Problem 5.4

I wrote three classes extending \textit{Planet} to perform the calculations assigned in problem 5.4. Each extension replaced only the \texttt{getRate} function of the superclass; a new input, \textit{“part”} was added to allow the user to select which part of the code to implement. When \textit{part} is 1, 2, or 3, the code implements uses the appropriate form for the gravitational force for part a, b, or c and d of the problem respectively. Any other value for \textit{part} implements a planet following a standard inverse square law.

Part a.:

Assuming the gravitational force is given by:

$$\vec{F} = -\frac{Cm}{r^{2+\delta}} \vec{r} = -\frac{Cm}{r^{3+\delta}} \vec{r}$$

This can be broken up into the x- and y-directions as:

$$F_x = -\frac{Cm}{r^{3+\delta}} x$$

$$F_y = -\frac{Cm}{r^{3+\delta}} y$$

These were then used to find the acceleration in the x- and y-directions as:

$$a_x = -\frac{C}{r^{3+\delta}} x$$

$$a_y = -\frac{C}{r^{3+\delta}} y$$

The new class \textit{PlanetDelta} was used to find the motion of a planet in the given gravitational field. The value of $\delta$ was added to the list of parameters the user can input.
to allow the code to be run for different values without recompiling the program. First, I tried using the Euler-Richardson (ER) algorithm to find the path of the planet. For all values of $\delta$ and $\Delta t$ that I tried, the orbit of the planet was a series of ellipses. The only time the orbits retraced themselves was for the initial conditions were altered to: $x(t = 0) = 1\text{AU}$, $y(t = 0) = 0\text{AU}$, $v_x(t = 0) = 0\text{AU/yr}$, and $v_y(t = 0) = 6.28\text{AU/yr}$. For these conditions, circular motion occurred; the analytical value for the necessary initial velocity for $v_y$ is the same as that found in part c below. This is because both force laws depend on a power of $r$ meaning that for a radius of one. Several combinations of $\delta$, $\Delta t$, and $r$ were tried to compare the results of the program. The orbits were always counterclockwise, i.e. $v_y(t = 0)$ was always positive. For larger values of $\Delta t$ and $\delta$, the orbits precessed clockwise. However, for smaller values of $\Delta t$ and $\delta$, the orbits precessed counterclockwise. I believe the reason for this is that when going at longer $\Delta t$, numerical errors will crop up more because more path is traveled between calculations. So in the beginning of the simulation, the planet may be “pulled” in more by the sun and any numerical errors than with a small $\Delta t$ for the same value of $\delta$ causing the first leg of the orbit to be shortened and the orbit to precess clockwise. However, larger disturbances to the inverse square law, larger $\delta$, also resulted in a clockwise precession. For larger $\delta$, at a given value of $r$, the force felt by the particle would be smaller, which would have implied to me that in the beginning of the orbit, the weaker force would result in the planet swinging outwards and preceeding counterclockwise, but my simulations do not agree with that. At this point, I cannot determine why a change to increase $\delta$ produces the change in the orbit it seems to be doing. Regardless of the numerical reasons behind it though, a change in the direction of precession from a change in $\Delta t$ must be incorrect.
So, I also tried using the Verlet (V) and Runga-Kutta (RK4) algorithms. The Verlet algorithm also resulted in different values of $\Delta t$ causing the orbit to precess in different directions. On the other hand, the Runga-Kutta algorithm seemed to be well-behaved in that regard. It did, however, result in the orbit spiraling inwards when smaller initial radii were tried which should not have occurred. I am unsure at this time which algorithm to try to improve the simulation. Both algorithms produced elliptical orbits which did not seem to retrace themselves for non-zero values of $\delta$.

With the Euler-Richardson algorithm for the smallest value of $\delta$ I used, the orbits stayed the same size even though they did not retrace themselves once they returned to the original orientation. For the larger values of $\delta$, the orbits quickly became quite large. However, using $\delta = 0.05$, the planet stayed confined to the system for at least 1,750 years of the solar system’s simulated lifetime though the last completed orbit was an order of magnitude larger than its predecessor and reached out more than 200 A.U. from the star. These variations make sense since for smaller values of $\delta$, the planet should behave closer to an inverse square law, whereas larger variations of $\delta$ would result in the force significantly weakening allowing the particle to move farther afield. When I tried the simulation for larger values of $r(t = 0)$, the orbits were precessing ellipses which did not retrace themselves but did not balloon in size either. However, I was unable to get well-behaved orbits with smaller radii. The aphelions for initial radii larger than one were opposite the starting location for the planet. I am unsure what caused this effect.

Various numbers of orbits for different values of $\delta$, $\Delta t$, and $r(t = 0)$ are shown below; the blue squares represent the aphelion of a particular orbit:
Figure 1: ER; \( r(t = 0) = 1 \text{ A.U.}; v_y(t = 0) = 5 \text{ A.U. / yr}; \delta = 0.01, \Delta t = 0.01 \text{ yrs}, t_{\text{final}} = 47.9 \text{ yr}; \) this plot shows the clockwise precession found for smaller values of \( \delta \) which I found counter-intuitive.
Figure 2: ER; $r(t = 0) = 1$ A.U.; $v_y(t = 0) = 5$ A.U. / yr; $\delta = 0.05$, $\Delta t = 0.01$ yrs, $t_{\text{final}} = 4.65$ yr; this shows the elliptical nature and counterclockwise precession of the orbits typical for these values of $\delta$ and $\Delta t$. 
Figure 3: ER; \( r(t = 0) = 1 \text{ A.U.}; v_y(t = 0) = 5 \text{ A.U.} / \text{yr}; \delta = 0.05, \Delta t = 0.01 \text{yrs}, t_{\text{final}} = 389 \text{yrs}; \) this plot shows the first 61 orbits of a run, continuing to show the elliptical orbits and counterclockwise precession.
Figure 4: ER; $r(t = 0) = 1$ A.U.; $v_y(t = 0) = 5$ A.U./yr; $\delta = 0.05$, $\Delta t = 0.01$ yrs, $t_{\text{final}} = 1,750$ yr; this plot shows the next completed orbit after Figure 3 showing the great increase in orbit size and time at this point; due to the length of time needed to reach this point, I could not continue the simulation to see the 63rd orbit.
Figure 5: ER; \( r(t = 0) = 1 \text{A.U.}; v_r(t = 0) = 5 \text{ A.U. / yr}; \delta = 0.05, \Delta t = 0.025 \text{yrs}, t_{\text{final}} = 9.50 \text{yr}; \) this plot shows the clockwise precession found for larger values of \( \Delta t \) which I think I understand.
Figure 6: ER; \( r(t = 0) = 1 \text{ A.U.} \); \( v_y(t = 0) = 5 \text{ A.U. / yr} \); \( \delta = 0.05 \); \( \Delta t = 0.001 \text{ yrs} \); \( t_{\text{final}} = 2 \text{ yr} \); this plot shows that even with a rather small \( \Delta t \), the orbits are elliptical. Here they are shown to be precessing counterclockwise.
Figure 7: ER; $r(t = 0) = 1$ A.U.; $v_y(t = 0) = 5$ A.U. / yr; $\delta = 0.05$ $\Delta t = 0.001$ yrs, $t_{\text{final}} = 25.2$ yr; this plot shows the well-behaved orbits generated for smaller values of $\Delta t$. Note that while bound, the orbit does not appear to be closed at this time.
Figure 8: \( r(t = 0) = 1 \text{A.U.}; \) \( v_r(t = 0) = 5 \text{A.U. / yr}; \) \( \delta = 0.05 \Delta t = 0.01 \text{yrs}, \) \( t_{\text{final}} = 2.60; \) this plot shows that using the initial conditions recommended in the problem statement with the Verlet algorithm the orbits are ellipses which precess counterclockwise.
Figure 9: \( r(t = 0) = 1\text{A.U.}; v_y(t = 0) = 5\text{ A.U. / yr}; \delta = 0.05 \Delta t = 0.03\text{yrs}, t_{\text{final}} = 3.30\text{yr}; \) when \( \Delta t \) is increased, the Verlet algorithm begins to precess counterclockwise showing that it has similar difficulties to the Euler-Richardson for this problem.
Figure 10: RK4; \( \mathbf{r}(t = 0) = 1\, \text{A.U.}; \mathbf{v}(t = 0) = 5\, \text{A.U.} / \text{yr}; \Delta = 0.05 \, \Delta t = 0.01\, \text{yrs}, \ t_{\text{final}} = 2.80\, \text{yr}; \) this plot shows that for the initial conditions recommended in the problem statement the Runge-Kutta algorithm generates elliptical orbits which precess counterclockwise.
Figure 11: RK4; $r(t = 0) = 1\text{A.U.}; \nu (t = 0) = 5\text{ A.U. / yr}; \delta = 0.05 \Delta t = 0.01 \text{yrs}, t_{\text{final}} = 25.4\text{yr};$ this is a continuation of the previous plot and shows that for these values of $\delta$ and $\Delta t$, the Runge-Kutta algorithm produces well-behaved orbits.
Figure 12: \( r(t = 0) = 1 \text{A.U.}; \) \( v_y(t = 0) = 5 \text{ A.U. / yr}; \) \( \delta = 0.05 \) \( \Delta t = 0.03 \text{yrs}, \) \( t_{\text{final}} = 2.80 \text{yr}; \) with larger values of \( \Delta t, \) the Runga-Kutta keeps the precession in the same direction unlike the Euler-Richardson and Verlet algorithms.
Figure 13: RK4; \( r(t = 0) = 1 \text{A.U.}; \ n_y(t = 0) = 5 \text{ A.U. / yr}; \ \delta = 0.01; \ \Delta t = 0.01\text{yrs}, t_{\text{final}} = 2.80\text{yr}; \) with a smaller value of \( \delta, \) the Runge-Kutta algorithm continues to produce elliptical orbits which precess counterclockwise.
Figure 14: RK4; $r(t = 0) = 1 \text{ A.U.}; v_y(t = 0) = 5 \text{ A.U. / yr}; \delta = 0.01$ $\Delta t = 0.01 \text{ yrs}, t_{final} = 125 \text{ yr};$ this is an extreme close up of the paths followed by the planet in the Runge-Kutta algorithm after the orbits have precessed a full $360^\circ$. From this zoom, it can be seen that these orbits are not retracing themselves.
Figure 15: ER; \( r(t = 0) = 2 \text{AU}; \ v_y(t = 0) = 5 \text{AU/yr}; \ \delta = 0.075, \ \Delta t = 0.01 \text{yrs}, \ t_{\text{final}} = 25.2\text{yr}; \) this plot shows that the orbits for starting further away from the sun are elliptical. These conditions resulted in a counterclockwise precession.
Figure 16: ER; $r(t = 0) = 2\text{A.U.}; v_y(t = 0) = 5\text{ A.U. / yr}; \delta = 0.075, \Delta t = 0.01\text{yrs}, t_{\text{final}} = 151\text{yr};$ this plot shows the well-behaved elliptical orbits generated by starting further away from the sun. The precession here was counterclockwise.
Figure 17: ER; \( r(t = 0) = 2 \text{A.U.}; v_y(t = 0) = 5 \text{A.U. / yr}; \delta = 0.05, \Delta t = 0.01 \text{yrs}, t_{\text{final}} = 21.7 \text{yrs} \); this plot shows another set of elliptical orbits which precess counterclockwise and are generated from an initial position further away from the sun. An interesting difference from the unity initial radius orbits is the fact that the aphelion was on the opposite side of the star from the initial position.
Figure 18: ER; $r(t = 0) = 2\text{A.U.}$; $v_y(t = 0) = 5\text{ A.U. / yr}$; $\delta = 0.05$, $\Delta t = 0.01\text{yrs}$, $t_{\text{final}} = 136$; this plot shows that even over an extended period, the orbits generated by starting further out from the sun for the tested values are well behaved.
Figure 19: RK4; \( r(t = 0) = 2 \text{ A.U.}; v_y(t = 0) = 5 \text{ A.U. / yr}; \delta = 0.05 \Delta t = 0.01 \text{ yrs}, t_{\text{final}} = 11.8 \text{ yr}; \) this plot shows that for these initial conditions the Runge-Kutta algorithm generates elliptical orbits which precess counterclockwise; as with the Euler-Richardson algorithm for the larger initial radius, the aphelions is opposite from the orbits’ starting position.
Figure 20: RK4; $r(t = 0) = 2$ A.U.; $v_y(t = 0) = 5$ A.U. / yr; $\delta = 0.05$; $\Delta t = 0.01$ yrs, $t_{\text{final}} = 208$ yrs; this plot is a continuation of the previous. It shows that the Runga-Kutta algorithm is well-behaved and does not appear to retrace orbits.
Figure 21: ER; $r(t = 0) = 0.5\text{A.U.}; v_y(t = 0) = 5\text{ A.U. / yr}; \delta = 0.05, \Delta t = 0.01\text{yrs}, t_{\text{final}} = 5.4\text{yr};$ this plot shows the ballooning elliptical orbits typical of using smaller initial radii.
Figure 22: RK4; \( r(t = 0) = 0.5 \text{ A.U.}; \) \( v_y(t = 0) = 7.55 \) A.U. / yr; \( \delta = 0.05 \) \( \Delta t = 0.01 \) yrs, \( t_{\text{final}} = 1.70 \) yr; this plot shows that for a smaller initial radius, the Runge-Kutta algorithm continues to generate counterclockwise precessing elliptical orbits.
Figure 23: RK4; \( r(t = 0) = 0.5 \text{A.U.}; v_y(t = 0) = 7.5 \text{A.U. / yr}; \delta = 0.05 \Delta t = 0.01 \text{yrs}, t_{\text{final}} = 10.7 \text{yr}; \) here we can see that the orbits for the Runga-Kutta begin to spiral inwards a problem which is dissimilar from that of the Euler-Richardson algorithm in which the orbits ballooned in size over time.

I added a function to find and plot the change in orientation of the major axis between successive orbits. The angle I used for this comparison, \( \Delta \phi_{\text{major}} \), was that between the line connecting the sun and planet’s original position, i.e. the x-axis, and the line connecting the sun with an orbit’s aphelion. For the first several orbits of the \( \delta = 0.05 \) case, I confirmed the change in angles I calculated and they seemed correct. Typical values of \( \Delta \phi_{\text{major}} \) had absolute values between 0° and 20°. Positive \( \Delta \phi_{\text{major}} \) indicates a counterclockwise precession; clockwise precessions generate negative \( \Delta \phi_{\text{major}} \). Charts of \( \Delta \phi_{\text{major}} \) in degrees versus time are shown below.
Figure 24: ER; \( r(t = 0) = 1 \text{ A.U.}; \) \( v_y(t = 0) = 5 \text{ A.U. / yr}; \) \( \delta = 0.05 \) \( \Delta t = 0.001 \text{ yrs}, t_{\text{final}} = 25.2 \text{ yr}; \) this corresponds to Figure 7. This shows that the precession is fairly constant at approximately \( 9.5^\circ - 9.75^\circ \) per orbit.

Figure 25: RK4; \( r(t = 0) = 1 \text{ A.U.}; \) \( v_y(t = 0) = 5 \text{ A.U. / yr}; \) \( \delta = 0.05 \) \( \Delta t = 0.01 \text{ yrs}, t_{\text{final}} = 208; \) this graph of \( \Delta \phi_{\text{major}} \) shows that for the Runge-Kutta algorithm, typical values for the precession are between \( 8^\circ \) and \( 11^\circ \) and that the magnitude seems to be oscillating about some linear function.
Figure 26: ER; $r(t = 0) = 2\text{A.U.}; \nu_i(t = 0) = 5\text{ A.U. / yr}; \delta = 0.05, \Delta t = 0.01\text{yrs}, t_{\text{final}} = 136$; this plot shows that the precession for the larger radius was even more uniform than that for the $r(t = 0) = 1\text{A.U.}$ orbits.
Figure 27: RK4; $r(t = 0) = 2\text{A.U.}$; $v_y(t = 0) = 5 \text{ A.U. / yr}; \delta = 0.05 \Delta t = 0.01\text{yrs}, t_{\text{final}} = 208$; the precession of the Runge-Kutta algorithm for an initial radius of 2A.U. is very regular at just under $10^\circ$ per orbit with one unusual result.
Figure 28: \( r(t = 0) = 0.5 \text{A.U.}; \; v_y(t = 0) = 5 \text{ A.U. / yr}; \; \delta = 0.05, \; \Delta t = 0.01 \text{ yrs}, \; t_{\text{final}} = 5.4 \text{ yr}; \) this corresponds to Figure 21, and the fact that it is negative shows the counterclockwise procession for these initial conditions. The fact that the magnitude of the precession decreases can be visually confirmed by examining Figure 11 where later orbits appear closer in orientation to their predecessors.
Figure 29: RK4; \( r(t = 0) = 0.5 \text{ A.U.}; \) \( v_y(t = 0) = 7.5 \text{ A.U. / yr}; \) \( \delta = 0.05 \Delta t = 0.01 \text{ yrs}, t_{\text{final}} = 10.7; \) the precession for the smaller initial radius with the Runge-Kutta method resulted in an interesting saw tooth pattern. I believe this is a result of the inward spiral causing discontinuities in the precession.

Part b.

From general relativity, there is a correction to the gravitational force between two objects making the net acceleration of an object in a gravitational field:

\[
\frac{d^2 \hat{r}}{dt^2} = -\frac{GM}{r^2} \left[ 1 + \alpha \left( \frac{GM}{c^2} \right)^2 \frac{1}{r^2} \right] \hat{r} = -\frac{GM}{r^3} \left[ 1 + \alpha \left( \frac{GM}{c^2} \right)^2 \frac{1}{r^2} \right] \hat{r}
\]

This can be used to find the accelerations in the x- and y-directions:

\[
a_x = -\frac{GM}{r^3} \left[ 1 + \alpha \left( \frac{GM}{c^2} \right)^2 \frac{1}{r^2} \right] x
\]

\[
a_y = -\frac{GM}{r^3} \left[ 1 + \alpha \left( \frac{GM}{c^2} \right)^2 \frac{1}{r^2} \right] y
\]
The parameter $\alpha$ is a dimensionless parameter that was added as a static double figure in the `Planet` class. The `PlanetRel` class was created to use the altered accelerations to find the rates of change for $v_x$ and $v_y$. When this class is used with the Euler-Richardson algorithm, elliptical, clockwise precessing orbits were generated. Using a relatively long $\Delta t = 0.01\text{yrs}$, these orbits began to grow rather quickly. The Verlet method also generated clockwise precessing elliptical orbits. The Runga-Kutta algorithm, however, did not have a steady precession, instead, the aphelion oscillated back and forth about $0^\circ$. A shorter $\Delta t$ has not been used yet.
Figure 30: ER; using relativistic correction, $r(t = 0) = 1\text{A.U.}; v_y(t = 0) = 5 \text{ A.U. / yr}; \Delta t = 0.001\text{yrs}, t_{\text{final}} = 12.7\text{yr};$ this plot shows the elliptical, clock-wise precessing behavior obtained using the relativistic correction to the gravitational force. The precession for this situation is minimal as shown by the fact that the orbits are tightly packed as are the aphelion markers.
Figure 31: ER: using relativistic correction, \( r(t = 0) = 1 \text{A.U.}; v_y(t = 0) = 5 \text{ A.U.} / \text{yr}; \Delta t = 0.001 \text{yrs}, t_{\text{final}} = 12.7 \text{yr}; \) this plot shows that the precession was clockwise for these initial conditions, and minimal, the largest magnitude reported being a change of less than 0.35°, this corresponds to the aphelion moving approximately 0.006 A.U., since the radial position of the aphelion was approximately 1 A.U. out the entire time.
Figure 32: Using relativistic correction $r(t = 0) = 1 \text{A.U.}; v_y(t = 0) = 5 \text{A.U. / yr}; \Delta t = 0.01 \text{yrs}; t_{\text{final}} = 1.45 \text{yrs};$ this plot shows the second orbit beginning to precess in the clockwise direction from the first from the fact that the blue dots are on the x-axis for the first orbit and below for the second.
Figure 33: Using relativistic correction $r(t = 0) = 1$ A.U.; $v_y(t = 0) = 5$ A.U. / yr; $\Delta t = 0.01$ yrs; $t_{\text{final}} = 160$; this plot shows that the precession for the Verlet algorithm with the relativistic correction is clockwise and varied from under $0.25^\circ$ to $3^\circ$ in magnitude.
Figure 34: RK4; using relativistic correction \( r(t = 0) = 1 \text{A.U.}; \) \( v_y(t = 0) = 5 \text{A.U. / yr}; \) \( \Delta t = 0.01 \text{yrs}; \) \( t_{\text{final}} = 12.9 \text{yrs}; \) from the cluster of blue dots around the x-axis and the fact that the paths lie atop one another, this plot shows that the orbits from the relativistic situation using the Runge-Kutta algorithm did not precess uniformly.
Figure 35: RK4; using relativistic correction \( r(t = 0) = 1 \text{A.U.} \); \( v_y(t = 0) = 5 \text{ A.U.} / \text{yr} \); \( \Delta t = 0.01 \text{yrs} \); \( t_{\text{final}} = 12.9 \text{yrs} \); the precessions oscillated between \(-360^\circ\) and \(360^\circ\); my precession finding code does not properly take into account the fact that \(0^\circ\) and \(360^\circ\) are the same thing, and this oscillation is really just about the positive x-axis.

Part c

If an inverse cube law is used instead of an inverse square law, then:

\[
\vec{F} = -\frac{C m}{r^3} \hat{r} = -\frac{C m}{r^4} \vec{r}
\]

The units of \( C \) for this case are:

\[
[C] = \left[ \frac{\text{force} \times \text{volume}}{\text{mass}} \right] = \left[ \frac{\text{length}^4}{\text{time}^2} \right]
\]

Using astronomical units, this gives \( C \) to be, using the same numerical value as earlier:

\[
C = 4\pi \frac{\text{A.U.}^4}{\text{yr}^2}
\]
The same process used above can be used to find the force and acceleration broken up into the x- and y-directions:

\[
F_x = -\frac{Cm}{r^4} x
\]

\[
F_y = -\frac{Cm}{r^4} y
\]

\[
a_x = -\frac{C}{r^4} x
\]

\[
a_y = -\frac{C}{r^4} y
\]

By setting the acceleration equal to the formula for centripetal acceleration, an expression for the initial velocity needed to form a circular orbit can be found:

\[
-m \frac{v_c^2}{r} \hat{r} = -\frac{Cm}{r^3} \hat{r}
\]

\[
v_c = \frac{\sqrt{C}}{r} \Rightarrow v_c = \frac{2\pi \text{ A.U.}}{r \text{ yr}}
\]

To obtain a circular orbit, a \(\Delta t\) smaller than 0.01yr must be used. The largest value of \(\Delta t\) I could find for which the orbit stayed circular for several periods was \(\Delta t = 0.0005\text{yr}\) for the Euler-Richardson algorithm. The Runga-Kutta algorithm and Verlet method both maintained circular periods with \(\Delta t = 0.001\text{yr}\). For an inverse square law, a \(\Delta t\) of 0.005yrs was sufficient to maintain a circular orbit for six orbits using the Euler-Richardson algorithm. The results of my trials to find an appropriate \(\Delta t\) are shown below.
Figure 36: ER; using inverse square law; $r(t = 0) = 1\text{ A.U.}; v_y(t = 0) = v_c = 6.2832\ \text{A.U. / yr}; \Delta t = 0.01\text{yrs}; t_{\text{final}} = 7.30\text{yr};$ for this large a $\Delta t$, the orbits are close but there is a clear precession and from this graph the thicker lines for the trail show that they are not quite lying on top of one another.
Figure 37: ER; using inverse square law; $r(t = 0) = 1$ A.U.; $v_y(t = 0) = v_c = 6.2832$ A.U./yr; $\Delta t = 0.005$ yrs, $t_{\text{final}} = 6.10$ yr; here the orbits appear to be lying atop one another rather well and the precession was minimal.
Figure 38: ER; using inverse square law; \( r(t = 0) = 1 \text{A.U.}; \) \( v_y(t = 0) = v_c = 6.2832 \text{ A.U. / yr}; \) \( \Delta t = 0.005 \text{yrs}, \) \( t_{\text{final}} = 6.10 \text{yr}; \) this shows that the precession for the first five orbits was less than half a degree implying that this \( \Delta t \) value is sufficient to get circular orbits for an inverse square law.
Figure 39: ER; using inverse cubic law; \( r(t = 0) = 1\text{ A.U.}; \) \( v_x(t = 0) = v_c = 6.2832 \text{ A.U. / yr}; \) \( \Delta t = 0.01\text{ yrs}, t_{\text{final}} = 3.75\text{ yr}; \) for this large a \( \Delta t, \) the orbits begin spiraling outwards immediately.
Figure 40: ER; using inverse cubic law; $r(t = 0) = 1$ A.U.; $v_y(t = 0) = v_c = 6.2832$ A.U. / yr; $\Delta t = 0.05$ yr, $t_{\text{final}} = 2.37$ yr; reducing $\Delta t$ by a factor of two still resulted in clearly outward spiraling orbits, though now the spirals are much more tightly packed than above.
Figure 41: ER; using inverse cubic law; \( r(t = 0) = 1 \text{ A.U.} \); \( v_y(t = 0) = v_c = 6.2832 \text{ A.U. / yr} \); \( \Delta t = 0.0025 \text{ yrs} \), \( t_{\text{final}} = 3.07 \text{ yr} \); from the figure above, \( \Delta t \) has been cut in half again, the orbits still spiral away from the sun, but now they are even more tightly packed.
Figure 42: ER; using inverse cubic law; \( r(t = 0) = 1 \text{ A.U.}; v_y(t = 0) = v_c = 6.2832 \text{ A.U. / yr}; \Delta t = 0.001 \text{ yrs}, t_{\text{final}} = 4.03 \text{ yr}; \) by reducing \( \Delta t \) by a factor of ten from its original value (in Figure 15), the orbits now appear to be closed as well as circular. However, from the thickness of the trails, I believe that they are not quite closed yet.
Figure 43: ER; using inverse cubic law; \( r(t = 0) = 1 \text{A.U.}; v_y(t = 0) = v_c = 6.2832 \text{ A.U.} / \text{yr}; \Delta t = 0.0005 \text{yrs}, t_{\text{final}} = 4.34 \text{yr}; \) by reducing \( \Delta t \) by a factor of twenty from its original value (in Figure 39), the orbits now appear to be closed as well as circular.
Figure 44: ER; using inverse cubic law; \( r(t = 0) = 1 \text{ A.U.}; v_y(t = 0) = v_c = 6.2832 \text{ A.U. / yr}; \Delta t = 0.0005 \text{ yrs}, \ t_{\text{final}} = 4.34 \text{ yr}; \) this shows that the orbits for this set of conditions barely precess since the largest magnitude of change in \( \Delta \phi_{\text{major}} \) is barely larger than 0.14°.
Figure 45: Using inverse cubic law; $r(t = 0) = 1\text{ A.U.}; v_{y}(t = 0) = v_c = 6.2832 \text{ A.U. / yr}; \\
\Delta t = 0.001\text{ yrs}; t_{\text{final}} = 2.43\text{ yr};$ this shows that the orbits for this set of conditions barely 
precess and are circular since the paths and blue dots lie on top of one another.
Figure 46: RK4sing inverse cubic law; \( r(t = 0) = 1 \text{A.U.}; v_y(t = 0) = v_c = 6.2832 \text{ A.U. / yr}; \)
\( \Delta t = 0.001 \text{yrs}; t_{\text{final}} = 7.07 \text{yrs}; \) as with the Verlet method, the Runga-Kutta algorithm maintained circular, non-spiraling orbits for a much larger \( \Delta t \) value than the Euler-Richardson’s method.

The analytic value of \( v_c \) was also tested for smaller and larger initial radii.
Figure 47: ER; using inverse cubic law; $r(t = 0) = 0.5\text{A.U.}; v_y(t = 0) = v_c = 12.5664\text{ A.U.}/\text{yr}$; $\Delta t = 0.0001\text{yrs}$, $t_{\text{final}} = 0.76\text{yr}$; this shows that the analytic formula for $v_c$ works for an initial radius smaller than 1A.U.
Figure 48: V; using inverse cubic law; \( r(t = 0) = 0.5 \text{A.U.}; \ v_y(t = 0) = v_c = 12.5664 \text{ A.U. / yr}; \ \Delta t = 0.001 \text{ yrs}; \ t_{\text{final}} = 0.542 \text{ yr}; \) with a smaller radius for the planet’s initial condition, the Verlet method still behaves better than the Euler-Richardson model.
Figure 49: ER; using inverse cubic law; \( r(t = 0) = 2 \text{ A.U.} \); \( v_y(t = 0) = v_c = 3.1416 \text{ A.U. / yr} \); \( \Delta t = 0.001 \text{ yrs} \); \( t_{\text{final}} = 4.45 \text{ yrs} \); this shows that the analytic formula for \( v_c \) works for an initial radius smaller than 1 A.U.

Part d.

From the relationship between potential energy and force, the total energy of a planet within this field can be found. Using the inverse cube force:

\[
\vec{F} = -\nabla U(r)
\]

\[
\therefore U(r) = -\frac{Cm}{2r^2}
\]

\[
E = T + U
\]

\[
\frac{E}{m} = \frac{v^2}{2} + \frac{C}{2r^2} = \frac{v_x^2 + v_y^2}{2} - \frac{C}{2r^2}
\]
This was used to generate a graph of energy versus time for the inverse cube law for initial velocities on either side of $v_c$ that used for circular motion. From the expression for $v_c \left( v_c = \sqrt{C/r} \right)$ is clear that for initial velocities less than $v_c$ the energy is negative meaning the particle should be bound while for initial velocities greater than $v_c$ the energy is positive and thus the particle should not be bound. The simulation behaved as expected for values of $v(t = 0)$ greater than $v_c$.

Figure 50: ER; using inverse cubic law; $r(t = 0) = 1$ A.U.; $v_y(t = 0) = 1.01v_c = 6.3460$ A.U./yr; $\Delta t = 0.001$ yrs; $t_{\text{final}} = 3.17$ yr; from the behavior of the planet in this plot, it is clearly unbound as it spirals away from the sun.
Figure 51: ER; using inverse cubic law; $r(t = 0) = 1$ A.U.; $v_y(t = 0) = 1.01 v_c = 6.3460$ A.U. / yr; $\Delta t = 0.001$ yrs, $t_{\text{final}} = 3.17$ yr; this is the energy versus time plot corresponding to Figure 50; here the energy is positive supporting the fact that the particle is unbound. The energy changes by approximately 5% of its initial value.
Figure 52: ER; using inverse cubic law; $r(t = 0) = 1\text{A.U.}; v_y(t = 0) = 1.02v_c = 6.4089\text{A.U./yr}; \Delta t = 0.001\text{yrs}, t_{\text{final}} = 3.35\text{yr};$ again, the planet is clearly unbound and it does not even come close to closing an orbit and is moving quickly away from the sun.
The simulation did not perform as expected for $v(t = 0)$ less than $v_c$. The Euler-Richardson and Runga-Kutta methods behaved similarly for this situation. For both, the energy fluctuated wildly throughout each run, though it started negative as expected. The position plot exposes some of the oddities of the simulation. For the Euler-Richardson and Runga-Kutta methods, the planet orbited the sun tightly before being ejected from the system. For the Verlet algorithm, though the planet spiraled tightly in towards the star, it remained bound though the orbits did not appear to be closed. I believe the tight loops around the star resulted in large variations of velocity which resulted in the odd behavior of the energy, though I am unsure why the planet ever spiraled that close to the sun.
Figure 53: ER; using inverse cubic law; \( r(t = 0) = 1 \text{A.U.} \); \( v_y(t = 0) = 0.99v_c = 6.2204 \) A.U. / yr; \( \Delta t = 0.001 \text{yrs} \); \( t_{\text{final}} = 2.47 \text{yr} \); this shows the planets path as it moved in made several tight orbits before flinging itself out of the system.
Figure 54: ER; using inverse cubic law; $r(t = 0) = 1\text{A.U.}; v_y(t = 0) = 0.99v_{c} = 6.2204\text{A.U./yr}; \Delta t = 0.001\text{yrs}; t_{\text{final}} = 2.47\text{yr};$ this shows the planet’s radial position as a function of time showing it made its closest approach to the star shortly after 1 year had passed.
Figure 55: ER; using inverse cubic law; $r(t = 0) = 1 \text{A.U.}$; $v_y(t = 0) = 0.99v_c = 6.2204 \text{A.U. / yr}$; $\Delta t = 0.001 \text{yrs}$, $t_{\text{final}} = 2.47 \text{yr}$; this shows the planet’s energy as a function of time; the massive spike shows that the simulation had a problem with this system though. Note, though that the spike occurs at roughly the same time as the closest approach shown in Figure 54 reinforcing my idea that these close approaches are what doom the simulation.
Figure 56: ER; using inverse cubic law; $r(t = 0) = 1 \text{A.U.}; v_y(t = 0) = 0.98v_y = 6.1575 \text{A.U./yr}; \Delta t = 0.001\text{yrs, } t_{\text{final}} = 1.36\text{yr};$ again the planet made several tight passes past the sun and was then ejected from the system.
Figure 57: ER; using inverse cubic law; \( r(t = 0) = 1 \text{ A.U.}; v_r(t = 0) = 0.98 v_c = 6.1575 \text{ A.U. / yr}; \Delta t = 0.001 \text{ yrs}, t_{\text{final}} = 1.36 \text{ yr}; \) radial position versus time for the path shown in Figure 56. Here the closest approach was at 0.8 years.
Figure 58: ER; using inverse cubic law; $r(t = 0) = 1\text{ A.U.}; \nu_y(t = 0) = 0.98\nu_c = 6.1575\text{ A.U.}/\text{yr}; \Delta t = 0.001\text{ yrs}, t_{\text{final}} = 1.36\text{ yr};$ this energy vs. time plot shows another extremely large spike showing that the simulation has problems dealing with this smaller velocity as well.
Figure 59: RK4; using inverse cubic law; $r(t = 0) = 1$ A.U.; $v_y(t = 0) = 0.99v_c = 6.22204$ A.U. / yr; $\Delta t = 0.001$ yrs; $t_{\text{final}} = 1.26$ yr; as with the Euler-Richardson method, after a tight pass, the planet was ejected from the system.
Figure 60: RK4; using inverse cubic law; \( r(t = 0) = 1\text{AU}; \) \( v_y(t = 0) = 0.99v_c = 6.22204\text{AU/yr}; \) \( \Delta t = 0.001\text{yrs}; t_{\text{final}} = 1.26\text{yr}; \) the position versus time graph corresponding to the previous figure shows that the close approach occurred slightly after 1.1yrs.
Figure 61: RK4; using inverse cubic law; \( r(t = 0) = 1 \text{A.U.}; v_y(t = 0) = 0.99v_c = 6.2204 \text{A.U. / yr}; \Delta t = 0.001 \text{yrs}; t_{\text{final}} = 1.26 \text{yr}; \) the spike in energy for the Runge-Kutta method is gigantic when compared to that in the Euler-Richardson results.
Figure 62: RK4; using inverse cubic law; $r(t = 0) = 1\text{A.U.}; \; v_y(t = 0) = 0.99v_c = 6.2204\text{A.U. / yr}; \; \Delta t = 0.001\text{yrs}; \; t_{\text{final}} = 1.26\text{yr};$ a detail of the energy versus time plot for the Runga-Kutta method shows that it’s energy started negative, as is expected, before the massive spike in energy magnitude and change of signs; this shows that the Runga-Kutta method did not succeed in simulating the inverse cube law as I programmed it.
Figure 63: Using inverse cubic law; $r(t = 0) = 1\text{A.U.}; v_y(t = 0) = 0.99v_c = 6.22204\text{A.U.}/\text{yr}; \Delta t = 0.001\text{yrs}; t_{\text{final}} = 1.29\text{yr};$ from the path at this point, the Verlet algorithm seems to have the same properties, and thus problems as the Euler-Richardson and Runga-Kutta algorithms.
Figure 64: Using inverse cubic law; \( r(t = 0) = 1 \text{A.U.}; v_y(t = 0) = 0.99v_c = 6.22204\text{A.U.} \) / yr; \( \Delta t = 0.001\text{yrs}; t_{\text{final}} = 10.0\text{yr} \); after a decade has passed though, it is clear that the Verlet algorithm has its own problems from the fact that while the path is bound, though not closed, it still spirals in much more closely to the star than expected.
Figure 65: Using inverse cubic law; \( r(t = 0) = 1 \text{A.U.} \); \( v_y(t = 0) = 0.99v_c = 6.22204 \text{A.U.} / \text{yr} \); \( \Delta t = 0.001 \text{yrs} \); \( t_{\text{final}} = 10.0 \text{yr} \); the position versus time graph for the proceeding orbits shows that the magnitude of the radial position has a periodicity to it.
Figure 66: Using inverse cubic law; \( r(t = 0) = 1 \text{ A.U.}; v_y(t = 0) = 0.99 v_c = 6.22204 \text{ A.U.} / \text{yr}; \Delta t = 0.001 \text{ yrs}; t_{\text{final}} = 10.0 \text{yr}; \) this energy versus time graph shows spikes in energy magnitude where expected from the position versus time graph. The changing value of energy emphasizes the Verlet method’s obvious incorrectness; however the fact that the energy was always negative shows that the planet is indeed bound.
Figure 67: ER; using inverse cubic law; $r(t = 0) = 2 \text{A.U.}$; $v_y(t = 0) = 0.99v_c = 3.1101 \text{A.U. / yr}$; $\Delta t = 0.001 \text{yrs}$, $t_{\text{final}} = 5.97 \text{yr}$; again the planet made several tight passes past the sun before leaving the system showing that the simulation had problems with starting further away but still below the calculated $v_c$. 
Figure 68: ER; using inverse cubic law; \( r(t = 0) = 2 \text{A.U.}; v_y(t = 0) = 0.99v_c = 3.1101 \text{A.U. / yr}; \Delta t = 0.001\text{yrs}, t_{\text{final}} = 5.97\text{yr}; \) this radial position versus time has its closest approach at 4.5 years corresponding to the tightest path shown in Figure 67.
Figure 69: ER; using inverse cubic law; $r(t = 0) = 2\text{A.U.}; v_y(t = 0) = 0.99v_c = 3.1101\text{ A.U. / yr}; \Delta t = 0.001\text{yrs, } t_{\text{final}} = 5.97\text{yr};$ once again the spike in the energy versus time graph occurs at the same time as the closest approach of the planet. The spike shows that the simulation could not handle the initial conditions starting further from the sun and moving slower.
Figure 70: ER; using inverse cubic law; \( r(t = 0) = 0.5 \text{A.U.}; \) \( v_y(t = 0) = 0.99v_c = 12.4407 \text{A.U. / yr}; \) \( \Delta t = 0.0001 \text{yrs}, t_{\text{final}} = 0.463 \text{yr}; \) the path for the simulation attempting this motion is by now familiar showing the close approach before the planet leaves the system.
Figure 71: ER; using inverse cubic law; $r(t = 0) = 0.5\text{A.U.}$; $v_y(t = 0) = 0.99v_c = 12.4407\text{A.U. / yr}$; $\Delta t = 0.0001\text{yrs}$, $t_{\text{final}} = 0.463\text{yr}$; this radial position versus time has its closest approach just before 0.30 years
Figure 72: ER; using inverse cubic law; \( r(t = 0) = 0.5 \text{A.U.}; v_y(t = 0) = 0.99v_c = 12.4407 \text{A.U. / yr}; \Delta t = 0.0001 \text{yrs}, t_{\text{final}} = 0.463 \text{yr}; \) as was anticipated, the spike in the energy versus time graph occurs at the same time as the closest approach of the planet for the conditions of starting at higher speed and smaller initial radius.
Figure 73: RK4; using inverse cubic law; \( r(t = 0) = 0.5 \text{A.U.}; v_y(t = 0) = 0.99v_c = 12.4407 \text{A.U. / yr}; \Delta t = 0.0001 \text{yrs}, t_{\text{final}} = 0.288 \text{yr}; \) the Runga-Kutta model fairs no better than the Euler-Richardson when the initial radius was shrunk. The position and energy versus time graphs for this situation follow the same pattern previously demonstrated ad naseam.
The orbits that should have been bound, the Euler-Richardson and Runge-Kutta orbits for \( v < v_c \) might not necessarily have been closed. The Verlet orbits using the cubic method and the orbits shown in, e.g., Figure 7 or 14, some bound orbits are not closed, and this is supported by [1] in which the authors write that only for inverse square law and harmonic oscillator are “all finite motions closed.” Before the large spikes in energy cause the simulation to collapse for the Euler-Richardson and Runge-Kutta models, the energy is negative supporting the idea that the simulation tried to portray bound motion but the numerical properties of the situation prevented it. However, I am unsure what caused the problem of changing energy for all three algorithms tried.

Reference: