# On the Higher-Order Sheffer Orthogonal Polynomial Sequences (Chapter Abstracts)

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## Chapter 1
## The Sheffer A-Type 0 Orthogonal Polynomial Sequences and Related Results

In this chapter, we present a rigorous development of I. M. Sheffer's characterization of the *A-Type 0* orthogonal polynomial sequences. We first develop the results that led to the main theorem that characterizes the general *A-Type 0* polynomial sequences via a linear generating function. From there, we develop the additional theory that Sheffer utilized in order to determine which *A-Type 0* polynomial sequences are also orthogonal. We then address Sheffer's additional characterizations of *B-Type* and *C-Type*, as well as E.D. Rainville's $\sigma$-*Type* classification. Lastly, we cover J. Meixner's approach to the same characterization problem studied by Sheffer and then discuss an extension of Meixner's analysis by W.A. Al-Salam. Portions of the analysis addressed throughout this chapter are supplemented with informative concrete examples.

## Chapter 2
## Some Applications of the Sheffer A-Type 0 Orthogonal Polynomial Sequences

In this chapter, we address several of the many applications of the classical orthogonal polynomial sequences. Theses applications include; first-order differential equations that characterize linear generating functions, additional first-order differential equations, second-order differential equations (with applications to quantum mechanics), difference equations and numerical integration (Gaussian Quadrature). We first develop each of these applications in a general context and then cover examples using specific Sheffer Sequences, i.e. the Laguerre, Hermite, Charlier, Meixner, Meixner-Pollaczek and Krawtchouk polynomials.

## Chapter 3
## A Method for Analyzing a Special Case of the Sheffer B-Type 1 Polynomial Sequences

In 1939, I.M. Sheffer proved that every polynomial sequence belongs to one and only one *Type*. He also extensively developed properties of the most basic *Type* set, entitled *B-Type 0*, or equivalently *A-Type 0*, ($k = 0$ below)

$$A(t)\exp\left[xH_1(t) + \cdots + x^{k+1}H_{k+1}(t)\right] = \sum_{n=0}^{\infty} P_n(x)t^n,$$
$$\text{with } \; H_i(t) = h_{i,i}t^i + h_{i,i+1}t^{i+1} + \cdots, \;\; h_{1,1} \neq 0, \;\; i = 1, 2, \ldots, k+1$$

and then determined which of these sets are also orthogonal. He subsequently generalized his classification by letting $k$ be arbitrary in the relation above, i.e. he defined the *B-Type* $k$ class.

Thus far, no research has been published that *specifically* analyzes the higher-order Sheffer classes ($k \geq 1$ above). Therefore, we present a preliminary analysis of a special case of the *B-Type* 1 ($k = 1$) class, in order to determine which sets, if any, are also orthogonal. Moreover, the method utilized herein is quite useful, as it can be applied to other types of characterization problems as well. This chapter also demonstrates how computer algebra packages, like Mathematica®, can aid in the development of rigorous results in orthogonal polynomials and special functions. We conclude this chapter by discussing some future research problems that can be solved by using the techniques of this chapter.