An Existence Theorem for a Nonlocal Global Pandemic Model for Insect-Borne Diseases

John R. Cannon and Daniel J. Galiffa

1 Mathematics Department, University of Central Florida, Orlando, FL 32816, USA
2 Department of Mathematics, Penn State Erie, The Behrend College, Erie, PA 16563, USA

Correspondence should be addressed to Daniel J. Galiffa; djg34@psu.edu

Received 6 May 2014; Revised 14 July 2014; Accepted 15 July 2014; Published 24 July 2014

Academic Editor: Kanishka Perera

We construct and analyze a nonlocal global pandemic model that comprises a system of two nonlocal integro-differential equations (functional differential equations) and an ordinary differential equation. This model was constructed by considering a spherical coordinate transformation of a previously established epidemiology model that can be applied to insect-borne diseases, like yellow fever. This transformation amounts to a nonlocal boundary value problem on the unit sphere and therefore can be interpreted as a global pandemic model for insect-borne diseases. We ultimately show that a weak solution to the weak formulation of this model exists using a fixed point argument, which calls upon the construction of a weak formulation and the existence and uniqueness of an auxiliary problem.

1. Introduction

In [1], Cannon and Galiffa analyzed the nonlocal model:

\[-\alpha \left( \int_0^1 u(t) dt \right) u''(x) + f(x) = 0, \quad x \in (0,1)\]

\[u(0) = a, \quad u(1) = b,\]

with \(\alpha = \alpha(q)\) being a positive function of \(q\) defined over \(-\infty < q < \infty\), \(f(x)\) defined over \(0 \leq x \leq 1\), and \(a\) and \(b\) real constants. The analysis contained in [1] included both existence and uniqueness theorems for the analytic solution, \(u\), to (1) and a discretization, as well as the construction and implementation of a finite-difference procedure. This model was a special case of [2], which has applications to thermodynamics.

The following homogeneous extension of model (1) was considered in [3]:

\[-\alpha \left( \int_0^1 u(t) dt \right) u''(x) + \left[ u(x) \right]^{2m+1} = 0, \quad x \in (0,1),\]

\[u(0) = a, \quad u(1) = b,\]

As in (1), the existence and uniqueness of the analytic solution, \(u\), to (2), as well as a discretization, were established and another finite-difference procedure was developed and executed; however, the analysis was significantly different than in [1].

Upon the completion of [3], the authors realized the potential for additional applications to physical problems via mathematical models consisting of variations of the nonlocal term \(\alpha(\int_0^1 u(t) dt)\). In particular, a model was constructed and analyzed in [4, 5] that can be applied to insect-borne diseases, for example, yellow fever, which is as follows:

\[u_t = u_{xx} - \alpha \left( \int_a^b w(x,t) dx \right) u + \gamma w, \quad -\infty < x < \infty, \quad t \leq T,\]

\[v_t = v_{xx} + \alpha \left( \int_a^b w(x,t) dx \right) u - \beta v, \quad -\infty < x < \infty, \quad t \leq T,\]

\[w_t = \beta v - \gamma w, \quad -\infty < x < \infty, \quad t \leq T,\]
\[ u(x,0) = f(x), \quad -\infty < x < \infty, \]
\[ v(x,0) = 0, \quad -\infty < x < \infty, \]
\[ w(x,0) = 0, \quad -\infty < x < \infty \]  

(3)

with \( \beta \) and \( \gamma \) arbitrary positive constants.

We next give an overview of this model, which is based on [5]. Biologically, \( \beta \) represents the rate at which mobile 
infecteds become immobile and \( \gamma \) represents the rate at which 
immobile infecteds recover. The unknowns \( u, v, \) and \( w, \) 
respectively, represent a susceptible class, a mobile infected 
class, and an immobile infected class. In addition, neither 
infected class is infective; that is, susceptibles become infected 
via contact with vectors (disease-spreading agents, e.g., *Aedes 
aegypti* mosquitoes), which was not modeled directly. The 
nonnegative function \( \alpha(q), \) \( 0 \leq q < \infty, \) represents the 
per capita rate at which susceptibles contract the disease 
in a disease-free population, which is decreasing with \( q. \) 
The argument \( \int_a^b w(x,t)\,dx, \) representing the number of the 
population immobilized from the disease, when substituted 
into \( \alpha \) brings about a reduction in the rate of infection via 
actions taken at large and at the individual level, to reduce 
the risk of infections. Even more specifically, \( \int_a^b w(x,t)\,dx \)
counts the number of immobile infecteds within a single fixed 
interval \([a,b]\) and influences the rate \( \alpha(q) \) from a biological 
standpoint. For example, if all news stories about immobile 
infecteds concern individuals living only in region \([a,b], \) then 
susceptibles will take precautions based only on the reported 
number \( \int_a^b w(x,t)\,dx. \)

In this paper, we consider an extension of model (3) 
via a spherical coordinate transformation that results in the 
following model on the unit sphere \( S: \)

\[

t_u = \frac{1}{\sin \phi} \frac{\partial}{\partial \phi} \left( \sin \phi \frac{\partial u}{\partial \phi} \right) + \csc^2 \phi \frac{\partial^2 u}{\partial \theta^2} \\
- \alpha \left( \int_{\Omega} w(\phi, \theta, t) \sin \phi \, d\phi \, d\theta \right) u + \gamma w, \\
	on S \times (0, T], \\

v_t = \frac{1}{\sin \phi} \frac{\partial}{\partial \phi} \left( \sin \phi \frac{\partial w}{\partial \phi} \right) + \csc^2 \phi \frac{\partial^2 v}{\partial \theta^2} \\
+ \alpha \left( \int_{\Omega} w(\phi, \theta, t) \sin \phi \, d\phi \, d\theta \right) u - \beta v, \\
	on S \times (0, T], \\

w_t = \beta v(\phi, \theta, t) - \gamma w(\phi, \theta, t), \\
	on S \times (0, T], \\

u(\phi, \theta, 0) = f(\phi, \theta) \quad \text{on } S, \\

\nu(\phi, \theta, 0) = 0 \quad \text{on } S, \\

w(\phi, \theta, 0) = 0 \quad \text{on } \Omega \subset S. \\

(4)

Physically, the system comprising (4) can be interpreted as a 
preliminary model for a global pandemic of an insect-borne 
disease. First, we note that the system of spherical coordinates 
is the natural and historical coordinate system for the surface 
of the Earth. Second, we note that diffusion has been used to 
describe epidemics in the past; compare [6–9] for a few 
of several such examples. Third, the system proposed herein 
is preliminary, since we take no account of the oceans, as 
to do so will require boundary conditions to be assigned 
to the coasts of the Earth’s land masses. Fourth, spherical 
coordinates come into play for relatively short distances 
on the Earth’s surface. For example, a Cartesian coordinate 
system \((x, y, z)\) with origin located at the John F. Kennedy 
Space Center will find the plane \((x, y, 0)\) approximately 
20,000 feet over Tampa, FL. Fifth, we have made no attempt 
to model air travel, which can be handled by the introduction 
of spatially located sources and sinks. Sixth, we note that 
mosquitoes have been found alive in shipping containers 
filled with used tires shipped to the United States from 
Asia, as well as other locations, which provides a vector 
for tropical Asian diseases on the American continent 
(cf. http://www.epa.state.il.us/land/tires/mosquito-borne 
ilnesses.html, for one such example). Moreover, we also 
ote note that passengers entering Australia are sprayed with 
insecticide disinfectant before they can be allowed to deplane 
(http://abcnews.go.com/).

Finally, our interest here is to show that a solution exists 
to (4) in order to enable the study of the possible effect of 
the death toll of a disease on the dynamics of the epidemic 
itself. Thus, the system (4), which incorporates a coefficient 
containing a functional of part of the solution of the system, 
requires a nonlinear analysis, which we will outline later in 
this introduction.

Several papers have recently been published regarding 
the mathematical analysis of pandemics, as well as the 
construction of mathematical models comprising a system of 
differential equations applicable to such biological phenom-
ena. Several of these papers influenced our current work and 
it is therefore worth briefly describing them before we further 
discuss (4).

For example, in [10] the authors explain how the utili-
zation of antiviral drugs was intended to moderate the 
severity of a new strain of an influenza pandemic and that 
the success of these drugs was dependent upon the timely 
onset of therapy (within 48 hours) after the appearance of 
clinical symptoms. They discuss further that this requirement 
can be understood by a compartmental model that examines 
the density of infected individuals in terms of the time 
elapsed since the onset of symptoms. Therefore, based on 
this compartmental model, a system of delay-differential 
equations, with both discrete and disturbed delays, was 
constructed and analyzed. This system is as follows:

\[
\frac{d}{dt} S(t) = -\beta S(t) Q(t), \\
\frac{d}{dt} E(t) = \beta S(t) Q(t) - \mu E(t),
\]

(5)
for $t \geq 0$, where susceptible and exposed classes are, respectively, denoted by $S$ and $E$, with $\beta$ representing the baseline transmission rate, $1/\mu_t$ representing the incubation period, and $Q(t)$ representing the force of infection yet to be formulated.

As a precaution to a potential H5N1 pandemic, the authors in [11] simulate a possible outbreak in Italy by analyzing the structure and the effect of including more specific details like heterogeneities and stochasticity. Moreover, analyzing the structure and the effect of including more authors in [11] simulate a possible outbreak in Italy by basing the SEIR model: dynamics when the number of the infected population is low.

We have a weak formulation of (4). Based on this analysis, an auxiliary problem is constructed in Section 3, from which we show that its solution exists and is unique. In Section 4, we prove the existence of a solution to the weak formulation of (4), using the analysis of Section 3. We conclude the paper with Section 5, wherein we briefly discuss future considerations.

2. A Weak Formulation

We begin by multiplying each equation in (4) by $\sin \phi$, resulting in

$$
u_i \sin \phi = \frac{\partial}{\partial \phi} \left( \sin \phi \frac{\partial u}{\partial \phi} \right) + \csc \phi \frac{\partial^2 u}{\partial \theta^2} - \alpha \left( \int_{\Omega} w (\phi, \theta, t) \sin \phi \, d\phi \, d\theta \right) u \sin \phi$$

$$+ \gamma \nu \sin \phi, \quad \text{on } \mathcal{S} \times (0, T], \label{eq:6}$$

$$
u_i \sin \phi = \frac{\partial}{\partial \phi} \left( \sin \phi \frac{\partial v}{\partial \phi} \right) + \csc \phi \frac{\partial^2 v}{\partial \theta^2} - \beta \nu \sin \phi, \quad \text{on } \mathcal{S} \times (0, T], \label{eq:7}$$

$$w_i = \mathbf{v} \left( \phi, \theta, t \right) - \gamma \mathbf{w} \left( \phi, \theta, t \right), \quad \text{on } \mathcal{S} \times (0, T], \label{eq:8}$$

$$u (\phi, \theta, 0) = f (\phi, \theta) \quad \text{on } \mathcal{S}, \quad \nu (\phi, \theta, 0) = g (\phi, \theta) \quad \text{on } \mathcal{S}, \label{eq:9}$$

Moreover, $u$, $v$, and $w$, as well as $\partial u / \partial \theta$, $\partial v / \partial \theta$, and $\partial w / \partial \theta$, are all continuous across the data line, and all of the derivatives with respect to $\phi$ of $u$, $v$, and $w$ exist for every direction $\theta$ when $\phi = 0, \pi$.

In considering a weak solution to (7)–(9), we need to establish appropriate test functions. Consider $\mathbf{g}(\phi, \theta, t) \in C^1$ in $\phi, \theta,$ and $t$ and let $g$ and $g_0$ be continuous across the data line. Now multiply (7) by $g(\phi, \theta, t)$ and let $g(\phi, \theta, T) = 0$; then

$$
\int_{\pi}^{\pi} \int_{-\pi}^{\pi} \left\{ \int_{0}^{T} \int_{0}^{0} g u_i \sin \phi \, d\phi \, d\theta \right\} \, dt \\
\quad = - \int_{\pi}^{\pi} \int_{-\pi}^{\pi} \left\{ \int_{0}^{T} \int_{0}^{0} g \left( \phi, \theta, 0 \right) f \left( \phi, \theta \right) \sin \phi \, d\phi \, d\theta \right\} \, dt \\
\quad - \int_{\pi}^{\pi} \int_{-\pi}^{\pi} \left\{ \int_{0}^{T} \int_{0}^{0} u g_i \sin \phi \, d\phi \, d\theta \right\} \, dt. \label{eq:10}
$$

The next term in (7) yields

$$
\int_{0}^{T} \int_{-\pi}^{\pi} \left\{ \int_{0}^{\phi} \frac{\partial}{\partial \phi} \left( \sin \phi u_{\phi} \right) \, d\phi \right\} \, d\theta \, dt \\
\quad = \int_{0}^{T} \int_{-\pi}^{\pi} \left\{ \int_{0}^{\phi} g u_{\phi} \sin \phi \, d\phi \right\} \, d\theta \, dt - \int_{0}^{T} \int_{-\pi}^{\pi} \left\{ \int_{0}^{\phi} g u_{\phi} \sin \phi \, d\phi \right\} \, d\theta \, dt. \label{eq:11}
$$

Our current model is quite different than the aforementioned ones, since it comprises two nonlocal integrodifferential equations, also often referred to as functional differential equations, and an ordinary differential equation. Moreover, in order to establish the existence of the weak solution to the weak formulation of (4), an interesting Hilbert space is constructed, as established in Section 3. In addition, several facets of the analysis needed in the study of Laplace’s equation on the unit sphere are used throughout. We also emphasize that, to the very best of our knowledge, there are currently no published results related to nonlocal global pandemic models.

This paper is organized as follows. In Section 2, we derive a weak formulation of (4). Based on this analysis, an auxiliary problem is constructed in Section 3, from which we show that its solution exists and is unique. In Section 4, we prove the existence of a solution to the weak formulation of model (4), using the analysis of Section 3. We conclude the paper with Section 5, wherein we briefly discuss future considerations.
We also have
\[ \int_0^T \int_{-\pi}^\pi \csc \phi g u_{\theta \phi} d\theta d\phi \, dt = \int_0^T \int_{-\pi}^\pi \csc \phi g u_{\theta \phi}^{\theta=\pi} d\theta d\phi \, dt \]
\[ - \int_0^T \int_{-\pi}^\pi \csc \phi g u_{\theta \phi} d\theta d\phi \, dt. \quad (13) \]

Rearranging terms we obtain
\[ - \int_0^T \int_{-\pi}^\pi u g_t \sin \phi d\theta \, dt \]
\[ - \int_0^T \int_{-\pi}^\pi g u_{\theta \phi} \sin \phi d\theta \, dt \]
\[ + \int_0^T \int_{-\pi}^\pi \csc \phi g u_{\theta \phi} d\theta d\phi \, dt, \]
\[ = \int_{-\pi}^\pi \int_0^\pi g(\phi, \theta, 0) f(\phi, \theta) \sin \phi d\phi d\theta \]
\[ - \int_0^T \int_{-\pi}^\pi \alpha \left( \int_\Omega w \sin \phi d\phi d\theta \right) g u \sin \phi d\phi d\theta \, dt \]
\[ + \gamma \int_0^T \int_{-\pi}^\pi g_w \sin \phi d\phi d\theta, \quad (14) \]

for all \( g \in C^1(\Omega_T) \) and \( \Omega_T = \{(\phi, \theta, t) \mid 0 \leq \phi \leq \pi, -\pi \leq \theta \leq \pi, 0 < t < T\}. \)

Now let \( D_t := \{(\phi, \theta, t) \mid 0 \leq \phi \leq \pi, -\pi \leq \theta \leq \pi, 0 < t < T\} \).

For the weak form of (8) we have
\[ - \int_{\Omega_T} v g_t \sin \phi d\Omega_T + \int_{\Omega_T} v g \sin \phi d\Omega_T \]
\[ + \int_{\Omega_T} v g \sin \phi d\Omega_T \quad (18) \]

On the other hand, the weak form of (9) is the pointwise integral representation for almost all \((\phi, \theta, t) \in \Omega_T\), which is
\[ w = \beta \int_0^t e^{-\gamma(t-t')} \, dt'. \quad (19) \]

The integral representation (19) of \( w \) is a nonlinear functional equation for \( w \) as \( \int_\Omega w \, d\Omega \) occurs in the function \( \alpha \) in both (16) and (18) for \( u \) and \( v \).

3. The Auxiliary Problem

We replace \( \int_\Omega w \, d\Omega \) in the function \( \alpha \) in (16) and (18) with the continuous function \( q(t) \) to yield the auxiliary problem of finding \( u(\phi, \theta, t; q), v(\phi, \theta, t; q), \) and \( w(\phi, \theta, t; q) \) satisfying
\[ - \int_{\Omega_T} u g_t \sin \phi d\Omega_T + \int_{\Omega_T} u g \sin \phi d\Omega_T \]
\[ + \int_{\Omega_T} u g \sin \phi d\Omega_T \]
\[ = \int_{\Omega_T} q \sin \phi d\Omega_T + \int_{\Omega_T} \alpha(q(t)) u \sin \phi d\Omega_T \]
\[ + \gamma \int_{\Omega_T} w \sin \phi d\Omega_T, \quad \forall g \in W^{1,1}. \]
for almost all $(\phi, \theta, t) \in Q_T$. It should also be noted that we have already assumed that $g$ is periodic in $\theta$. Also, we can extend $g$ as an odd function in $\phi$ with $\phi < \pi \leq \phi \leq 0$ and regard it as periodic in $\phi$. Therefore, we can utilize the functions $\sin n\phi \sin m\theta$ and $\sin n\phi \cos m\theta$ in order to generate an indexed set of functions $h_k(\phi, \theta)$ by first counting the functions $\sin n\phi \cos m\theta$ according to an orderly method of counting the ordered pairs $(n, m)$ for $n = 1, 2, \ldots$ and $m = 0, 1, 2, \ldots$ in the closure of the $x$-$y$ plane via the diagonal pattern

\begin{align*}
(1, 0) \rightarrow (1, 1) & \rightarrow (2, 0) \rightarrow (3, 0) \rightarrow (2, 1) \rightarrow (1, 2) \\
& \rightarrow (1, 3) \rightarrow (2, 2) \rightarrow (3, 1) \rightarrow (4, 0) \rightarrow (5, 0) \\
& \rightarrow (4, 1) \rightarrow (3, 2) \rightarrow (2, 3) \rightarrow (1, 4) \rightarrow (1, 5) \\
& \rightarrow (2, 4) \cdots .
\end{align*}

(21)

Next, take the number index assigned to each pair $(n, m)$ and double it, yielding the $h_k$-terms for even integers. Then, for the odd integers, as we did above, count the functions $\sin n\phi \cos m\theta$ according to an orderly method of counting the ordered pairs $(n, m)$ for $n = 1, 2, \ldots$ and $m = 1, 2, \ldots$ in the closure of the $x$-$y$ plane via the diagonal pattern

\begin{align*}
(1, 1) \rightarrow (1, 2) & \rightarrow (2, 1) \rightarrow (3, 1) \rightarrow (2, 2) \\
& \rightarrow (1, 3) \rightarrow (1, 4) \rightarrow (2, 3) \rightarrow (3, 2) \\
& \rightarrow (4, 1) \rightarrow (5, 1) \rightarrow (4, 2) \rightarrow (3, 3) \\
& \rightarrow (2, 4) \rightarrow (1, 5) \rightarrow (1, 6) \rightarrow (2, 5) \cdots .
\end{align*}

(22)

So, we have a list of $h_k(\phi, \theta)$ for $k = 1, 2, \ldots$. The first six of such are as follows:

\begin{align*}
h_1 &= \sin(\phi) \sin(\theta), & h_2 &= \sin(\phi), \\
h_3 &= \sin(\phi) \sin(2\theta), & h_4 &= \sin(\phi) \cos(\theta), \\
h_5 &= \sin(2\phi) \sin(\theta), & h_6 &= \sin(2\phi), \ldots 
\end{align*}

(23)

Since the integrals involving the $h_k^2$-term, given by

\begin{align*}
\int_{-\pi}^{\pi} \int_{0}^{\pi} \sin^2 n\phi \cos^2 m\theta \sin\phi \, d\phi \, d\theta, \\
\int_{-\pi}^{\pi} \int_{0}^{\pi} \sin^2 n\phi \sin^2 m\theta \sin\phi \, d\phi \, d\theta,
\end{align*}

(24)

converge, and the integrals involving the $\partial h_k^2/\partial \phi$-term, which are

\begin{align*}
\int_{-\pi}^{\pi} \int_{0}^{\pi} n^2 \cos^2 n\phi \cos^2 m\theta \sin\phi \, d\phi \, d\theta, \\
\int_{-\pi}^{\pi} \int_{0}^{\pi} n^2 \cos^2 n\phi \sin^2 m\theta \sin\phi \, d\phi \, d\theta,
\end{align*}

(25)

also converge, we then consider the integrals involving the $\partial h_k^2/\partial \theta$-term below:

\begin{align*}
\int_{-\pi}^{\pi} \int_{0}^{\pi} m^2 \csc^2 \sin^2 n\phi \sin^2 m\theta \sin\phi \, d\phi \, d\theta, \\
\int_{-\pi}^{\pi} \int_{0}^{\pi} m^2 \csc^2 \sin^2 n\phi \cos^2 m\theta \sin\phi \, d\phi \, d\theta
\end{align*}

(26)

and remark that

\begin{align*}
\int_{0}^{\pi} m^2 \csc^2 \sin^2 n\phi \sin\phi < \frac{n^2 \pi^4}{8}.
\end{align*}

(27)

Hence,

\begin{align*}
\int_{-\pi}^{\pi} \int_{0}^{\pi} h_k^2 \sin\phi \, d\phi \, d\theta + \int_{-\pi}^{\pi} \int_{0}^{\pi} \left( \frac{\partial h_k}{\partial \phi} \right)^2 \sin\phi \, d\phi \, d\theta \\
+ \int_{-\pi}^{\pi} \int_{0}^{\pi} \left( \frac{\partial h_k}{\partial \theta} \right)^2 \csc^2 \sin\phi \sin\phi \, d\phi \, d\theta < \infty,
\end{align*}

(28)

for $k = 1, 2, \ldots$. Thus, via the Gram-Schmidt process, we can generate a countable orthogonal basis for the Hilbert space $\mathcal{T}^2(D_0)$ with the inner product

\begin{align*}
\langle f(\phi, \theta), g(\phi, \theta) \rangle_{D_0} &= \int_{D_0} fg \sin\phi \, dD_0 \\
&\quad + \int_{D_0} \frac{\partial f}{\partial \phi} \frac{\partial g}{\partial \phi} \sin\phi \, dD_0 \\
&\quad + \int_{D_0} \frac{\partial f}{\partial \theta} \frac{\partial g}{\partial \theta} \csc^2 \sin\phi \, dD_0,
\end{align*}

(29)

where $D_0 = \{(\phi, \theta) \mid 0 \leq \phi \leq \pi, -\pi \leq \theta \leq \pi\}$. Henceforth, we denote the orthonormal basis formed from $h_k(\phi, \theta)$ via the Gram-Schmidt process by $\{h_k(\phi, \theta)\}$.

Now, define

\begin{align*}
u_N(\phi, \theta, t) &= \sum_{k=1}^{N} \alpha_k h_k(\phi, \theta), \\
\nu_N(\phi, \theta, t) &= \sum_{k=1}^{N} \alpha_k^N h_k(\phi, \theta),
\end{align*}

(30)

\begin{align*}
\psi_N(\phi, \theta, t) &= \sum_{k=1}^{N} \beta_k h_k(\phi, \theta).
\end{align*}
From there, we construct a system of ODEs as follows:

\[
\int_{D_0} u_i h_i \sin \phi \, dD_0 + \int_{D_0} \frac{\partial u_N}{\partial \phi} \frac{\partial h_i}{\partial \phi} \sin \phi \, dD_0
\]

\[+ \int_{D_0} \frac{\partial u_N}{\partial \theta} \frac{\partial h_i}{\partial \theta} \csc \phi \, dD_0\]

\[= - \int_{D_0} \alpha (q(t)) u_N h_i \sin \phi \, dD_0 + \gamma \int_{D_0} w_N h_i \sin \phi \, dD_0,\]

\[
\int_{D_0} v_i h_i \sin \phi \, dD_0 + \int_{D_0} \frac{\partial v_N}{\partial \phi} \frac{\partial h_i}{\partial \phi} \sin \phi \, dD_0
\]

\[+ \int_{D_0} \frac{\partial v_N}{\partial \theta} \frac{\partial h_i}{\partial \theta} \csc \phi \, dD_0\]

\[= \int_{D_0} \alpha (q(t)) u_N h_i \sin \phi \, dD_0 - \beta \int_{D_0} v_N h_i \sin \phi \, dD_0,\]

\[
\int_{D_0} w_i h_i \sin \phi \, dD_0 = \beta \int_{D_0} v_N h_i \sin \phi \, dD_0
\]

\[\quad - \gamma \int_{D_0} w_N h_i \sin \phi \, dD_0,\]

\[a_i^N (0) = \langle f, h_i \rangle_{D_0}, \quad b_i^N (0) = c_i^N (0) = 0, \quad i = 1, 2, \ldots, N.\]  

The equations above become

\[
\frac{d a_i^N (t)}{dt} + \sum_{l=1}^N A_{il}^N a_l^N (t) = -\alpha (q(t)) a_i^N (t) + \gamma c_i^N (t),
\]

\[
\frac{d b_i^N (t)}{dt} + \sum_{l=1}^N B_{il}^N b_l^N (t) = \alpha (q(t)) a_i^N (t) - \beta b_i^N (t),
\]

\[
\frac{d c_i^N (t)}{dt} = \beta b_i^N (t) - \gamma c_i^N (t),\]

\[a_i^N (0) = \langle f, h_i \rangle_{D_0}, \quad b_i^N (0) = c_i^N (0) = 0, \quad i = 1, 2, \ldots, N.\]

As the above equations are linear, for each positive integer \(N\) there exists a solution on \([0, T]\) since \(\alpha(q(t))\) is bounded for \(q(t) \geq 0\). Hence, for each \(N, u_N, v_N,\) and \(w_N\) are determined uniquely. We note that

\[c_i^N (t) = \beta \int_0^t b_i^N (\tau) e^{\gamma(t-\tau)} \, d\tau.\]  

Upon multiplying each of the equations in the system (32) by the coefficient corresponding to the \(t\)-derivative in the equation and summing each equation over \(i = 1, \ldots, N\), we achieve

\[
\left\langle \frac{du_N}{dt}, u_N \right\rangle_{D_0} + \int_{Q_1} \left( \frac{\partial u_N}{\partial \phi} \right)^2 \sin \phi \, dQ_t
\]

\[+ \int_{Q_1} \left( \frac{\partial u_N}{\partial \theta} \right)^2 \csc \phi \, dQ_t = - \int_{D_0} \alpha (q(t)) u_N^2 \sin \phi \, dD_0,
\]

\[
+ \gamma \int_{D_0} w_N u_N \sin \phi \, dD_0,
\]

\[
\left\langle \frac{dv_N}{dt}, v_N \right\rangle_{D_0} + \int_{Q_1} \left( \frac{\partial v_N}{\partial \phi} \right)^2 \sin \phi \, dQ_t
\]

\[+ \int_{Q_1} \left( \frac{\partial v_N}{\partial \theta} \right)^2 \csc \phi \, dQ_t = \int_{D_0} \alpha (q(t)) u_N v_N \sin \phi \, dD_0,
\]

\[
+ \beta \int_{D_0} v_N^2 \sin \phi \, dD_0,
\]

\[
\left\langle \frac{dw_N}{dt}, w_N \right\rangle_{D_0} = \beta \int_{D_0} v_N w_N \sin \phi \, dD_0
\]

\[- \gamma \int_{D_0} w_N^2 \sin \phi \, dD_0\]

and integrating each of the above equations from 0 to \(t\) we obtain

\[
\frac{1}{2} \left\| u_N (\phi, \theta, t) \right\|_{D_0}^2 + \int_{Q_1} \left( \frac{\partial u_N}{\partial \phi} \right)^2 \sin \phi \, dQ_t
\]

\[+ \int_{Q_1} \left( \frac{\partial u_N}{\partial \theta} \right)^2 \csc \phi \, dQ_t = \frac{1}{2} \left\| u_N (\phi, \theta, 0) \right\|_{D_0}^2 - \int_{Q_1} \alpha (q(t)) u_N^2 \sin \phi \, dQ_t,
\]

\[+ \gamma \int_{Q_1} w_N u_N \sin \phi \, dQ_t,
\]

\[
\frac{1}{2} \left\| v_N (\phi, \theta, t) \right\|_{D_0}^2 + \int_{Q_1} \left( \frac{\partial v_N}{\partial \phi} \right)^2 \sin \phi \, dQ_t
\]

\[+ \int_{Q_1} \left( \frac{\partial v_N}{\partial \theta} \right)^2 \csc \phi \, dQ_t = \int_{D_0} \alpha (q(t)) u_N v_N \sin \phi \, dD_0 - \beta \int_{Q_1} v_N^2 \sin \phi \, dQ_t,
\]

\[
+ \gamma \int_{Q_1} v_N w_N \sin \phi \, dQ_t,
\]

\[
\frac{1}{2} \left\| w_N (\phi, \theta, t) \right\|_{D_0}^2 = \beta \int_{Q_1} v_N w_N \sin \phi \, dQ_t
\]

\[- \gamma \int_{Q_1} w_N^2 \sin \phi \, dQ_t.
\]
Multiplying the above equations by 2, utilizing the inequality $2ab \leq a^2 + b^2$ on the right-hand sides of each equation, adding the inequalities, and dropping the $Q_i$ integrals on the left-hand side of the resulting inequalities, we have

$$F(t) \leq \|f\|^2_{D_b} + C_1 \int_0^t F(r) \, dr,$$

where

$$F(t) := \|u_N(\phi, \theta, t)\|_{D_b}^2 + \|v_N(\phi, \theta, t)\|_{D_b}^2$$

$$+ \|w_N(\phi, \theta, t)\|_{D_b}^2,$$

$$C_1 = 3\|\alpha\|_{\infty} + 3\beta + 2\gamma + 1$$

and $\|u_N(\phi, \theta, 0)\|_{D_b}^2 \leq \|f\|^2_{D_b}$.

Then, by Gronwall's lemma we see that

$$y'(t) \leq \|f\|^2_{D_b} + C_1 y$$

$$\frac{d}{dt} \left(e^{-C_1 t} y(t)\right) \leq \|f\|^2_{D_b} e^{-C_1 t}$$

$$y(t) \leq \frac{\|f\|^2_{D_b}}{C_1} \left[e^{C_1 t} - 1\right].$$

Therefore,

$$F(t) \leq \|f\|^2_{D_b} e^{C_1 t}.$$

In addition, we have

$$F(t) + 2 \int_{Q_r} \left(\frac{\partial u_N}{\partial \phi}\right)^2 \sin \phi \, dQ_r + 2 \int_{Q_r} \left(\frac{\partial u_N}{\partial \theta}\right)^2 \csc \phi \, dQ_t$$

$$+ 2 \int_{Q_r} \left(\frac{\partial v_N}{\partial \phi}\right)^2 \sin \phi \, dQ_r + 2 \int_{Q_r} \left(\frac{\partial v_N}{\partial \theta}\right)^2 \csc \phi \, dQ_t$$

$$\leq \|f\|^2_{D_b} + C_1 \int_0^t F(r) \, dr$$

and we observe that the entire left-hand side of the above inequality is also bounded by $\|f\|^2_{D_b} e^{C_1 t}$ for each $t \in [0, T]$ and so is each individual term on the left-hand side.

From there, we see that

$$\int_{Q_r} u_N^2 \sin \phi \, dQ_r, \quad \int_{Q_r} v_N^2 \sin \phi \, dQ_r,$$

$$\int_{Q_r} w_N^2 \sin \phi \, dQ_r,$$

are all bounded uniformly with respect to $N$. Likewise, $u_N$ and $v_N$ are also uniformly bounded in the Hilbert space $\mathcal{V}$ with inner product

$$\langle h, g \rangle_{\mathcal{V}} = \int_{Q_r} h \phi \sin \phi \, dQ_r + \int_{Q_r} h \phi g_\phi \sin \phi \, dQ_r$$

$$+ \int_{Q_r} h \phi g_\theta \csc \phi \, dQ_r,$$

where $h_\phi$, $g_\phi$, $h_\theta$, and $g_\theta$ are the first partial derivatives of $h$ and $g$, with respect to the subscripted variables.

Next, we take (33) and multiply it by $h_i(\phi, \theta)$ and sum with $i$ from 1 to $N$ and obtain

$$w_N(\phi, \theta, t) = \beta \int_0^t v_N(\phi, \theta, r) e^{-\gamma t - r} \, dr,$$

$$\forall (\phi, \theta, t) \in Q_r.$$

Thus, by a Cantor diagonalization process, there exist sequences $\{u_N\}$, $\{v_N\}$, and $\{w_N\}$, which converge weakly in $L^2(Q_r)$ and $\{u_N\}$ and $\{v_N\}$ also converging weakly in $\mathcal{V}$. Let $u$, $v$, and $w$ denote the weak limits and let

$$g_m(\phi, \theta, t) = \sum_{i=1}^m e_i(t) h_i(\phi, \theta)$$

denote an approximation in $W^{1,1}_2(Q_r)$ to $g(\phi, \theta, t) \in C_1(Q_r)$ with $e_i(t) = 0$ for all $i = 1, 2, \ldots$; that is, $g_m \rightarrow g$ strongly in $W^{1,1}_2(Q_r)$ and $g_m(\phi, \theta, T) = g(\phi, \theta, T) = 0$. Taking the system of ODEs and multiplying each by $e_i(t)$ and summing each block of equations with respect to $i$ transform the system as follows:

$$\left\langle \frac{\partial u_N}{\partial t}, g_m \right\rangle_{D_b} + \int_{D_b} \frac{\partial u_N}{\partial \phi} \frac{\partial u_N}{\partial \phi} g_m \sin \phi \, dD_0$$

$$+ \int_{D_b} \frac{\partial u_N}{\partial \theta} \frac{\partial u_N}{\partial \theta} g_m \csc \phi \, dD_0$$

$$= \int_{D_b} a(q(t)) u_N g_m \sin \phi \, dD_0$$

$$+ \gamma \int_{D_b} u_N g_m \sin \phi \, dD_0,$$

$$\left\langle \frac{\partial v_N}{\partial t}, g_m \right\rangle_{D_b} + \int_{D_b} \frac{\partial v_N}{\partial \phi} \frac{\partial v_N}{\partial \phi} g_m \sin \phi \, dD_0$$

$$+ \int_{D_b} \frac{\partial v_N}{\partial \theta} \frac{\partial v_N}{\partial \theta} g_m \csc \phi \, dD_0$$

$$= \int_{D_b} a(q(t)) v_N g_m \sin \phi \, dD_0$$

$$+ \gamma \int_{D_b} v_N g_m \sin \phi \, dD_0,$$

$$\int_{Q_r} \left(\omega_N - \beta \int_0^t v_N e^{-\gamma t - r} \, dr\right) g_m \, dQ_r = 0.$$
Upon considering the energy inequality for parabolic equations (cf. [18]), it can be shown that any weak solution for \( u, v, \) and \( w \) satisfies an energy inequality similar to the one for \( \{u_N, v_N, w_N\} \). Hence, \( u, v, \) and \( w \) satisfy the same bounds as \( u_N, v_N, \) and \( w_N \). Consequently, by the linearity of the system \( f(\phi, \theta) \equiv 0 \), we have \( u = v = w = 0 \) and thus, the weak solution is unique.

### 4. Existence of a Solution to the Auxiliary Problem

As the auxiliary problem possesses a unique weak solution for each continuous \( q(t) \), we can now define a mapping \( \mathcal{F} : C([0, T]) \to C([0, T]) \) via

\[
(\mathcal{F}q)(t) = \int_\Omega w(\phi, \theta; t; q) \, d\Omega.
\]

First, we show that \((\mathcal{F}q)(t)\) is bounded and continuous on \([0, T]\). From integrating the pointwise representation for \( w(\phi, \theta; t; q) \), we see that

\[
\int_\Omega w \, d\Omega = \beta \int_0^T \int_\Omega e^{-\gamma(t_1-t)} \, d\Omega
d\Omega
\]

and, taking into account the fact that \( 0 < e^{-\gamma(t_1-t)} \leq 1 \), we have

\[
\int_\Omega w \, d\Omega \leq \beta \int_0^T \int_\Omega [v] \, d\Omega \, d\tau
\]

\[
\leq \beta \left( \int_0^T \int_\Omega \sin \phi d\theta d\phi d\tau \right)^{1/2}
\]

\[
\times \left( \int_0^T \int_\Omega v^2 \sin \phi d\theta d\phi d\tau \right)^{1/2}
\]

\[
\leq \beta T^{1/2} \left( \text{mes} (\Omega) \right)^{1/2} \left( \int_{Q_T} v^2 \sin \phi dQ_T \right)^{1/2}
\]

\[
\leq C_2,
\]

via the Schwartz inequality, where \( C_2 \) is a positive constant that depends on \( C_1 \) of Section 3, \( T \), and \( \text{mes}(\Omega) \).

For the continuity of \((\mathcal{F}q)(t)\), we consider \( 0 \leq t_1 < t_2 \leq T \), and thus

\[
\int_\Omega w(\phi, \theta, t_1; q) \, d\Omega - \int_\Omega w(\phi, \theta, t_2; q) \, d\Omega
\]

\[
\leq \beta \int_0^T \left[ \int_0^{t_1} v(\phi, \theta, \tau; q) e^{-\gamma(t_1-\tau)} \, d\tau \right] \, d\Omega
\]

\[
- \beta \int_0^T \left[ \int_0^{t_2} v(\phi, \theta, \tau; q) e^{-\gamma(t_2-\tau)} \, d\tau \right] \, d\Omega
\]

\[
\leq \beta \int_\Omega \left[ \int_0^{t_2} v(\phi, \theta, \tau; q) \, d\tau \right] \, d\Omega
\]

where we used the fact that \( 0 < e^{-\gamma(t_1-t)} \leq 1 \), estimated \( 1 - e^{-\gamma(t_1-t)} \) via the mean value theorem, and estimated the last two integrals exactly as the integral in (48). We note that \( C_3 \) and \( C_4 \) have the same dependence as \( C_2 \), which is independent of \( q \) since \( C_1 \) is independent of \( q \).

It therefore follows from (48) and (49) that \( \mathcal{F} \) maps \( C([0, T]) \) into a convex subset of itself. All that remains for an application of Schauder's fixed point theorem is to show that \( \mathcal{F} \) is continuous in the norm \( \| \cdot \|_\infty \) of the separable Banach space \( \mathcal{B}(C([0, T])); \| \cdot \|_\infty \).

To show that \((\mathcal{F}q)(t)\) is continuous in \( \| \cdot \|_\infty \), we consider

\[
(\mathcal{F}q_1)(t) - (\mathcal{F}q_2)(t)
\]

\[
= \beta \int_\Omega \int_0^T [v(\phi, \theta, \tau; q_1) - v(\phi, \theta, \tau; q_2)] e^{-\gamma(t_2-t_1)} \, d\tau \, d\Omega,
\]

where \( v(\phi, \theta, \tau; q) \) is the weak solution of (20) for continuous \( q(t), 0 \leq t \leq T \), for \( i = 1, 2 \). Forming the difference of (20) for \( q_i(t), i = 1, 2 \), we have

\[
- \int_{Q_T} (u_1 - u_2) g_\phi \sin \phi \, dQ_T + \int_{Q_T} (u_1 \phi - u_2 \phi) \sin \phi \, dQ_T
\]

\[
+ \int_{Q_T} (u_1 \phi - u_2 \phi) g_\phi \csc \phi \, dQ_T
\]

\[
= - \int_{Q_T} [\alpha(q_1) u_1 - \alpha(q_2) u_2] g \sin \phi \, dQ_T
\]

\[
+ \gamma \int_{Q_T} (w_1 - w_2) g \sin \phi \, dQ_T, \quad \forall g \in W^{1,1}
\]

(51)
\[-\int_{Q_T} (v_1 - v_2) g_1 \sin \phi \, dQ_T + \int_{Q_T} (v_1 - v_2) g_2 \sin \phi \, dQ_T \]
\[+ \int_{Q_T} (v_1 - v_2) g_0 \csc \phi \, dQ_T \]
\[= \int_{Q_T} [\alpha(q_1) u_1 - \alpha(q_2) u_2] \sin \phi \, dQ_T \]
\[+ \beta \int_{Q_T} (v_1 - v_2) g \sin \phi \, dQ_T, \quad \forall g \in W^{1,1}, \]
\[\int_{Q_T} (v_1 - v_2) \sin \phi \, dQ_T = 0 \]
\[\int_{Q_T} \sin \phi \, dQ_T = \beta \int_{Q_T} (v_1 - v_2) g \sin \phi \, dQ_T \]
\[(52)\]
\[
\int_{Q_T} \sin \phi \, dQ_T = \beta \int_{Q_T} (v_1 - v_2) g \sin \phi \, dQ_T, \quad \forall g \in W^{1,1},
\]
\[\int_{Q_T} (\sin \phi \, dQ_T)^2 \leq \|\alpha\|_{\infty}^2 \int_{Q_T} u_1^2 \sin \phi \, dQ_T \]
\[+ \frac{1}{2} \int_{Q_T} g^2 \sin \phi \, dQ_T \]
\[\leq C_{\beta} \|q_1 - q_2\|_{\infty} + \frac{1}{2} \int_{Q_T} g^2 \sin \phi \, dQ_T, \]
\[(56)\]

where \(C_{\beta}\) depends on the bound for \(\|u_1\|_{Q_T}\) which is given by (39) for the weak solution of (20) and is independent of \(q_1\). The second term on the right-hand side of (54) and (56) involving the \(g\) becomes \(u_1 - u_2\) in the energy inequality associated with (52). We now have a replay of (37), (38), and (39), with \(\|f\|_{T_0}^2\) replaced with \(C_{\beta}\|q_1 - q_2\|_{\infty}\). From the estimate of \(\|v_1 - v_2\|_{Q_T}\) resulting from the Galerkin estimate for \(\|u_1 - u_2\|_{Q_T}\), \(\|v_1 - v_2\|_{Q_T}^2\), \(\|w_1 - w_2\|_{Q_T}\), in terms of \(C_{\beta}\|q_1 - q_2\|_{\infty}\), we obtain the estimation of (50) below:
\[\|\bar{\mathcal{F}}(q_1) - \bar{\mathcal{F}}(q_2)\|_{\infty} \leq C_{\beta} \|q_1 - q_2\|_{\infty}, \]
\[(57)\]

This leads to the following statement.

**Theorem 1.** From (57), one infers the continuity of the mapping \(\bar{\mathcal{F}}\) and the validity of the application of the Schauder fixed point theorem, yielding the existence of a weak solution of the weak formulation of (7), (8), and (9), given by (16), (18), and (19).

**Proof.** See the preceding analysis. \(\square\)

## 5. Future Considerations

We leave open for consideration the establishment of a unique solution to the weak formulation developed in this paper. This will entail a numerous amount of estimates, which are too tedious to be included in the present work. We refer the interested reader to Chapter 3, Section 2, of [18], as we anticipate that analogues of the energy inequalities derived therein can be generalized to ones that apply to our current problem. We also anticipate that these inequalities will ultimately produce a contraction mapping via a bootstrapping procedure. The constants in these inequalities will have to be estimated closely enough to show the dependence on (small) \(t\), in order to permit the construction of the aforementioned contraction mapping.

**Conflict of Interests**

The authors declare that there is no conflict of interests regarding the publication of this paper.

**Acknowledgments**

The authors thank the referee for the timely insights, suggestions, and inquiries. In addition, they appreciate the assistance of Frank B. Jones (Rice University) for sharing with them his unpublished lecture notes pertaining to the heat equation on the unit sphere.
References


Submit your manuscripts at http://www.hindawi.com