Series Testing

a. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n^2+1}$

n-th term test

$\displaystyle\lim_{n\to\infty} \frac{n^2}{n^2+1} = 1 \neq 0$ series div.

b. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{5/2}}$

p-series $p = 5/2 > 1$ conv.

c $\displaystyle\sum_{n=0}^{\infty} \frac{n^2}{3^{n+1}}$

Ratio Test

$a_n = \dfrac{n^2}{3^{n+1}}$ $\quad a_{n+1} = \dfrac{(n+1)^2}{3^{n+1+1}}$

$$\lim_{n\to\infty} \left| \frac{\frac{(n+1)^2}{3^{n+2}}}{\frac{n^2}{3^{n+1}}} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^2}{3^{n+2}} \cdot \frac{3^{n+1}}{n^2} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(n+1)^2}{n^2} \cdot \frac{3^{n+1}}{3^{n+2}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(n+1)^2}{n^2} \cdot \frac{1}{3} \right| = (1)(1/3) = 1/3 < 1$$

conv. absolutely

d. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{3^n}{5^{n+1}} = \sum_{n=1}^{\infty}\left(-\frac{1}{5}\right)\left(\frac{-3}{5}\right)^n$ geometric

$r = -3/5$, $|-3/5| < 1$, conv.

$$S = \frac{3/25}{1-(-3/5)} = \frac{3/25}{8/5} = \frac{3}{25}\cdot\frac{5}{8}$$

$$= \frac{3}{40}$$

e. $\sum_{n=1}^{\infty}\frac{2n}{n^2+1}$ LCT to div. p-series $\sum_{n=1}^{\infty}\frac{1}{n}$

$$\lim_{n\to\infty}\left(\frac{\frac{2n}{n^2+1}}{\frac{1}{n}}\right) = \lim_{n\to\infty}\left(\frac{2n}{n^2+1}\cdot\frac{n}{1}\right) = \lim_{n\to\infty}\frac{2n^2}{n^2+1} = 2$$

pos and finite

both series div.

5. $\sum_{n=1}^{\infty} (-1)^n \frac{n!}{(2n+1)!}$

Ratio Test $\quad a_n = \frac{n!}{(2n+1)!} \qquad a_{n+1} = \frac{(n+1)!}{(2(n+1)+1)!} = \frac{(n+1)!}{(2n+3)!}$

$$\lim_{n\to\infty} \left| \frac{\frac{(n+1)!}{(2n+3)!}}{\frac{n!}{(2n+1)!}} \right| = \lim_{n\to\infty} \left| \frac{(n+1)!}{(2n+3)!} \cdot \frac{(2n+1)!}{n!} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(n+1)!}{n!} \cdot \frac{(2n+1)!}{(2n+3)!} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(n+1)}{1} \cdot \frac{1}{(2n+3)(2n+2)} \right| = 0 < 1$$

series conv. abs.

g. $\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+2}\right)=\left(1-\frac{1}{3}\right)+\left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\cdots$

$S_3 = 1+\frac{1}{2}-\frac{1}{4}-\frac{1}{5}$

$S_n = 1+\frac{1}{2}-\frac{1}{n+1}-\frac{1}{n+2}$

$S=\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1+1/2-\frac{1}{n+1}-\frac{1}{n+2}\right)=3/2$

conv.

h. $4-1+\frac{1}{4}-\frac{1}{16}+\frac{1}{64}-\cdots$

$\sum_{n=0}^{\infty} 4\left(-\frac{1}{4}\right)^n$ geometric

$r=-1/4$, $|-1/4|<1$, $S=\frac{4}{1-(-1/4)}=\frac{4}{5/4}$

conv. $=16/5$

i. $\sum_{n=1}^{\infty}\frac{(-1)^n}{n+2}$ AST conv. (conditionally)

$a_n=\frac{1}{n+2}$ $a_{n+1}=\frac{1}{(n+1)+2}=\frac{1}{n+3}$

1) $\lim_{n\to\infty}\frac{1}{n+2}=0$

2) $a_{n+1}\le a_n$

$\frac{1}{n+3}\le\frac{1}{n+2}$

$\sum_{n=1}^{\infty}\frac{1}{n+2}$ div. LCT to $\sum_{n=1}^{\infty}\frac{1}{n}$

j. $\sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$ div. p-series $p = 1 \leq 1$