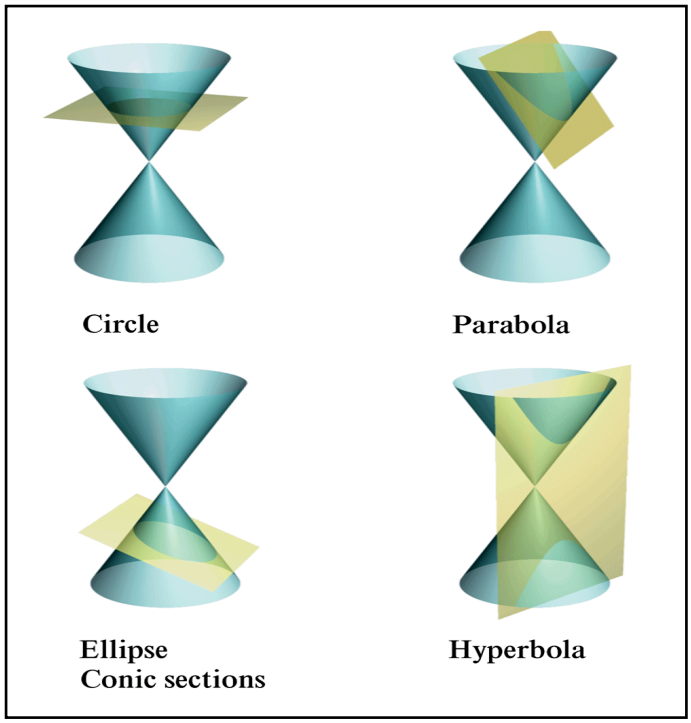
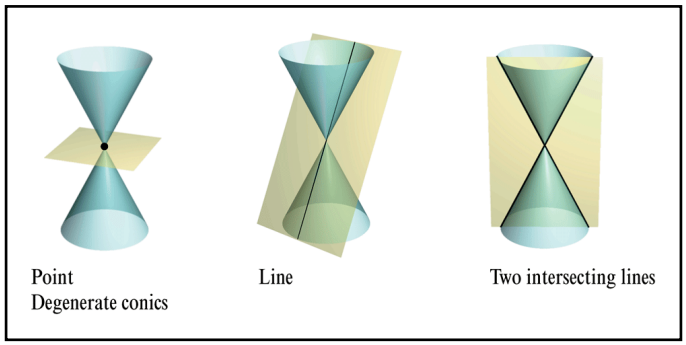
**Section 10.1 Conic Sections**
**A conic (section) is the intersection of a plane and a double napped cone.**




*Degenrate Conics*




***General Second Degree Equation***

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$
$$B = 0, \quad no \;\; rotation$$

$$If \quad A = 0 \;\; or \;\; C = 0 \;\; but \;\; not \;\; both \rightarrow parabola$$
$$If \quad A = C \neq 0 \rightarrow circle$$
$$If \quad AC > 0 \rightarrow ellipse \quad (A \neq C, \quad if \quad A = C \neq 0, \quad circle^*)$$
$$If \;\; AC < 0 \rightarrow hyperbola$$

$assumingno \quad degenerateconics$

## Circles

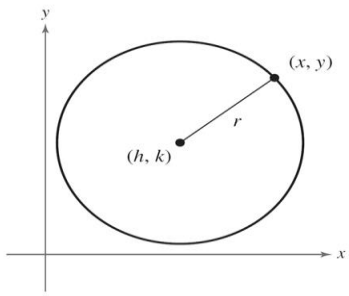



Figure 9.3

**Definition of a Circle**

A **circle** is the set of all points $(x, y)$ in a plane that are equidistant from a fixed point $(h, k)$, called the **center** of the circle. (See Figure 9.3.) The distance $r$ between the center and any point $(x, y)$ on the circle is the **radius.**

**Standard Form of the Equation of a Circle**

The **standard form of the equation of a circle** is

$$(x - h)^2 + (y - k)^2 = r^2.$$

The point $(h, k)$ is the center of the circle, and the positive number $r$ is the radius of the circle. The standard form of the equation of a circle whose center is the origin, $(h, k) = (0, 0)$, is

$$x^2 + y^2 = r^2.$$

## Parabolas:

**Standard Equation of a Parabola** (See the proof on page 737.)

The **standard form of the equation of a parabola** with vertex at $(h, k)$ is as follows.

$(x - h)^2 = 4p(y - k),\ p \neq 0$ — Vertical axis; directrix: $y = k - p$

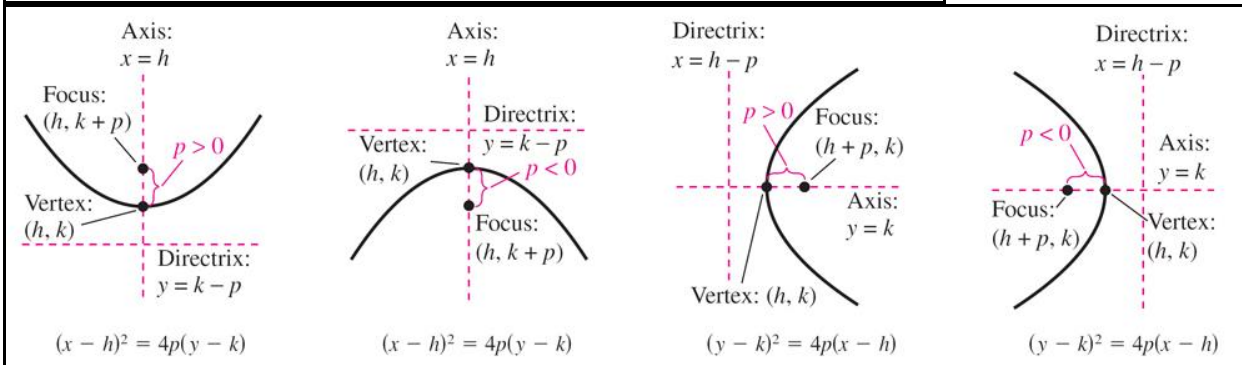
$(y - k)^2 = 4p(x - h),\ p \neq 0$ — Horizontal axis; directrix: $x = h - p$

The focus lies on the axis $p$ units (*directed distance*) from the vertex. If the vertex is at the origin $(0, 0)$, the equation takes one of the following forms.

$x^2 = 4py$ — Vertical axis

$y^2 = 4px$ — Horizontal axis

See Figure 9.8.




$(x - h)^2 = 4p(y - k)$
**(a) Vertical axis: $p > 0$**

$(x - h)^2 = 4p(y - k)$
**(b) Vertical axis: $p < 0$**

$(y - k)^2 = 4p(x - h)$
**(c) Horizontal axis: $p > 0$**

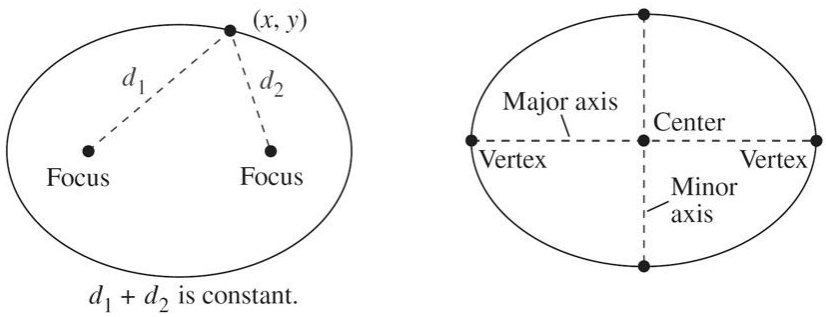
$(y - k)^2 = 4p(x - h)$
**(d) Horizontal axis: $p < 0$**

**Figure 9.8**

**Ellipses:**

### Definition of an Ellipse

An **ellipse** is the set of all points $(x, y)$ in a plane, the sum of whose distances from two distinct fixed points (**foci**) is constant. [See Figure 9.15(a).]




(a) (b)

**Figure 9.15**

### Standard Equation of an Ellipse

The **standard form of the equation of an ellipse** with center $(h, k)$ and major and minor axes of lengths $2a$ and $2b$, respectively, where $0 < b < a$, is

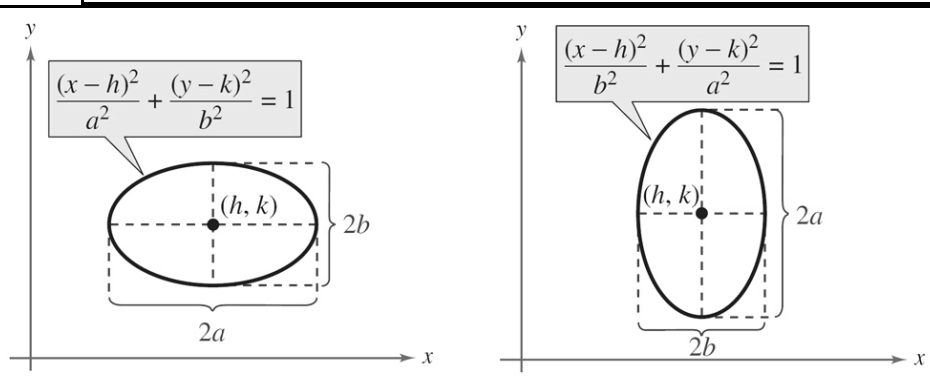
$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 \qquad \text{Major axis is horizontal.}$$

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1. \qquad \text{Major axis is vertical.}$$

The foci lie on the major axis, $c$ units from the center, with $c^2 = a^2 - b^2$. If the center is at the origin $(0, 0)$, the equation takes one of the following forms.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \text{Major axis is horizontal.}$$

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \qquad \text{Major axis is vertical.}$$




***Major axis is horizontal.*** ***Major axis is vertical.***

**Figure 9.18**

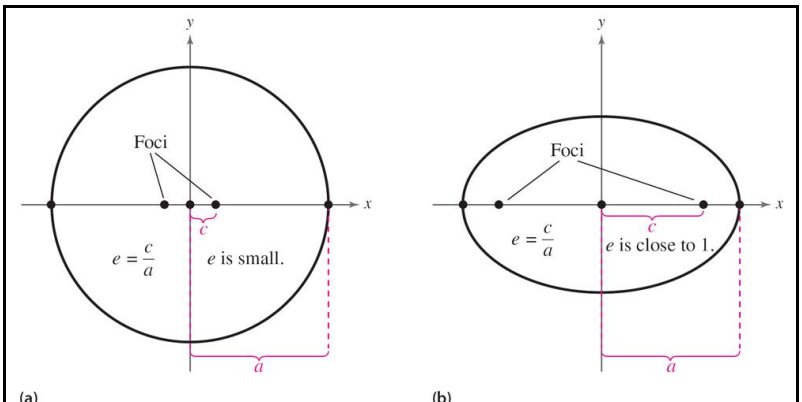



Figure 9.25

**Definition of Eccentricity of an Ellipse**

The **eccentricity** $e$ of an ellipse is given by the ratio

$$e = \frac{c}{a}.$$

**Hyperbolas:**

Definition of a Hyperbola

A **hyperbola** is the set of all points $(x, y)$ in a plane, the difference of whose distances from two distinct fixed points, the **foci,** is a positive constant. [See Figure 9.26(a).]

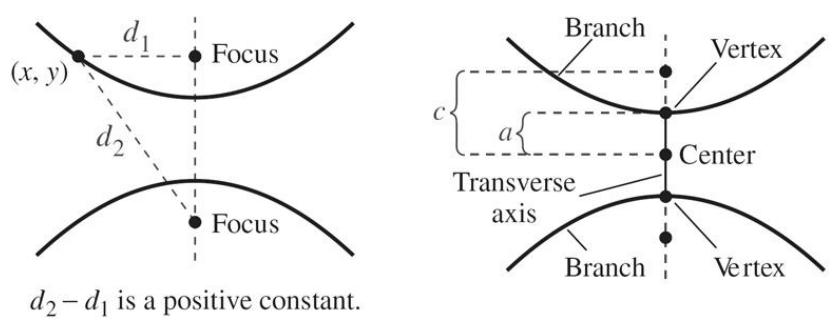



Figure 9.26

## Standard Equation of a Hyperbola

The **standard form of the equation of a hyperbola** with center at $(h, k)$ is

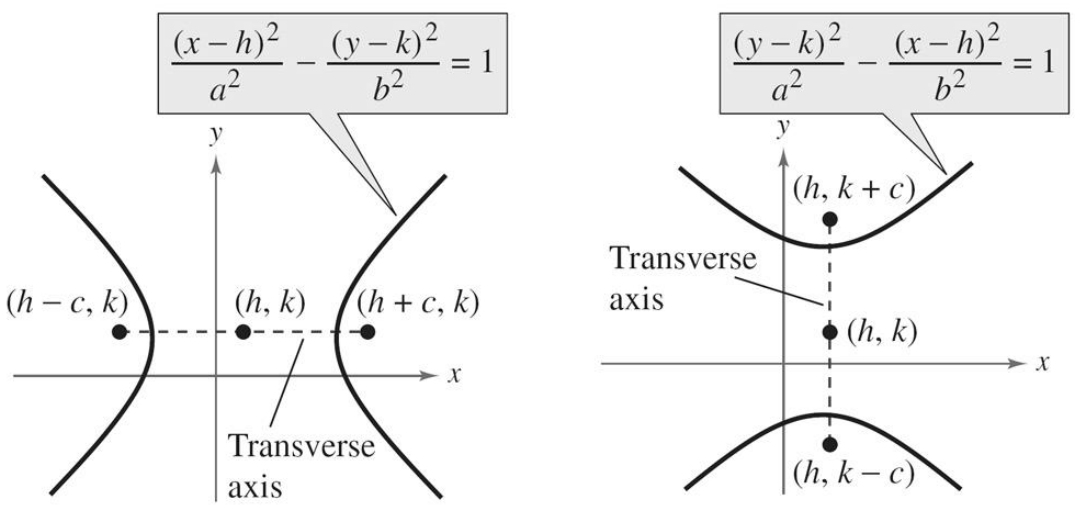
$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \qquad \text{Transverse axis is horizontal.}$$

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1. \qquad \text{Transverse axis is vertical.}$$

The vertices are $a$ units from the center, and the foci are $c$ units from the center. Moreover, $c^2 = a^2 + b^2$. If the center of the hyperbola is at the origin $(0, 0)$, the equation takes one of the following forms.

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad \text{Transverse axis is horizontal.}$$

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \qquad \text{Transverse axis is vertical.}$$




***Transverse axis is horizontal.*** ***Transverse axis is vertical.***

**Figure 9.27**

**THEOREM 10.6 Asymptotes of a Hyperbola**

For a *horizontal* transverse axis, the equations of the asymptotes are

$$y = k + \frac{b}{a}(x-h) \qquad \text{and} \qquad y = k - \frac{b}{a}(x-h).$$

For a *vertical* transverse axis, the equations of the asymptotes are

$$y = k + \frac{a}{b}(x-h) \qquad \text{and} \qquad y = k - \frac{a}{b}(x-h).$$

## Section 10.2 Plane Curve and Parametric Equations

### Definition of a Plane Curve

If $f$ and $g$ are continuous functions of $t$ on an interval $I$, then the equations

$$x = f(t) \quad \text{and} \quad y = g(t)$$

are called **parametric equations** and $t$ is called the **parameter.** The set of points $(x, y)$ obtained as $t$ varies over the interval $I$ is called the **graph** of the parametric equations. Taken together, the parametric equations and the graph are called a **plane curve,** denoted by $C$.

### Definition of a Smooth Curve

A curve $C$ represented by $x = f(t)$ and $y = g(t)$ on an interval $I$ is called **smooth** if $f'$ and $g'$ are continuous on $I$ and not simultaneously 0, except possibly at the endpoints of $I$. The curve $C$ is called **piecewise smooth** if it is smooth on each subinterval of some partition of $I$.

## Section 10.3 Parametric Equations and Calculus

### THEOREM 10.7 Parametric Form of the Derivative

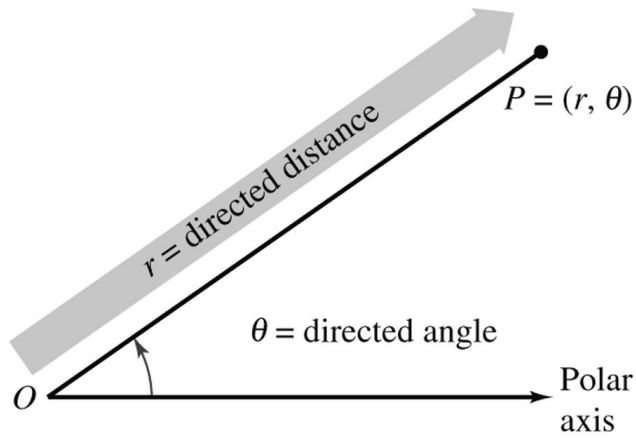
If a smooth curve $C$ is given by the equations $x = f(t)$ and $y = g(t)$, then the slope of $C$ at $(x, y)$ is

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}, \qquad \frac{dx}{dt} \neq 0.$$

**THEOREM 10.8 Arc Length in**

If a smooth curve $C$ is given by $x = f($
intersect itself on the interval $a \le t \le$
the arc length of $C$ over the interval is

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt =$$



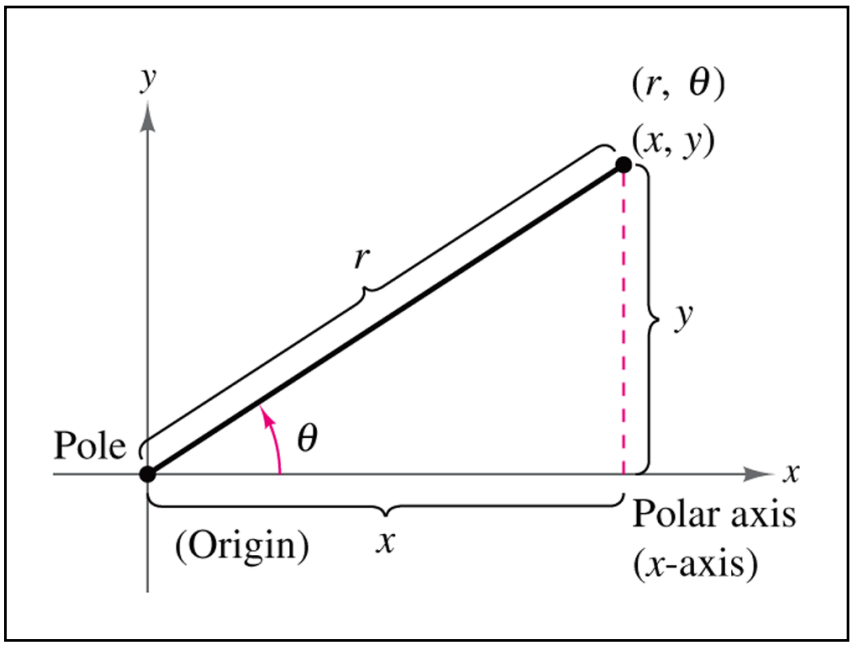

**THEOREM 10.9 Area of a Surface of Revolution**

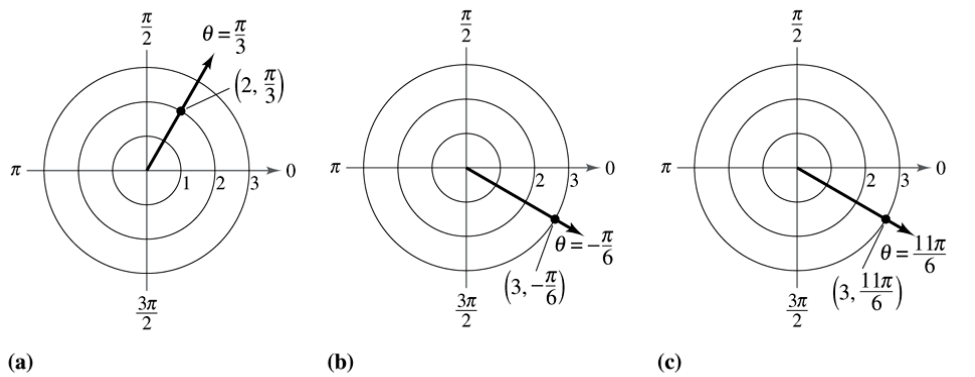
If a smooth curve $C$ given by $x = f(t)$ and $y = g(t)$ does not cross itself on an interval $a \le t \le b$, then the area $S$ of the surface of revolution formed by revolving $C$ about the coordinate axes is given by the following.

**1.** $S = 2\pi \int_a^b g(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$ Revolution about the $x$-axis: $g(t) \ge 0$

**2.** $S = 2\pi \int_a^b f(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$ Revolution about the $y$-axis: $f(t) \ge 0$

## Section 10.4 Polar Coordinates






**THEOREM 10.10 Coordinate Conversion**

The polar coordinates $(r, \theta)$ of a point are related to the rectangular coordinates $(x, y)$ of the point as follows.

**1.** $x = r\cos\theta$

$y = r\sin\theta$

**2.** $\tan\theta = \dfrac{y}{x}$

$r^2 = x^2 + y^2$

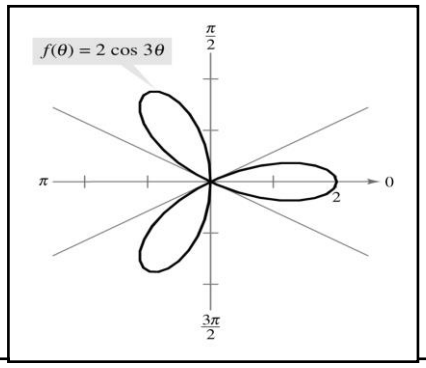
**THEOREM 10.11 Slope in Polar Form**

If $f$ is a differentiable function of $\theta$, then the *slope* of the tangent line to the graph of $r = f(\theta)$ at the point $(r, \theta)$ is

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f(\theta)\cos\theta + f'(\theta)\sin\theta}{-f(\theta)\sin\theta + f'(\theta)\cos\theta}$$

provided that $dx/d\theta \neq 0$ at $(r, \theta)$. (See Figure 10.45.)

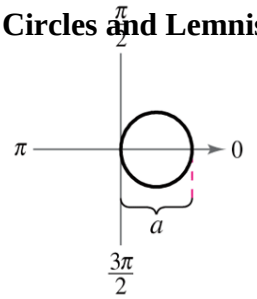
# Section 10.5 Graphs of Polar Equations



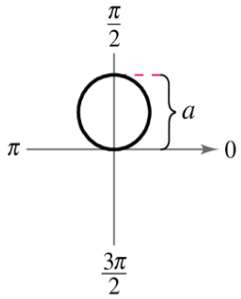

**THEOREM 10.12 Tangent Lines at the Pole**

If $f(\alpha) = 0$ and $f'(\alpha) \neq 0$, then the line $\theta = \alpha$ is tangent at the pole to the graph of $r = f(\theta)$.

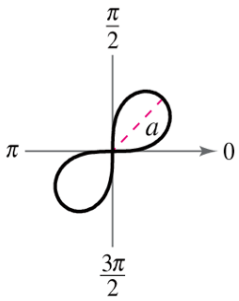
## Circles and Lemniscates



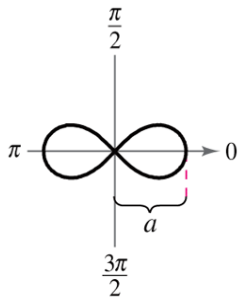

$r = a\cos\theta$
Circle




$r = a\sin\theta$
Circle




$r^2 = a^2\sin 2\theta$
Lemniscate




$r^2 = a^2\cos 2\theta$
Lemniscate

*Circles and Lemniscates*

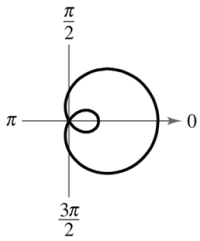
## Limacons

*Limaçons*

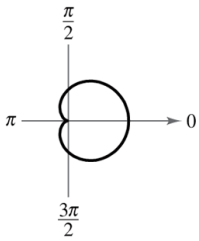
$r = a \pm b\cos\theta$

$r = a \pm b\sin\theta$

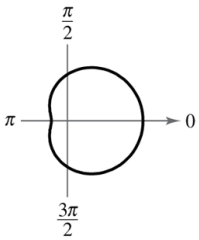
$(a > 0, b > 0)$



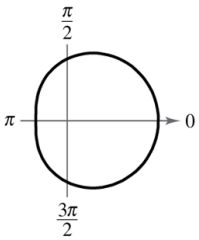

$\frac{a}{b} < 1$
Limaçon with inner loop




$\frac{a}{b} = 1$
Cardioid (heart-shaped)




$1 < \frac{a}{b} < 2$
Dimpled limaçon



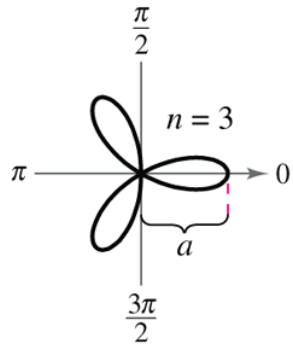

$\frac{a}{b} \geq 2$
Convex limaçon

**Rose Curves**

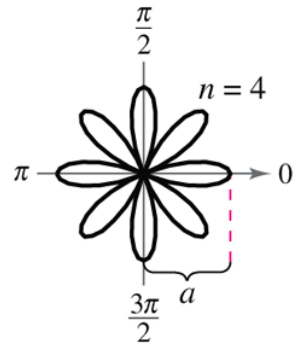
*Rose Curves*

$n$ petals if $n$ is odd

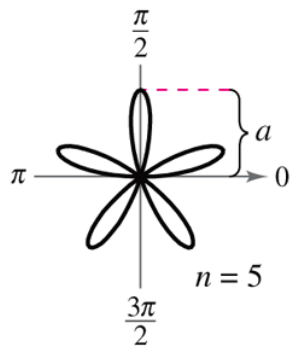
$2n$ petals if $n$ is even ($n \geq 2$)



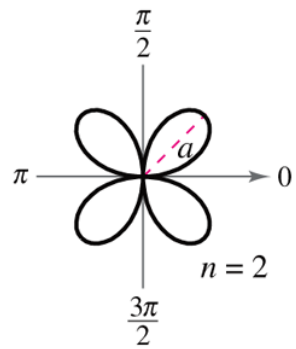

$r = a \cos n\theta$
Rose curve




$r = a \cos n\theta$
Rose curve

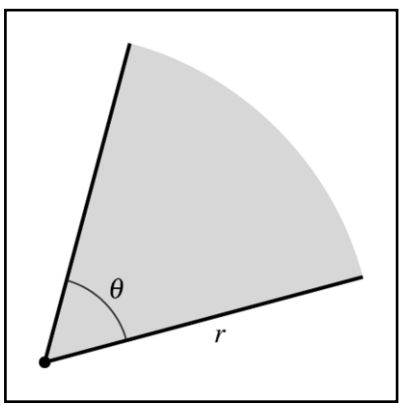


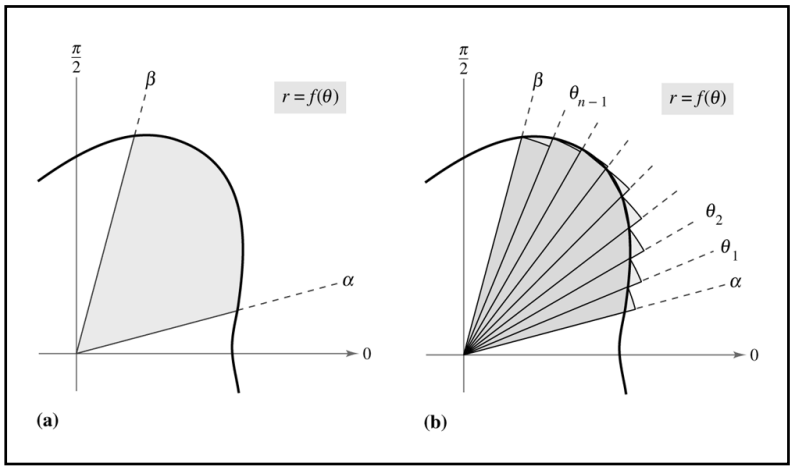

$r = a \sin n\theta$
Rose curve




$r = a \sin n\theta$
Rose curve

## Section 9.6 Area of a region bounded by the graph of a Polar Curve



**THEOREM 10.13 Area in Polar Coordinates**

If $f$ is continuous and nonnegative on the interval $[\alpha, \beta]$, $0 < \beta - \alpha \leq 2\pi$, then the area of the region bounded by the graph of $r = f(\theta)$ between the radial lines $\theta = \alpha$ and $\theta = \beta$ is given by

$$A = \frac{1}{2}\int_{\alpha}^{\beta} [f(\theta)]^2\, d\theta$$

$$= \frac{1}{2}\int_{\alpha}^{\beta} r^2\, d\theta. \qquad 0 < \beta - \alpha \leq 2\pi$$

**THEOREM 10.14 Arc Length of a Polar Curve**

Let $f$ be a function whose derivative is continuous on an interval $\alpha \leq \theta \leq \beta$. The length of the graph of $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ is

$$s = \int_{\alpha}^{\beta} \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

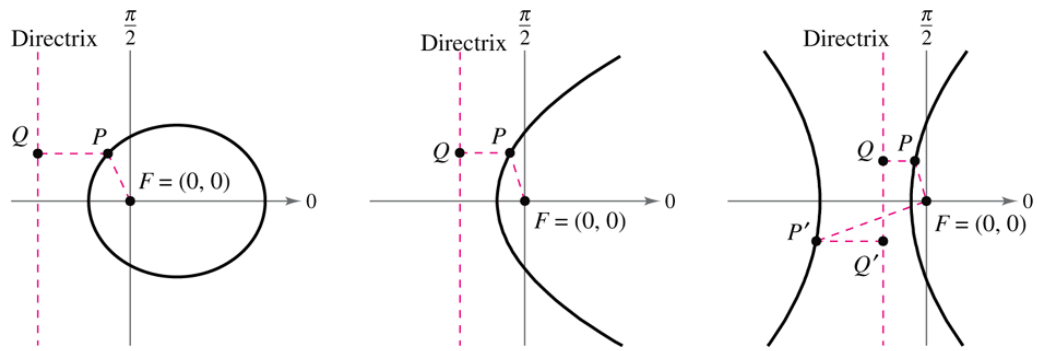
**THEOREM 10.15 Area of a Surface of Revolution**

Let $f$ be a function whose derivative is continuous on an interval $\alpha \leq \theta \leq \beta$. The area of the surface formed by revolving the graph of $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ about the indicated line is as follows.

**1.** $S = 2\pi \int_{\alpha}^{\beta} f(\theta) \sin\theta \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta$ About the polar axis

**2.** $S = 2\pi \int_{\alpha}^{\beta} f(\theta) \cos\theta \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta$ About the line $\theta = \frac{\pi}{2}$

# Section 9.7 Polar Equations of Conics




Ellipse: $0 < e < 1$
$\frac{PF}{PQ} < 1$

Parabola: $e = 1$
$PF = PQ$

Hyperbola: $e > 1$
$\frac{PF}{PQ} = \frac{P'F}{P'Q'} > 1$

**THEOREM 10.16 Classification of Conics by Eccentricity**

Let $F$ be a fixed point (*focus*) and $D$ be a fixed line (*directrix*) in the plane. Let $P$ be another point in the plane and let $e$ (*eccentricity*) be the ratio of the distance between $P$ and $F$ to the distance between $P$ and $D$. The collection of all points $P$ with a given eccentricity is a conic.

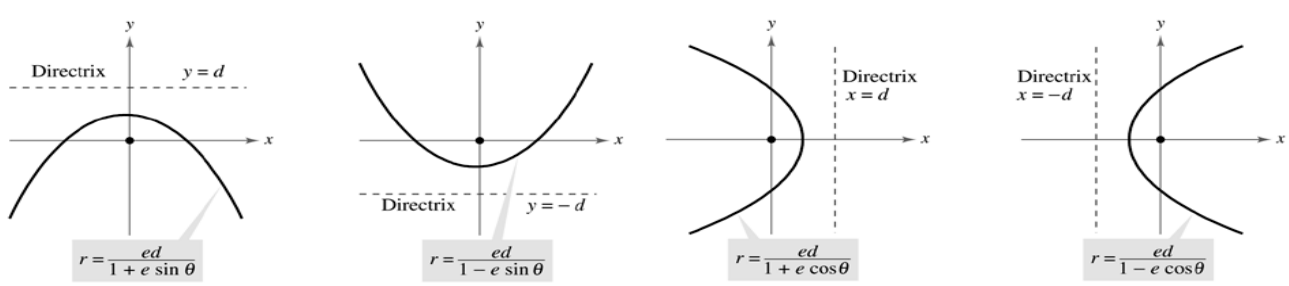
1. The conic is an ellipse if $0 < e < 1$.
2. The conic is a parabola if $e = 1$.
3. The conic is a hyperbola if $e > 1$.

**THEOREM 10.17 Polar Equations of Conics**

The graph of a polar equation of the form

$$r = \frac{ed}{1 \pm e\cos\theta} \quad \text{or} \quad r = \frac{ed}{1 \pm e\sin\theta}$$

is a conic, where $e > 0$ is the eccentricity and $|d|$ is the distance between the focus at the pole and its corresponding directrix.




**(a)** **(b)** **(c)** **(d)**

The four types of polar equations for a parabola