\mathcal{F}q)'(t)\right| \leq C_3(t)[b-a]$. Hence, $\mathcal{F}$ maps the convex set $\mathcal{K}$ into a compact subset of itself.

Next, we show that $\mathcal{F}$ is continuous. First, consider

$$(\mathcal{F}q_1)(t) - (\mathcal{F}q_2)(t) = \int_a^b [w(x,t;q_1) - w(x,t;q_2)]\mathrm{d}x$$
$$= \beta \int_a^b \int_0^t [v(x,t;q_1) - v(x,t;q_2)] \exp[-\gamma(t-\tau)]\,\mathrm{d}\tau\mathrm{d}x.$$

Upon considering the representations (13)–(16) while displaying the dependence of u and v upon q, we have

$$v(x,t;q_1) - v(x,t;q_2) = \int_0^t \int_{-\infty}^{\infty} K(x-\xi, t-\tau)[\alpha(q_1(\tau))u(\xi,\tau;q_1) - \alpha(q_2(\tau))u(\xi,\tau;q_2)]$$
$$\times \exp[-\gamma(t-\tau)]\,\mathrm{d}\xi\mathrm{d}\tau$$
$$= \int_0^t \int_{-\infty}^{\infty} K(x-\xi, t-\tau)\,[[\alpha(q_1(t)) - \alpha(q_2(t))]u(\xi,\tau;q_1)$$
$$+\alpha(q_2(t))[u(\xi,\tau;q_2) - u(\xi,\tau;q_1)]] \exp[-\gamma(t-\tau)]\,\mathrm{d}\xi\mathrm{d}\tau \tag{31}$$

and

$$u(x,t;q_1) - u(x,t;q_2) = \left[\exp\left[-\int_0^t \alpha(q_1(\tau))d\tau\right] - \exp\left[-\int_0^t \alpha(q_2(\tau))\mathrm{d}\tau\right]\right] \int_{-\infty}^{\infty} K(x-\xi,t)f(\xi)\mathrm{d}\xi$$
$$+\gamma \int_0^t \int_{-\infty}^{\infty} K(x-\xi,t-\tau)\left[\exp\left[-\int_\tau^t \alpha(q_1(\eta))\mathrm{d}\eta\right] - \exp\left[-\int_\tau^t \alpha(q_2(\eta))\mathrm{d}\eta\right]\right]$$
$$\times w(\xi,\tau;q_1) + \exp\left[-\int_\tau^t \alpha(q_2(\eta))\mathrm{d}\eta\right][w(\xi,\tau;q_1) - w(\xi,\tau;q_2)]\mathrm{d}\xi\mathrm{d}\tau. \tag{32}$$

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Via (32), we obtain

$$\begin{aligned}|u(x,t;q_1)-u(x,t;q_2)| \le \|f\| &\left[\int_0^t |\alpha(q_1(\tau))-\alpha(q_2(\tau))|\,\mathrm{d}\tau\right]\\ &+\gamma\int_0^t \|w\|_\tau \left|\int_0^t [\alpha(q_1(\eta))-\alpha(q_2(\eta))]\mathrm{d}\eta\right|\mathrm{d}\tau\\ &+\gamma\int_0^t \|w(x,\tau;q_1)-w(x,\tau;q_2)\|_\tau\,\mathrm{d}\tau,\end{aligned}$$

which implies by the Mean Value Theorem and the existence and boundedness of α' that

$$\begin{aligned}|u(x,t;q_1)-u(x,t;q_2)| \le& t\|f\|\,\|\alpha'\|\,\|q_1-q_2\|_t+\gamma t\|w\|_t\,\|\alpha'\|\,\|q_1-q_2\|_t\\ &+\gamma\int_0^t \|w(x,t;q_1)-w(x,t;q_2)\|_\tau\,\mathrm{d}\tau\\ =&\left[\|f\|+\gamma\|w\|_t\right]\|\alpha'\|\,\|q_1-q_2\|_t\,t\\ &+\gamma t\|w(x,t;q_1)-w(x,t;q_2)\|_t.\end{aligned}\tag{33}$$

Now, (31) yields

$$\begin{aligned}|v(x,t;q_1)-v(x,t;q_2)| \le& t\|\alpha'\|\,\|u\|_t\,\|q_1-q_2\|_t\\ &+\|\alpha\|\int_0^t \|u(x,\tau;q_1)-u(x,\tau;q_2)\|_\tau\,\mathrm{d}\tau\end{aligned}\tag{34}$$

and

$$|w(x,t;q_1)-w(x,t;q_2)| \le \beta\int_0^t \|v(x,t;q_1)-v(x,t;q_2)\|_\tau\,\mathrm{d}\tau,$$

which leads to

$$\|w(x,t;q_1)-w(x,t;q_2)\|_t \le C_4(t)\,\|q_1-q_2\|_t+\gamma\beta\int_0^t \|w(x,t;q_1)-w(x,t;q_2)\|_\tau\,\mathrm{d}\tau.\tag{35}$$

Then, applying Gronwall's inequality to (35), we have

$$\|w(x,t;q_1)-w(x,t;q_2)\|_T \le C_5(T)\,\|q_1-q_2\|_T\,e^{\gamma\beta T},\quad 0<t<T.\tag{36}$$

Finally, as

$$(\mathcal{F}q_1)(t)-(\mathcal{F}q_2)(t)=\int_a^b [w(x,t;q_1)-w(x,t;q_2)]\mathrm{d}x,$$

we see that

$$\|(\mathcal{F}q_1)-(\mathcal{F}q_2)\|_T \le C_5(t)e^{\gamma\beta T}(b-a)\,\|q_1-q_2\|_T,$$

which demonstrates the continuity of the mapping $\mathcal{F}:\mathcal{K}\to\mathcal{K}$. We conclude via the Schauder fixed point theorem [12] that (5)–(8) has a classical solution. We then have the following statement.

Theorem 4.1
If $f(x)$ is continuous and bounded on $-\infty<x<\infty$, $\alpha(q)>0$ is bounded with bounded first derivatives on $0\le q<\infty$, β and γ are positive constants, and a and b are real numbers, then there exists a unique classical solution to the problem defined by (5)–(8).

Proof
See the above analysis for the existence. Next, for uniqueness, we write (29) as

$$(\mathcal{F}q)(t):=\int_a^b w(x,t;q)\mathrm{d}x=q$$

and assume that the mapping defined by (29) has two fixed points q_1 and q_2. Then, upon substituting (33) into (34), we obtain

$$\begin{aligned}|v(x,t;q_1)-v(x,t;q_2)| \le& t\|\alpha'\|\,\|u\|_t\,\|q_1-q_2\|_t\\ &+\|\alpha\|\int_0^t \left([\|f\|+\gamma\|w\|_\tau]\,\|\alpha'\|\,\|q_1-q_2\|_\tau\,\tau\right.\\ &\left.+\gamma\tau\,\|w(x,\tau;q_1)-w(x,\tau;q_2)\|_\tau\right)\mathrm{d}\tau.\end{aligned}\tag{37}$$

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Expanding the right-hand side of (37) via (36) leads to

$$|v(x,t;q_1)-v(x,t;q_2)| \le C_6(t)\,\|q_1-q_2\|_t\,.$$

Then,

$$\begin{aligned}|q_1(t)-q_2(t)| &\le \int_a^b |w(x,t;q_1)-w(x,t;q_2)|\mathrm{d}x\\ &= \beta\int_a^b\int_0^t |v(x,\tau;w(x,\tau;q_1))-v(x,\tau;w(x,\tau;q_2))|\,e^{-\gamma(t-\tau)}\mathrm{d}x\mathrm{d}\tau\\ &\le C_7(t)\int_0^t \|q_1-q_2\|_\tau\,d\tau,\end{aligned}$$

from which it follows that

$$\|q_1-q_2\|_t \le C_7(t)\int_0^t \|q_1-q_2\|_\tau\,d\tau. \tag{38}$$

Because $q_1(t)$ and $q_2(t)$ are bounded by $C_2(t)$ via (30), we know that

$$\|q_1-q_2\|_t \le 2C_2(t)C_7(t)t. \tag{39}$$

Then, substituting (39) into (38), we have

$$\|q_1-q_2\|_t \le 2C_2(t)C_7(t)^2\frac{t^2}{2}.$$

Iterating the above result, we inductively conclude that

$$\|q_1-q_2\|_T \le 2C_2(T)C_7(T)^n\frac{T^n}{n!},\quad n=1,2,3,\ldots \tag{40}$$

As the right-hand side of (40) tends to zero as n tends to infinity, we see that $\|q_1-q_2\|_T=0$, which implies $q_1(t)\equiv q_2(t)$.

Hence, the solution to (5)–(8) is unique. □

5. A finite-difference approximation

Let R be a *large* positive number. Then, set $\Delta x = R/N$, where N is a positive integer. Next, let $\Delta t = T/M$, for M a positive integer. Then, we set

$$\begin{aligned}x_i &= -R+i\Delta x,\quad i=0,1,2,\ldots,2N+1,\\ t_j &= j\Delta t,\quad j=0,1,\ldots,M.\end{aligned}$$

We next define $u_{i,j} := u(x_i,t_j)$, $v_{i,j} := v(x_i,t_j)$, and $w_{i,j} := w(x_i,t_j)$ for each i and j. Now, we can replace the differential equations at (x_i,t_j) with the finite-difference equations:

$$\left.\begin{aligned}&(u_{i,j+1}-u_{i,j})/\Delta t = \Delta_x^2 u_{i,j+1} - \alpha\left(ETrw_{j+1}\right)u_{i,j+1} + \gamma Ew_{i,j+1} + \mathcal{O}((\Delta x)^2+\Delta t)\\ &(v_{i,j+1}-v_{i,j})/\Delta t = \Delta_x^2 v_{i,j+1} + \alpha\left(ETrw_{j+1}\right)u_{i,j+1} - \beta v_{i,j+1} + \mathcal{O}((\Delta x)^2+\Delta t)\\ &(w_{i,j+1}-w_{i,j})/\Delta t = \beta v_{i,j+1} - \gamma w_{i,j+1} + \mathcal{O}(\Delta t),\\ &\qquad \text{for } i=1,2,\ldots,2N-1,\ \ j=1,2,\ldots,M,\\ &u_{0,j}=u_{2N,j}=v_{0,j}=v_{2N,j}=w_{0,j}=w_{2N,j}=0,\\ &\qquad \text{for } j=0,1,2,\ldots,M.\\ &u_{i0}=f_i,\quad v_{i0}=w_{i0}=0,\quad i=1,2,\ldots,2N-1\end{aligned}\right\} \tag{41}$$

with the following:

$$\begin{aligned}\Delta_x^2 u_{i,j+1} &:= \frac{u_{i+1,j+1}-2u_{i,j+1}+u_{i-1,j+1}}{(\Delta x)^2},\\ Trw_{j+1} &:= \sum_{l=k}^{k+p}\frac{w_{l,j+1}+w_{l+1,j+1}}{2}(\Delta x),\\ Eq_{j+1} &:= 2q_j - q_{j-1},\\ a &= k\Delta x,\ \ b=(k+p+1)\Delta x.\end{aligned}$$

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By respectively replacing the $u_{i,j}$, $v_{i,j}$, and $w_{i,j}$ in the system (41) with $U_{i,j}$, $V_{i,j}$, and $W_{i,j}$ for all i and j and deleting the $\mathcal{O}((\Delta x)^2 + \Delta t)$ and $\mathcal{O}(\Delta t)$ terms, we obtain our approximation:

$$\left.\begin{array}{l}
(U_{i,j+1} - U_{i,j})/\Delta t = \Delta_x^2 U_{i,j+1} - \alpha\left(ETrW_{j+1}\right) U_{i,j+1} + \gamma E W_{i,j+1} \\
(V_{i,j+1} - V_{i,j})/\Delta t = \Delta_x^2 V_{i,j+1} + \alpha\left(ETrW_{j+1}\right) U_{i,j+1} - \beta V_{i,j+1} \\
(W_{i,j+1} - W_{i,j})/\Delta t = \beta V_{i,j+1} - \gamma W_{i,j+1}, \\
\qquad \text{for } \ i = 1, 2, \ldots, 2N-1, \ \ j = 1, 2, \ldots, M, \\
U_{0,j} = U_{2N,j} = V_{0,j} = V_{2N,j} = W_{0,j} = W_{2N,j} = 0, \\
\qquad \text{for } \ j = 0, 1, 2, \ldots, M. \\
U_{i0} = f_i, \ \ V_{i0} = W_{i0} = 0, \ \ i = 1, 2, \ldots, 2N-1
\end{array}\right\}. \tag{42}$$

We note that the system (42) is a linear system of equations because of the extrapolation operator

$$Eq(t+1) = 2q(t) - q(t-1) = q(t + \Delta t) + \mathcal{O}((\Delta t))^2.$$

We also observe that this scheme requires using two levels of data for $W_{i,j}$ to advance. The first level, $j = 1$, can be computed either by integrating for a solution of the nonlinear system or by simply setting $W_{i,1} = 0$, $i = 1, 2, \ldots, 2N-1$ and $\mathrm{Tr}W_1 = 0$ in the top equation of (42) as $W_{i0} = 0$ initially and the local truncation error is $\mathcal{O}(\Delta t(\Delta x)^2 + (\Delta t)^2)$.

We employ the above procedure to compute the solution of (42) for three examples of $\alpha(q)$, $0 \le q < \infty$ to gain some intuition regarding the overall behavior of the system and the effect of a decreasing $\alpha(q)$ versus $\alpha(q) \equiv$ constant.

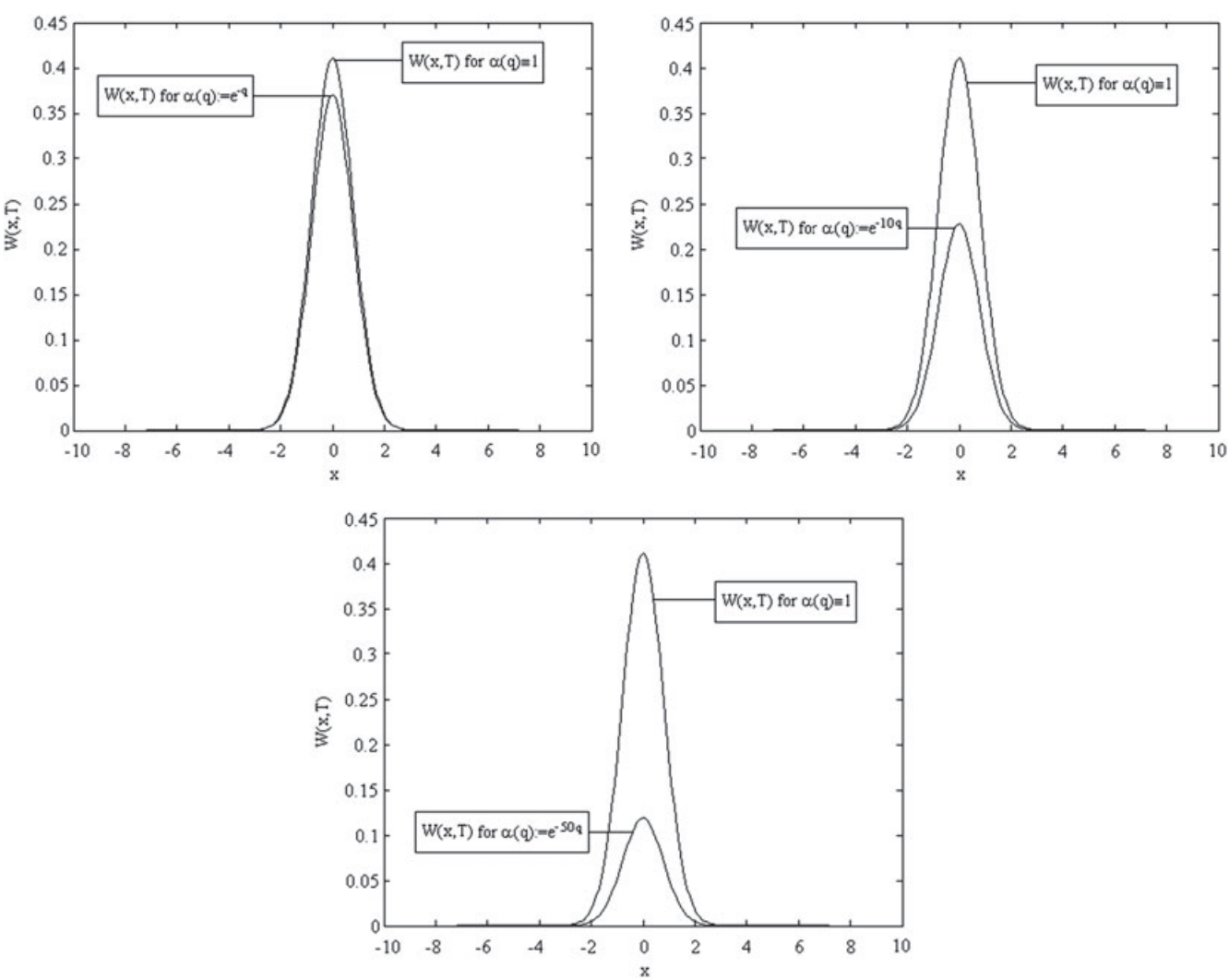

Figure 1. Plot of $W(x, T)$ for $\alpha(q) := e^{-\mu q}$ and $\alpha(q) \equiv 1$ with $\mu = 1, 10, 50$.

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6. Some numerical experiments

Herein, we implement the numerical procedure outlined in Section 5 using the initial function $f(x) := 100e^{-x^2}$. We choose $R = 10$, $N = 100$, $T = 1$, $M = 1000$, $\beta = 1$ and $\gamma = 1/2$. Below, we display the graphical results of $\alpha(q) := e^{-\mu q}$ versus $\alpha(q) \equiv 1$ on the one-hundredth iteration of our program for $\mu = 1, 10, 50$.

As evidenced from the graphical analysis displayed in Figure 1, we see that each plot $W_{i,j}$ for $\alpha(q) := e^{-\mu q}$ is lower than the graph of $W_{i,j}$ for $\alpha(q) \equiv 1$ and that these results become more apparent as the μ-value increases. This is tantamount to saying that, as precautionary measures are taken, the number of infected people diminishes, which agrees with intuition.

7. Concluding remarks and future research

In considering the model defined by (5)–(8) with compact support for initial u and zero initial values for v and w, one immediately sees two defects. One, the u, v and w each tend to zero as $t \to \infty$. Two, in this model everyone recovers and can be reinfected as well. The first problem can be alleviated by assigning finite domains and boundary conditions. The second problem can be resolved by adding a fourth variable, say z, for the diseased population.

In addition, to handle the case of conferred immunity via the disease, the variable u can be split into two variables: one for the susceptible population and one for the immune population, which will add another equation to the system. Additionally, to account for the incubation time, one would need to add a time-delay in the v-term in the w-equation.

Lastly, we mention that in addition to the nonlocality of the models discussed in (2)–(4) is the traveling wave nature of the solutions to these models. In fact, much attention has been paid to traveling waves, and spreading speeds for that matter, for reaction–diffusion epidemiological models as in [7]. The theory of traveling waves and spreading speeds was initially established in [13] and [14] by Diekmann, and Thieme utilized integral equations in [15] and [16]. Analogous to these works, our current model could incorporate a transport term that would permit movement of the infected population.

In conclusion, we emphasize that the model analyzed herein is designed to function a preliminary model for which consequent and increasingly more sophisticated models can be based. For example, as explained in this section, our current model has certain limitations, which can each be resolved by appropriate modifications. These modified models will give rise to new problems and new analyses that will naturally extend the results of this paper.

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