

Chapter 9 Infinite Series

Section 9.1 Sequences and the Convergence of a Sequence

A sequence is an ordered LIST. (note: a series in the next section is a SUM)

Definition of Sequence

A **sequence** $\{a_n\}$ is a function whose domain is the set of positive integers. The function values $a_1, a_2, a_3, \dots, a_n, \dots$ are the **terms** of the sequence. The number a_n is the ***n*th term** of the sequence.

THEOREM 9.1 Limit of a Sequence

Let L be a real number. Let f be a function of a real variable such that

$$\lim_{x \rightarrow \infty} f(x) = L.$$

If $\{a_n\}$ is a sequence such that $f(n) = a_n$ for every positive integer n , then

$$\lim_{n \rightarrow \infty} a_n = L.$$

THEOREM 9.2 Properties of Limits of Sequences

Let $\lim_{n \rightarrow \infty} a_n = L$ and $\lim_{n \rightarrow \infty} b_n = K$.

- | | |
|--|--|
| 1. $\lim_{n \rightarrow \infty} (a_n \pm b_n) = L \pm K$ | 2. $\lim_{n \rightarrow \infty} ca_n = cL$, c is any real number |
| 3. $\lim_{n \rightarrow \infty} (a_n b_n) = LK$ | 4. $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{L}{K}$, $b_n \neq 0$ and $K \neq 0$ |

Definition of the Limit of a Sequence

Let L be a real number. The **limit** of a sequence $\{a_n\}$ is L , written as

$$\lim_{n \rightarrow \infty} a_n = L$$

If the limit L of a sequence exists, then the sequence **converges** to L . If the limit of a sequence does not exist, then the sequence **diverges**.

THEOREM 9.4 Absolute Value Theorem

For the sequence $\{a_n\}$, if

$$\lim_{n \rightarrow \infty} |a_n| = 0 \quad \text{then} \quad \lim_{n \rightarrow \infty} a_n = 0.$$

Definition of a Monotonic Sequence

A sequence $\{a_n\}$ is **monotonic** if its terms are nondecreasing

$$a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_n \leq \cdots$$

or if its terms are nonincreasing

$$a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_n \geq \cdots.$$

A famous *recursive* sequence:

The **Fibonacci numbers** or **Fibonacci sequence** are the numbers in the following [integer sequence](#):

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ...

By definition, the first two Fibonacci numbers are 0 and 1, and each subsequent number is the sum of the previous two. Some sources omit the initial 0, instead beginning the sequence with two 1s.

In mathematical terms, the sequence F_n of Fibonacci numbers is defined by the [recurrence relation](#) or recursive formula:

$$F_n = F_{n-1} + F_{n-2},$$

With initial values

$$F_0 = 0 \quad \text{and} \quad F_1 = 1.$$

The Fibonacci sequence is named after [Leonardo of Pisa](#), who was known as Fibonacci

Definition of a Bounded Sequence

1. A sequence $\{a_n\}$ is **bounded above** if there is a real number M such that $a_n \leq M$ for all n . The number M is called an **upper bound** of the sequence.
2. A sequence $\{a_n\}$ is **bounded below** if there is a real number N such that $N \leq a_n$ for all n . The number N is called a **lower bound** of the sequence.
3. A sequence $\{a_n\}$ is **bounded** if it is bounded above and bounded below.

THEOREM 9.5 Bounded Monotonic Sequences

If a sequence $\{a_n\}$ is bounded and monotonic, then it converges.

Section 9.2 Series and Infinite Series

A series (infinite series) is a sum or summation (sigma notation).

Definitions of Convergent and Divergent Series

For the infinite series $\sum_{n=1}^{\infty} a_n$, the **n th partial sum** is given by

$$S_n = a_1 + a_2 + \cdots + a_n.$$

If the sequence of partial sums $\{S_n\}$ converges to S , then the series $\sum_{n=1}^{\infty} a_n$ **converges**. The limit S is called the **sum of the series**.

$$S = a_1 + a_2 + \cdots + a_n + \cdots$$

If $\{S_n\}$ diverges, then the series **diverges**.

The question “*Does a (infinite) series converge (does it sum to a finite value?) and if so, can that sum be determined?*”

Types of series: Geometric Series, Telescoping Series and Convergence

Geometric Series

The sum of a *finite* geometric series:

***n*th Partial Sum of a Geometric Series**

The *n*th partial sum of the geometric series $\sum_{n=0}^{\infty} ar^n$ is

$$S_n = \frac{a(1 - r^{n+1})}{1 - r}, \quad r \neq 1.$$

The sum of an *infinite* convergent geometric series:

THEOREM 9.6 Convergence of a Geometric Series

A geometric series with ratio r diverges if $|r| \geq 1$. If $0 < |r| < 1$, then the series converges to the sum

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1 - r}, \quad 0 < |r| < 1.$$

Telescoping Series

A **telescoping series** is a series such that

$$\sum_{n=1}^{\infty} (a_n - a_{n+1}) = (a_1 - a_2) + (a_2 - a_3) + (a_3 - a_4) + \dots$$

THEOREM 9.7 Properties of Infinite Series

If $\sum a_n = A$, $\sum b_n = B$, and c is a real number, then the following series converge to the indicated sums.

1. $\sum_{n=1}^{\infty} c a_n = cA$

2. $\sum_{n=1}^{\infty} (a_n + b_n) = A + B$

3. $\sum_{n=1}^{\infty} (a_n - b_n) = A - B$

THEOREM 9.8 Limit of n th Term of a Convergent Series

If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n \rightarrow \infty} a_n = 0$.

However, the converse is not necessarily true!

**** BUT**, the *contrapositive* is true, and follows:

(an important and useful theorem to show DIVERGENCE only)

THEOREM 9.9 n th-Term Test for Divergence

If $\lim_{n \rightarrow \infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Section 9.3 The Integral Test and p -series

THEOREM 9.10 The Integral Test

If f is positive, continuous, and decreasing for $x \geq 1$ and $a_n = f(n)$, then

$$\sum_{n=1}^{\infty} a_n \quad \text{and} \quad \int_1^{\infty} f(x) dx$$

either both converge or both diverge.

Note: Useful for positive termed series, and easily integrated functions. If the improper integral converges, that only states that the series also converges and does NOT imply the value of the improper integral is the sum of the series.

THEOREM 9.11 Convergence of p -Series

The p -series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \dots$$

1. converges if $p > 1$, and
2. diverges if $0 < p \leq 1$.

Section 9.4 Comparison Tests

Direct Comparison Test (DCT)

THEOREM 9.12 Direct Comparison Test

Let $0 < a_n \leq b_n$ for all n .

1. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
2. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

The Limit Comparison Test (LCT)

(Note: useful for positive termed series that are polynomial-like)

THEOREM 9.13 Limit Comparison Test

Suppose that $a_n > 0$, $b_n > 0$, and

$$\lim_{n \rightarrow \infty} \left(\frac{a_n}{b_n} \right) = L$$

where L is *finite and positive*. Then the two series $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

Section 9.5 Alternating Series and Convergence

THEOREM 9.14 Alternating Series Test

Let $a_n > 0$. The alternating series

$$\sum_{n=1}^{\infty} (-1)^n a_n \quad \text{and} \quad \sum_{n=1}^{\infty} (-1)^{n+1} a_n$$

converge if the following two conditions are met.

1. $\lim_{n \rightarrow \infty} a_n = 0$
2. $a_{n+1} \leq a_n$, for all n

THEOREM 9.15 Alternating Series Remainder

If a convergent alternating series satisfies the condition $a_{n+1} \leq a_n$, then the absolute value of the remainder R_N involved in approximating the sum S by S_N is less than (or equal to) the first neglected term. That is,

$$|S - S_N| = |R_N| \leq a_{N+1}.$$

Absolute Convergence

THEOREM 9.16 Absolute Convergence

If the series $\sum |a_n|$ converges, then the series $\sum a_n$ also converges.

Definitions of Absolute and Conditional Convergence

1. $\sum a_n$ is **absolutely convergent** if $\sum |a_n|$ converges.
2. $\sum a_n$ is **conditionally convergent** if $\sum a_n$ converges but $\sum |a_n|$ diverges.

Section 9.6 Ratio and Root Test

THEOREM 9.17 Ratio Test

Let $\sum a_n$ be a series with nonzero terms.

1. $\sum a_n$ converges absolutely if $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$.
2. $\sum a_n$ diverges if $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| > 1$ or $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty$.
3. The Ratio Test is inconclusive if $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$.

(Note: useful for “quickly converging series”, factorials and exponentials)

THEOREM 9.18 Root Test

Let $\sum a_n$ be a series.

1. $\sum a_n$ converges absolutely if $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} < 1$.
2. $\sum a_n$ diverges if $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} > 1$ or $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \infty$.
3. The Root Test is inconclusive if $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = 1$.

Summary of Testing Convergence and Divergence for a Series

Guidelines for Testing a Series for Convergence or Divergence

1. Does the n th term approach 0? If not, the series diverges.
2. Is the series one of the special types—geometric, p -series, telescoping, or alternating?
3. Can the Integral Test, the Root Test, or the Ratio Test be applied?
4. Can the series be compared favorably to one of the special types?

Summary of Tests for Series

Test	Series	Condition(s) of Convergence	Condition(s) of Divergence	Comment
n th-Term	$\sum_{n=1}^{\infty} a_n$		$\lim_{n \rightarrow \infty} a_n \neq 0$	This test cannot be used to show convergence.
Geometric Series	$\sum_{n=0}^{\infty} ar^n$	$ r < 1$	$ r \geq 1$	Sum: $S = \frac{a}{1-r}$
Telescoping Series	$\sum_{n=1}^{\infty} (b_n - b_{n+1})$	$\lim_{n \rightarrow \infty} b_n = L$		Sum: $S = b_1 - L$
p -Series	$\sum_{n=1}^{\infty} \frac{1}{n^p}$	$p > 1$	$p \leq 1$	
Alternating Series	$\sum_{n=1}^{\infty} (-1)^{n-1} a_n$	$0 < a_{n+1} \leq a_n$ and $\lim_{n \rightarrow \infty} a_n = 0$		Remainder: $ R_N \leq a_{N+1}$

Summary of Tests for Series

Test	Series	Condition(s) of Convergence	Condition(s) of Divergence	Comment
Integral (f is continuous, positive, and decreasing)	$\sum_{n=1}^{\infty} a_n$, $a_n = f(n) \geq 0$	$\int_1^{\infty} f(x) dx$ converges	$\int_1^{\infty} f(x) dx$ diverges	Remainder: $0 < R_N < \int_N^{\infty} f(x) dx$
Root	$\sum_{n=1}^{\infty} a_n$	$\lim_{n \rightarrow \infty} \sqrt[n]{ a_n } < 1$	$\lim_{n \rightarrow \infty} \sqrt[n]{ a_n } > 1$	Test is inconclusive if $\lim_{n \rightarrow \infty} \sqrt[n]{ a_n } = 1$.
Ratio	$\sum_{n=1}^{\infty} a_n$	$\lim_{n \rightarrow \infty} \left \frac{a_{n+1}}{a_n} \right < 1$	$\lim_{n \rightarrow \infty} \left \frac{a_{n+1}}{a_n} \right > 1$	Test is inconclusive if $\lim_{n \rightarrow \infty} \left \frac{a_{n+1}}{a_n} \right = 1$.
Direct Comparison ($a_n, b_n > 0$)	$\sum_{n=1}^{\infty} a_n$	$0 < a_n \leq b_n$ and $\sum_{n=1}^{\infty} b_n$ converges	$0 < b_n \leq a_n$ and $\sum_{n=1}^{\infty} b_n$ diverges	
Limit Comparison ($a_n, b_n > 0$)	$\sum_{n=1}^{\infty} a_n$	$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = L > 0$ and $\sum_{n=1}^{\infty} b_n$ converges	$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = L > 0$ and $\sum_{n=1}^{\infty} b_n$ diverges	

Section 9.7 Taylor and Maclaurin Polynomials

Definitions of n th Taylor Polynomial and n th Maclaurin Polynomial

If f has n derivatives at c , then the polynomial

$$P_n(x) = f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x - c)^n$$

is called the **n th Taylor polynomial for f at c** . If $c = 0$, then

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n$$

is also called the **n th Maclaurin polynomial for f** .

THEOREM 9.19 Taylor's Theorem

If a function f is differentiable through order $n + 1$ in an interval I containing c , then, for each x in I , there exists z between x and c such that

$$f(x) = f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x - c)^n + R_n(x)$$

where

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n + 1)!}(x - c)^{n+1}.$$

Section 9.8 Power Series

Definition of Power Series

If x is a variable, then an infinite series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots$$

is called a **power series**. More generally, an infinite series of the form

$$\sum_{n=0}^{\infty} a_n (x - c)^n = a_0 + a_1 (x - c) + a_2 (x - c)^2 + \cdots + a_n (x - c)^n + \cdots$$

is called a **power series centered at c** , where c is a constant.

THEOREM 9.20 Convergence of a Power Series

For a power series centered at c , precisely one of the following is true.

1. The series converges only at c .
2. There exists a real number $R > 0$ such that the series converges absolutely for $|x - c| < R$, and diverges for $|x - c| > R$.
3. The series converges absolutely for all x .

The number R is the **radius of convergence** of the power series. If the series converges only at c , the radius of convergence is $R = 0$, and if the series converges for all x , the radius of convergence is $R = \infty$. The set of all values of x for which the power series converges is the **interval of convergence** of the power series.

THEOREM 9.21 Properties of Functions Defined by Power Series

If the function given by

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} a_n(x - c)^n \\ &= a_0 + a_1(x - c) + a_2(x - c)^2 + a_3(x - c)^3 + \cdots \end{aligned}$$

has a radius of convergence of $R > 0$, then, on the interval $(c - R, c + R)$, f is differentiable (and therefore continuous). Moreover, the derivative and anti-derivative of f are as follows.

1.
$$\begin{aligned} f'(x) &= \sum_{n=1}^{\infty} n a_n(x - c)^{n-1} \\ &= a_1 + 2a_2(x - c) + 3a_3(x - c)^2 + \cdots \end{aligned}$$
2.
$$\begin{aligned} \int f(x) dx &= C + \sum_{n=0}^{\infty} a_n \frac{(x - c)^{n+1}}{n + 1} \\ &= C + a_0(x - c) + a_1 \frac{(x - c)^2}{2} + a_2 \frac{(x - c)^3}{3} + \cdots \end{aligned}$$

The *radius of convergence* of the series obtained by differentiating or integrating a power series is the same as that of the original power series. The *interval of convergence*, however, may differ as a result of the behavior at the endpoints.

Operations with Power Series

Let $f(x) = \sum a_n x^n$ and $g(x) = \sum b_n x^n$.

1. $f(kx) = \sum_{n=0}^{\infty} a_n k^n x^n$

2. $f(x^N) = \sum_{n=0}^{\infty} a_n x^{nN}$

3. $f(x) \pm g(x) = \sum_{n=0}^{\infty} (a_n \pm b_n) x^n$

THEOREM 9.22 The Form of a Convergent Power Series

If f is represented by a power series $f(x) = \sum a_n (x - c)^n$ for all x in an open interval I containing c , then $a_n = f^{(n)}(c)/n!$ and

$$f(x) = f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x - c)^n + \cdots$$

Section 9.10 Taylor and Maclaurin Series

Definitions of Taylor and Maclaurin Series

If a function f has derivatives of all orders at $x = c$, then the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x - c)^n = f(c) + f'(c)(x - c) + \cdots + \frac{f^{(n)}(c)}{n!} (x - c)^n + \cdots$$

is called the **Taylor series for $f(x)$ at c** . Moreover, if $c = 0$, then the series is the **Maclaurin series for f** .

THEOREM 9.23 Convergence of Taylor Series

If $\lim_{n \rightarrow \infty} R_n = 0$ for all x in the interval I , then the Taylor series for f converges and equals $f(x)$,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x - c)^n.$$

Guidelines for Finding a Taylor Series

1. Differentiate $f(x)$ several times and evaluate each derivative at c .

$$f(c), f'(c), f''(c), f'''(c), \cdots, f^{(n)}(c), \cdots$$

Try to recognize a pattern in these numbers.

2. Use the sequence developed in the first step to form the Taylor coefficients $a_n = f^{(n)}(c)/n!$, and determine the interval of convergence for the resulting power series

$$f(c) + f'(c)(x - c) + \frac{f''(c)}{2!} (x - c)^2 + \cdots + \frac{f^{(n)}(c)}{n!} (x - c)^n + \cdots$$

3. Within this interval of convergence, determine whether or not the series converges to $f(x)$.

Power Series for Elementary Functions

Function	Interval of Convergence
$\frac{1}{x} = 1 - (x - 1) + (x - 1)^2 - (x - 1)^3 + (x - 1)^4 - \dots + (-1)^n (x - 1)^n + \dots$	$0 < x < 2$
$\frac{1}{1 + x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \dots + (-1)^n x^n + \dots$	$-1 < x < 1$
$\ln x = (x - 1) - \frac{(x - 1)^2}{2} + \frac{(x - 1)^3}{3} - \frac{(x - 1)^4}{4} + \dots + \frac{(-1)^{n-1} (x - 1)^n}{n} + \dots$	$0 < x \leq 2$
$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots + \frac{x^n}{n!} + \dots$	$-\infty < x < \infty$
$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \dots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \dots$	$-\infty < x < \infty$
$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots + \frac{(-1)^n x^{2n}}{(2n)!} + \dots$	$-\infty < x < \infty$
$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \dots + \frac{(-1)^n x^{2n+1}}{2n+1} + \dots$	$-1 \leq x \leq 1$
$\arcsin x = x + \frac{x^3}{2 \cdot 3} + \frac{1 \cdot 3x^5}{2 \cdot 4 \cdot 5} + \frac{1 \cdot 3 \cdot 5x^7}{2 \cdot 4 \cdot 6 \cdot 7} + \dots + \frac{(2n)!x^{2n+1}}{(2^n n!)^2 (2n+1)} + \dots$	$-1 \leq x \leq 1$
$(1 + x)^k = 1 + kx + \frac{k(k-1)x^2}{2!} + \frac{k(k-1)(k-2)x^3}{3!} + \frac{k(k-1)(k-2)(k-3)x^4}{4!} + \dots$	$-1 < x < 1^*$

* The convergence at $x = \pm 1$ depends on the value of k .