Chapter 2 Sections: 1-7 Polynomial and Rational Functions

Section 2.1 Quadratic Functions

Constant Function: $f(x) = c$ or $y = c$ horizontal line

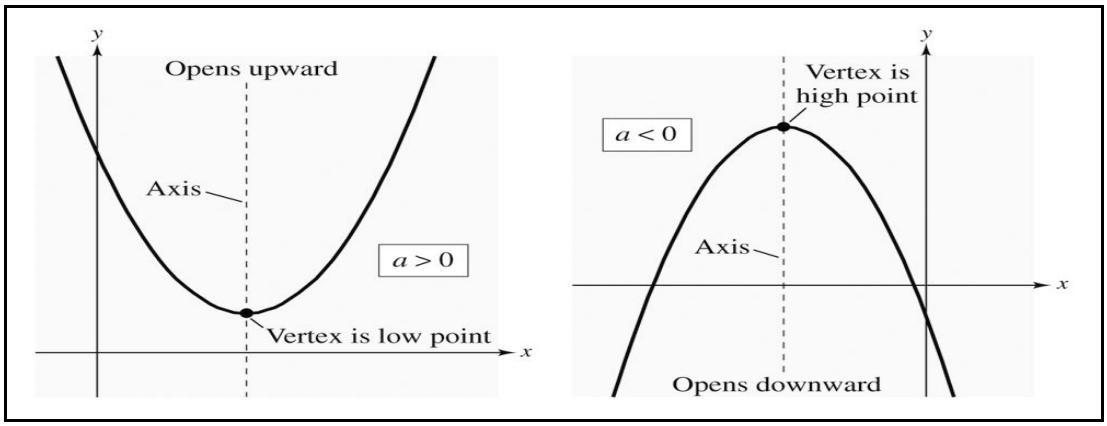
Linear Function: $f(x) = ax + b$ or $y = mx + b$ line with slope = m, $m \neq 0$

Definition of a Quadratic Function

Let a, b, and c be real numbers with $a \neq 0$. The function of x given by

$$f(x) = ax^2 + bx + c \qquad \text{Quadratic function}$$

is called a quadratic function.



Standard Form of a Quadratic Function

The quadratic function given by

$$f(x) = a(x - h)^2 + k, \quad a \neq 0$$

is said to be in **standard form.** The graph of f is a parabola whose axis is the vertical line $x = h$ and whose vertex is the point (h, k). If $a > 0$, the parabola opens upward, and if $a < 0$, the parabola opens downward.

Vertex of a parabola or quadratic function

Vertex Formula #2	Vertex Formula #1
$\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$	$\left(\frac{-b}{2a}, \frac{4ac - b^2}{4a}\right)$

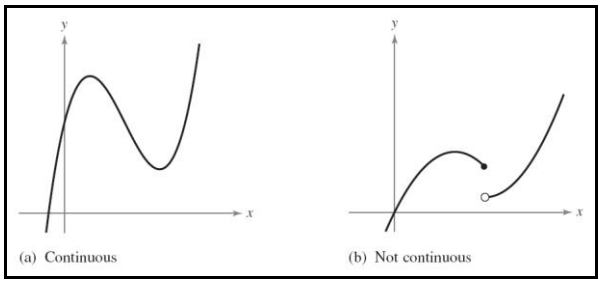
Section 2.2 Higher Degree Polynomials and Polynomial Functions

Definition of a Polynomial Function

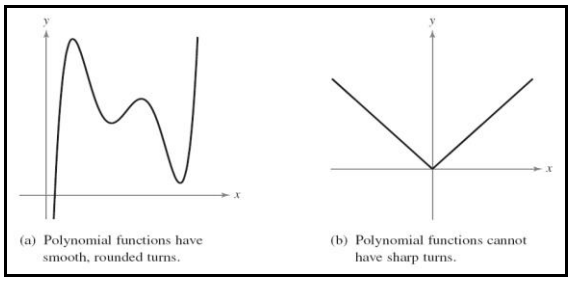
Let n be a nonnegative integer and let $a_n, a_{n-1}, \ldots, a_2, a_1, a_0$ be real numbers with $a_n \neq 0$. The function given by

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$$

is called a **polynomial function of x with degree n.**



(a) Continuous (b) Not continuous



(a) Polynomial functions have smooth, rounded turns. (b) Polynomial functions cannot have sharp turns.

Characteristics of Polynomial functions

- A polynomial function is of the form $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$
- The degree is the highest exponent, n, and the leading coefficient is a_n .
- The graph of a polynomial function is continuous with no breaks.
- The graph of a polynomial function is smooth (rounded) with no sharp turns.
- The function has at most n real zeros.
- The graph has at most $n-1$ turning points; relative extrema

Real Zeros of Polynomial Functions

If f is a polynomial function and a is a real number, then the following statements are equivalent.

1. $x = a$ is a zero of the function f.

2. $x = a$ is a solution of the polynomial equation $f(x) = 0$.

3. $(x - a)$ is a factor of the polynomial $f(x)$.

4. $(a, 0)$ is an x-intercept of the graph of f.

Multiplicity of Zeros

If a real zero occurs k times for a function, it is called a **repeated zero** of **multiplicity** k.

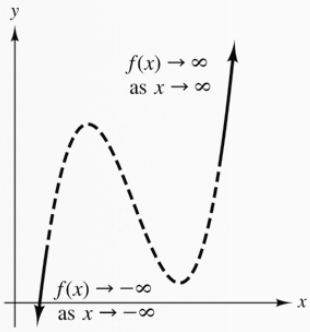
k = odd #, then the graph crosses through the x-axis.

k = even #, then the graph touches the x-axis.

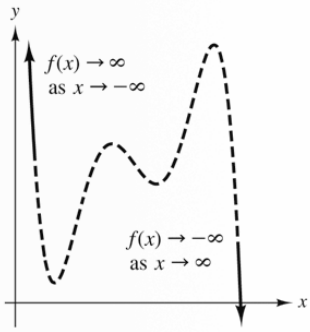
The Leading Coefficient Test

As x moves without bound to the left or to the right, the graph of the polynomial function $f(x) = a_n x^n + \cdot \cdot \cdot + a_1 x + a_0$ eventually rises or falls in the following manner.

1. When n is *odd:*

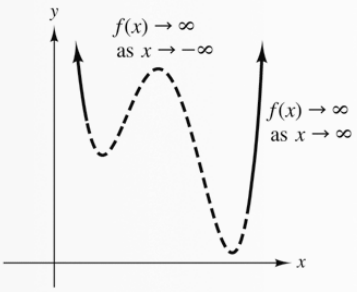


If the leading coefficient is positive ($a_n > 0$), the graph falls to the left and rises to the right.

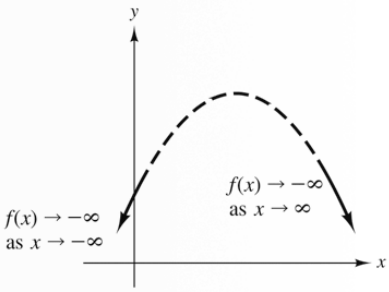


If the leading coefficient is negative ($a_n < 0$), the graph rises to the left and falls to the right.

2. When n is *even:*



If the leading coefficient is positive ($a_n > 0$), the graph rises to the left and right.



If the leading coefficient is negative ($a_n < 0$), the graph falls to the left and right.

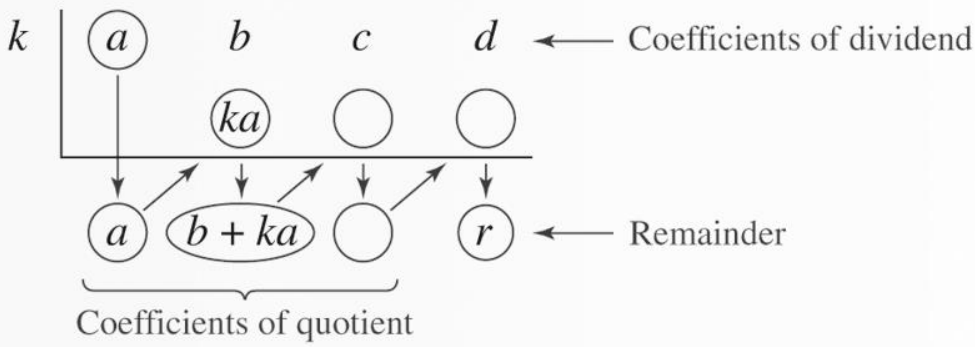
The dashed portions of the graphs indicate that the test determines *only* the right-hand and left-hand behavior of the graph.

Section 2.4: Real Zeros of Polynomial Functions

Polynomial Division: Long Division and Synthetic Division

Synthetic Division

To divide $ax^3 + bx^2 + cx + d$ by $x - k$, use the following pattern.



Vertical pattern: Add terms in columns.

Diagonal pattern: Multiply results by k.

The Division Algorithm

If $f(x)$ and $d(x)$ are polynomials such that $d(x) \neq 0$, and the degree of $d(x)$ is less than or equal to the degree of $f(x)$, there exist unique polynomials $q(x)$ and $r(x)$ such that

$$f(x) = d(x)q(x) + r(x)$$

Dividend, Divisor, Quotient, Remainder

where $r(x) = 0$ or the degree of $r(x)$ is less than the degree of $d(x)$. If the remainder $r(x)$ is zero, $d(x)$ **divides evenly** into $f(x)$.

Remainder Theorem and the Factor Theorem

The remainder r obtained in the synthetic division of $f(x)$ by $x - k$ provides the following information.

1. The remainder r gives the value of f at $x = k$. That is, $r = f(k)$.
2. If $r = 0$, $(x - k)$ is a factor of $f(x)$.
3. If $r = 0$, $(k, 0)$ is an x-intercept of the graph of f.

Real Zeros of Polynomial Functions

The Rational Zero Test

If the polynomial function given by

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdot \cdot \cdot + a_2 x^2 + a_1 x + a_0$$

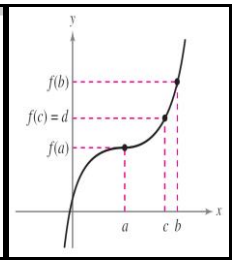
has *integer* coefficients, then every rational zero of f has the form

$$\text{Rational zeros} = \frac{\text{a factor of the constant term } a_0}{\text{a factor of the leading coefficient } a_n} = \frac{p}{q}$$

where p and q have no common factors other than 1.

Intermediate Value Theorem

Let a and b be real numbers such that $a < b$. If f is a polynomial function such that $f(a) \neq f(b)$, then, in the interval $[a, b]$, f takes on every value between $f(a)$ and $f(b)$.



Graphing Calculator Techniques

Zoom-and-Trace Technique

To approximate a real zero of a function with a graphing utility, use the following steps.

1. Graph the function so that the real zero you want to approximate appears as an x-intercept on the screen.
2. Move the cursor near the x-intercept and use the *zoom* feature to zoom in to get a better look at the intercept.
3. Use the *trace* feature to find the x-values that occur just before and just after the x-intercept. If the difference between these values is sufficiently small, use their average as the approximation. If not, continue zooming in until the approximation reaches the desired level of accuracy.

Section 2.4 Complex Numbers

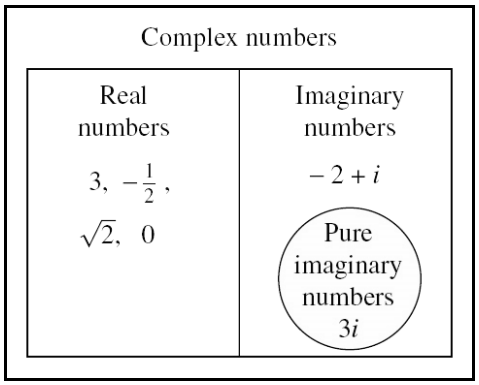
Definition of a Complex Number

If a and b are real numbers, the number $a + bi$ is called a **complex number,** and it is said to be written in **standard form.** If $b = 0$, the number $a + bi = a$ is a real number. If $b \neq 0$, the number $a + bi$ is called an **imaginary number.** A number of the form bi, where $b \neq 0$, is called a **pure imaginary number.**

Principal Square Root of a Negative Number

If a is a positive number, the **principal square root** of the negative number $-a$ is defined as

$$\sqrt{-a} = \sqrt{a}\,i.$$



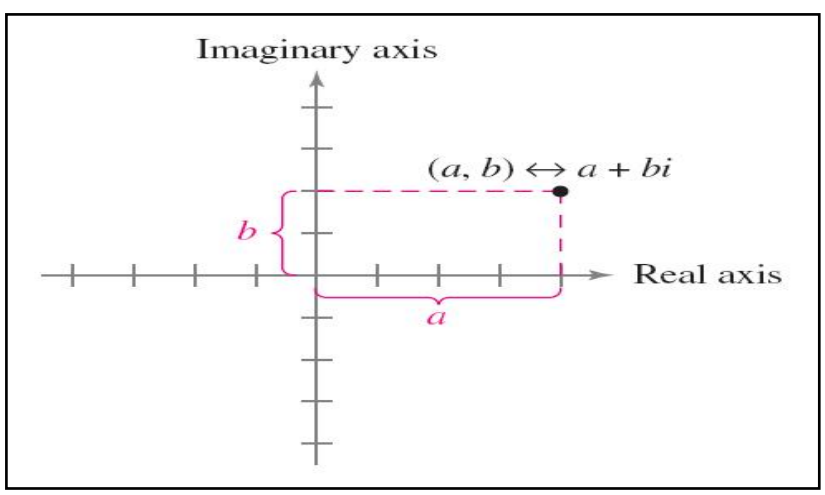
Addition and Subtraction of Complex Numbers

If $a + bi$ and $c + di$ are two complex numbers written in standard form, their sum and difference are defined as follows.

Sum: $(a + bi) + (c + di) = (a + c) + (b + d)i$

Difference: $(a + bi) - (c + di) = (a - c) + (b - d)i$

Complex Plane:



Section 2.5 Fundamental Theorem of Algebra

The Fundamental Theorem of Algebra

If $f(x)$ is a polynomial of degree n, where $n > 0$, then f has at least one zero in the complex number system.

Linear Factorization Theorem

If $f(x)$ is a polynomial of degree n

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where $n > 0$, then $f(x)$ has precisely n linear factors

$$f(x) = a_n(x - c_1)(x - c_2) \cdots (x - c_n)$$

where $c_1, c_2, \ldots, c_n$ are complex numbers and a_n is the leading coefficient of $f(x)$.

Complex Zeros Occur in Conjugate Pairs

Let f be a polynomial function that has *real coefficients*. If $a + bi$, where $b \neq 0$, is a zero of the function, then the conjugate $a - bi$ is also a zero of the function.

Factors of a Polynomial

Every polynomial of degree $n > 0$ with real coefficients can be written as the product of linear and quadratic factors with real coefficients, where the quadratic factors have no real zeros.

Sections 2.6 - 2.7 Rational Functions: graphs and asymptotes.

Let $f(x) = \dfrac{p(x)}{q(x)}$, where p and q are polynomials, and p and q have no common factors,

then f is called a ***rational function***.

Definition of Vertical and Horizontal Asymptotes

1. The line $x = a$ is a **vertical asymptote** of the graph of f if

$$f(x) \to \infty \quad \text{or} \quad f(x) \to -\infty$$

as $x \to a$, either from the right or from the left.

2. The line $y = b$ is a **horizontal asymptote** of the graph of f if

$$f(x) \to b$$

as $x \to \infty$ or $x \to -\infty$.

Asymptotes of a Rational Function

Let f be the rational function given by

$$f(x) = \frac{p(x)}{q(x)} = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0}, \; a_n \neq 0, \; b_m \neq 0.$$

1. The graph of f has *vertical* asymptotes at the zeros of $q(x)$.
2. The graph of f has one or no *horizontal* asymptote determined by comparing the degrees of $p(x)$ and $q(x)$.
   - **a.** If $n < m$, the graph of f has the line $y = 0$ (the x-axis) as a horizontal asymptote.
   - **b.** If $n = m$, the graph of f has the line $y = a_n/b_m$ (ratio of the leading coefficients) as a horizontal asymptote.
   - **c.** If $n > m$, the graph of f has no horizontal asymptote.

If the degree of the numerator is exactly one more than the degree of the denominator,

then there exists a ***slant asymptote*** or an ***oblique asymptote***.

Guidelines for Graphing Rational Functions

Let $f(x) = p(x)/q(x)$, where $p(x)$ and $q(x)$ are polynomials with no common factors.

1. Find and plot the y-intercept (if any) by evaluating $f(0)$.
2. Find the zeros of the numerator (if any) by solving the equation $p(x) = 0$. Then plot the corresponding x-intercepts.
3. Find the zeros of the denominator (if any) by solving the equation $q(x) = 0$. Then sketch the corresponding vertical asymptotes.
4. Find and sketch the horizontal asymptote (if any) by using the rule for finding the horizontal asymptote of a rational function.
5. Test for symmetry.
6. Plot at least one point both *between and beyond* each x-intercept and vertical asymptote.
7. Use smooth curves to complete the graph between and beyond the vertical asymptotes.