Chapter 3: Exponential and Logarithmic Functions

Section 1: Exponential Functions

Definition of Exponential Function

If $a > 0$ and $a \neq 1$, then the **exponential function** with base a is given by

$$f(x) = a^x.$$

Properties of Exponents

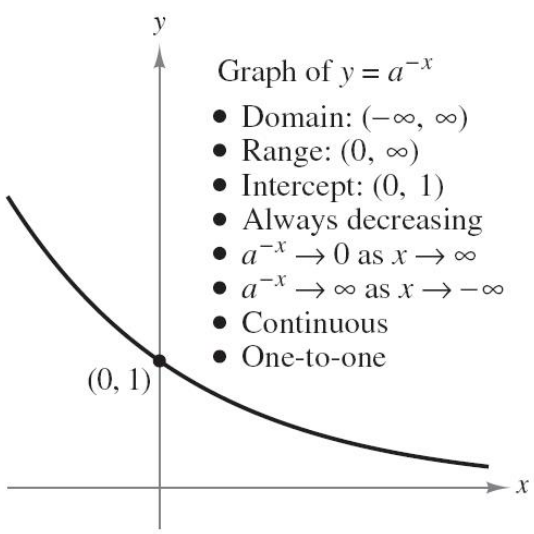
Let a and b be positive numbers.

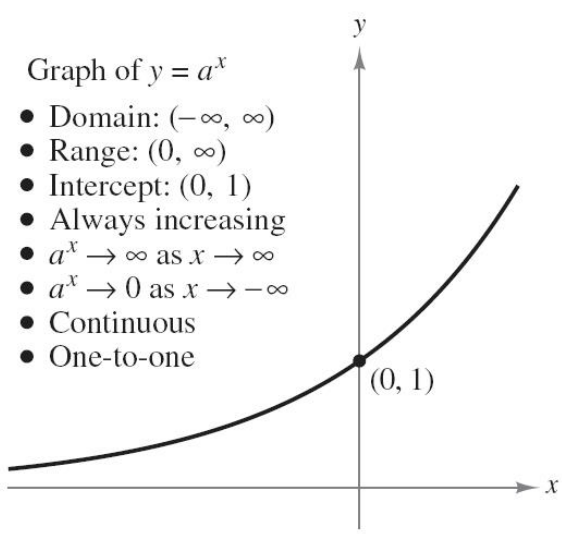
1. $a^0 = 1$ **2.** $a^x a^y = a^{x+y}$ **3.** $\dfrac{a^x}{a^y} = a^{x-y}$

4. $(a^x)^y = a^{xy}$ **5.** $(ab)^x = a^x b^x$ **6.** $\left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}$

7. $a^{-x} = \dfrac{1}{a^x}$

Graphs of exponential functions: (basic)





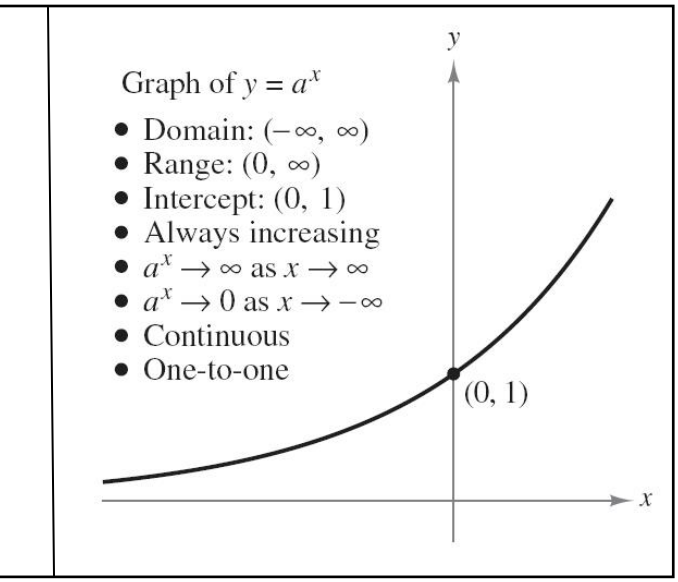
Characteristics of Exponential Functions

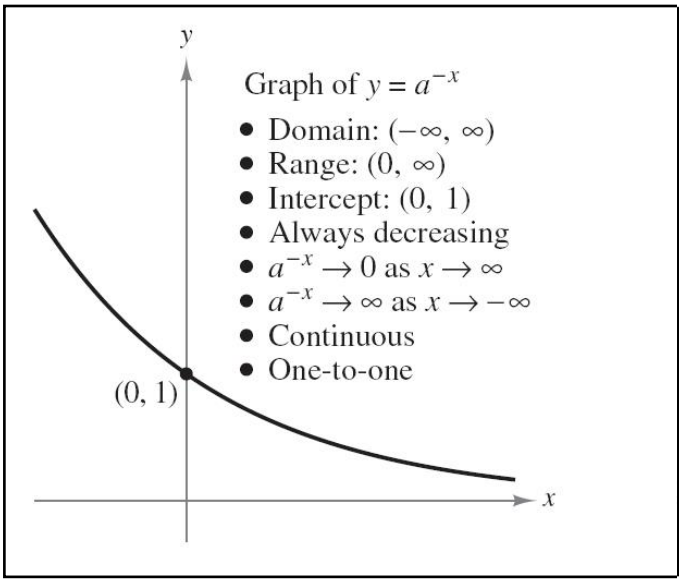
Graph of $y = a^x, a > 1$

- Domain: $(-\infty, \infty)$
- Range: $(0, \infty)$
- Intercept: $(0, 1)$
- Increasing
- x-axis is a horizontal asymptote ($a^x \to 0$ as $x \to -\infty$)
- Continuous

Graph of $y = a^{-x}, a > 1$

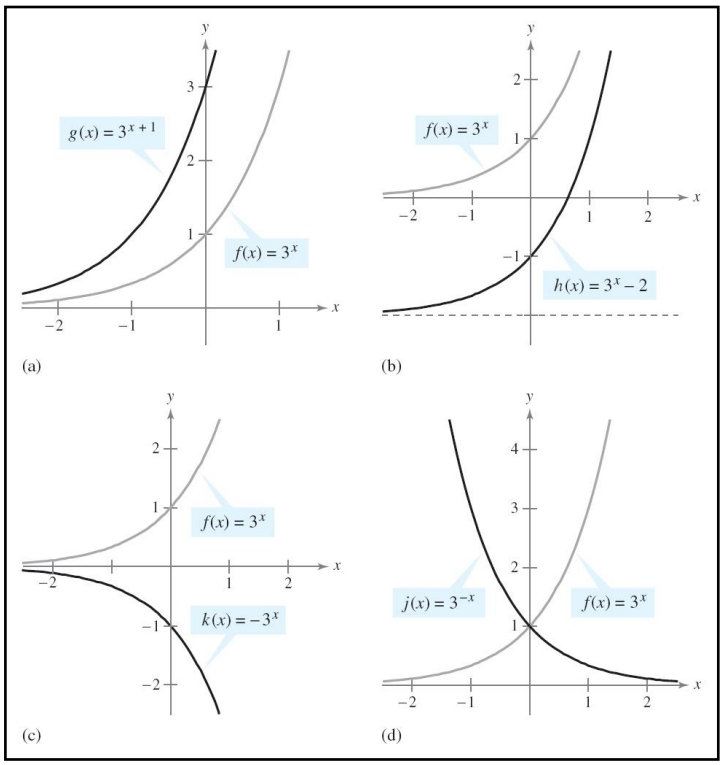
- Domain: $(-\infty, \infty)$
- Range: $(0, \infty)$
- Intercept: $(0, 1)$
- Decreasing
- x-axis is a horizontal asymptote ($a^{-x} \to 0$ as $x \to \infty$)
- Continuous
- Reflection of graph of $y = a^x$ about y-axis





Example:

transformations of $f(x) = 3^x$

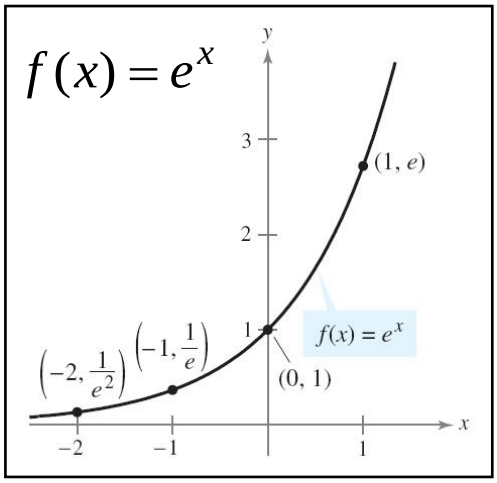


The natural base, exponential function: $f(x) = e^x$

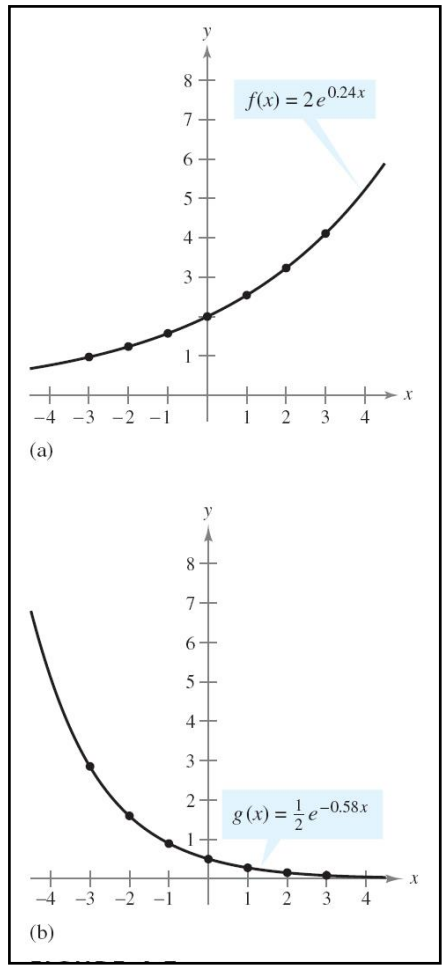
Limit Definition of e

The irrational number e is defined to be the limit of $(1 + x)^{1/x}$ as $x \to 0$. That is,

$$\lim_{x \to 0} (1 + x)^{1/x} = e.$$



examples :



Applications of exponential functions:

Formulas for Compound Interest

After t years, the balance A in an account with principal P and annual interest rate r (in decimal form) is given by the following formulas.

1. For n compoundings per year: $A = P\left(1 + \frac{r}{n}\right)^{nt}$

2. For continuous compounding: $A = Pe^{rt}$

Section 2: Logarithmic Functions and their graphs

Definition of a Logarithmic Function

Definition of a Logarithmic Function

For $x > 0$, $a > 0$, and $a \neq 1$,

$$y = \log_a x \text{ if and only if } x = a^y.$$

The function given by

$$f(x) = \log_a x$$

is called the **logarithmic function with base *a*.**

Properties of Logarithms

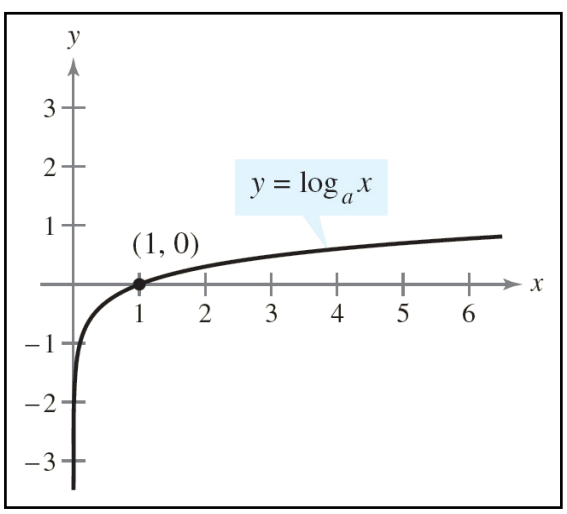
1. $\log_a 1 = 0$ because $a^0 = 1$.

2. $\log_a a = 1$ because $a^1 = a$.

3. $\log_a a^x = x$ and $a^{\log_a x} = x$ — Inverse Properties

4. If $\log_a x = \log_a y$, then $x = y$. — One-to-One Property

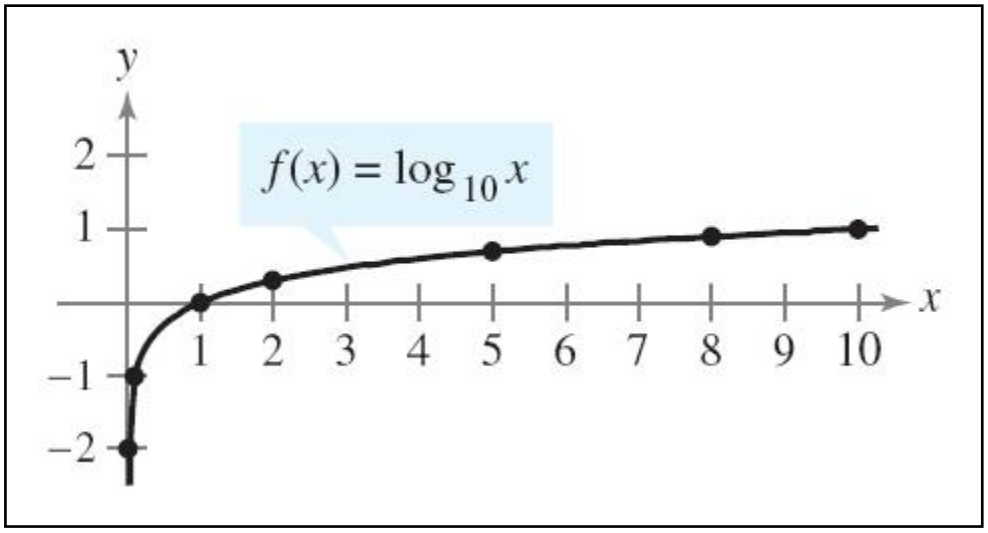
Characteristics of Logarithmic Functions



Graph of $y = \log_a x$, $a > 1$

- Domain: $(0, \infty)$
- Range: $(-\infty, \infty)$
- Intercept: $(1, 0)$
- Increasing
- One-to-one; therefore has an inverse function
- y-axis is a vertical asymptote ($\log_a x \to -\infty$ as $x \to 0^+$)
- Continuous
- Reflection of graph of $y = a^x$ about the line $y = x$

Example: base 10 : *the common log,* $f(x) = \log(x)$ *or* $f(x) = \log_{10} x$



<u>The Natural Logarithmic Function,</u> $f(x) = \ln(x)$,

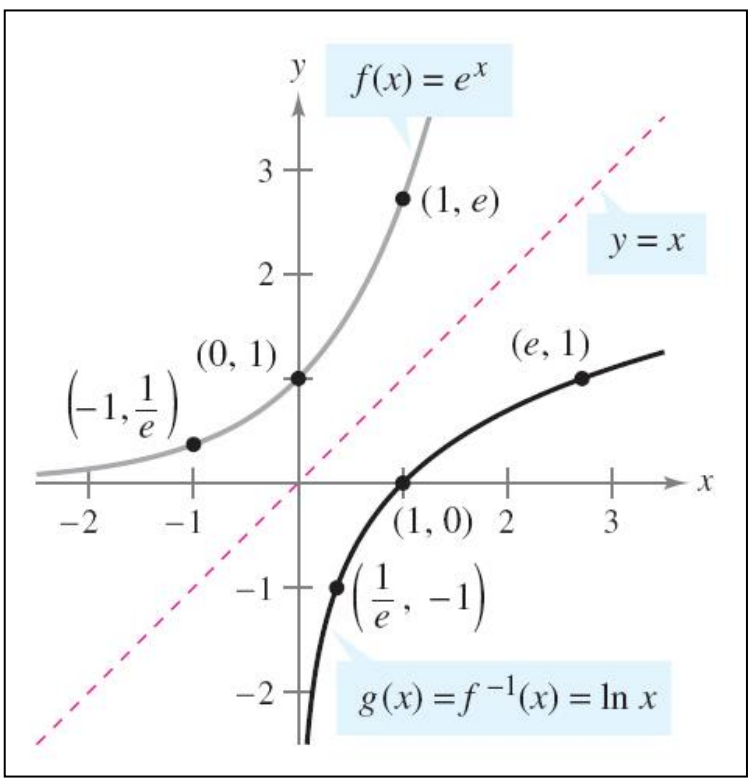
$f(x) = \log_e x \quad \Rightarrow \quad f(x) = \ln x$, ***" log of x with base e" or "natural log of x" or "ln x"***

Definition of the Natural Logarithmic Function

The **natural logarithmic function,** denoted by $\ln x$, is defined as

$$\ln x = b \quad \text{if and only if} \quad e^b = x.$$

$\ln x$ is read as "el en of x" or as "the natural log of x."



Properties of Natural Logarithms

1. $\ln 1 = 0$ because $e^0 = 1$.
2. $\ln e = 1$ because $e^1 = e$.
3. $\ln e^x = x$ and $e^{\ln x} = x$ — Inverse Properties
4. If $\ln x = \ln y$, then $x = y$. — One-to-One Property

Logarithmic form:	*Exponential form:*
$\ln 1 = 0$	$e^0 = 1$
$\ln e = 1$	$e^1 = e$
$\ln \frac{1}{e} = -1$	$e^{-1} = \frac{1}{e}$
$\ln 2 \approx 0.693$	$e^{0.693} \approx 2$

Section 3: Properties of Logarithms

Change-of-Base Formula

Let a, b, and x be positive real numbers such that $a \neq 1$ and $b \neq 1$. Then $\log_a x$ can be converted to a different base as follows.

Base b	*Base 10*	*Base e*
$\log_a x = \frac{\log_b x}{\log_b a}$	$\log_a x = \frac{\log_{10} x}{\log_{10} a}$	$\log_a x = \frac{\ln x}{\ln a}$

Properties of Logarithms

Let a be a positive number such that $a \neq 1$, and let n be a real number. If u and v are positive real numbers, then the following properties are true.

Logarithm with Base a	*Natural Logarithm*	
$\log_a(uv) = \log_a u + \log_a v$	$\ln(uv) = \ln u + \ln v$	Product Rule
$\log_a \frac{u}{v} = \log_a u - \log_a v$	$\ln \frac{u}{v} = \ln u - \ln v$	Quotient Rule
$\log_a u^n = n \log_a u$	$\ln u^n = n \ln u$	Power Rule

Properties of Logarithms

1. $\ln xy = \ln x + \ln y$
2. $\ln \frac{x}{y} = \ln x - \ln y$
3. $\ln x^n = n \ln x$

Section 4: Solving Exponential and Logarithmic Equations

Strategies for Solving Exponential and Logarithmic Equations

1. Rewrite the original equation in a form that allows the use of the One-to-One Property of exponential or logarithmic functions.
2. Rewrite an *exponential* equation in logarithmic form and apply the Inverse Property of logarithmic functions.
3. Rewrite a *logarithmic* equation in exponential form and apply the Inverse Property of exponential functions.

Inverse Properties of Exponential and Logarithms:

Base a	*Base e*
$a^{\log_a x} = x$	$e^{\ln x} = x$
$\log_a a^x = x$	$\ln e^x = x$
$a^{\log_a u} = u$	$e^{\ln u} = u$
$\log_a a^u = u$	$\ln e^u = u$

*Section 5: Exponential and Logarithmic Models *(covered in section 4)*

$$\textit{Exponential Growth}: y = a \cdot e^{bx}, \quad b > 0$$

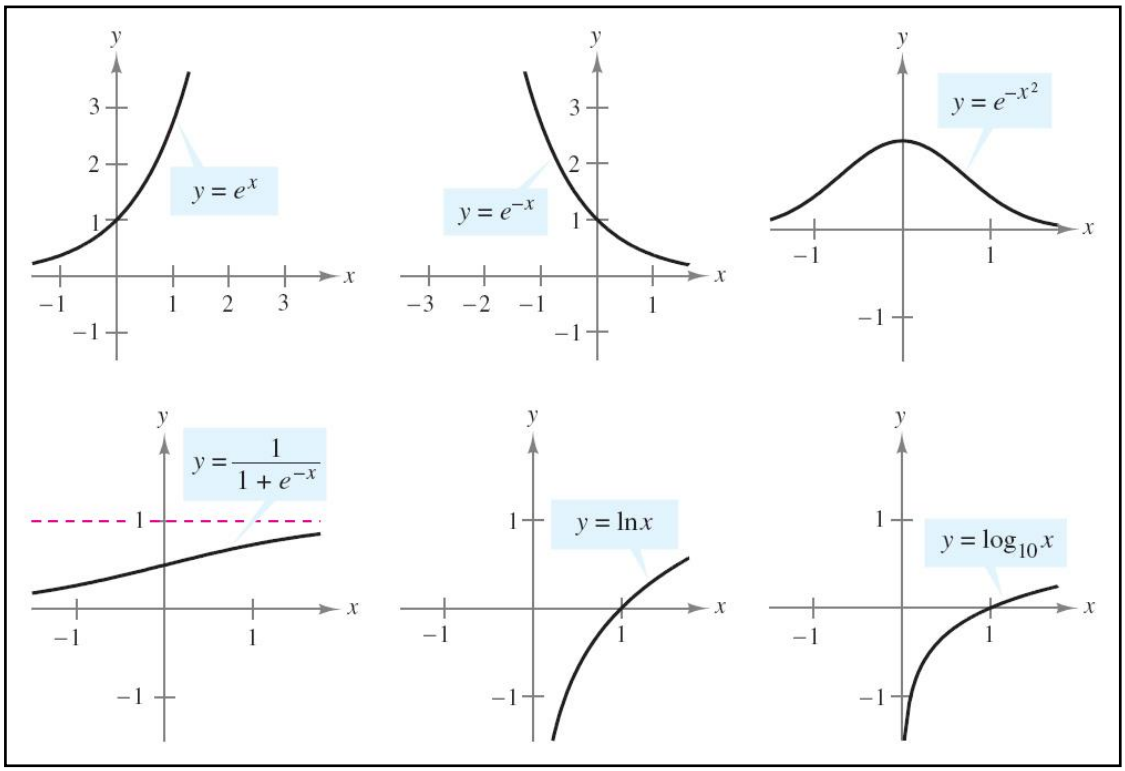
$$\textit{Exponatial Decay}: \quad y = a \cdot e^{bx}, \quad b < 0$$

$$\textit{Gaussian Model}: \quad y = a \cdot e^{-\frac{(x-b)^2}{c}}$$

$$\textit{Logistic Growth Model}: \; y = \frac{a}{1 + b \cdot e^{-kx}}, \; k > 0$$

$$\textit{Logarithmic Model}: \quad y = a + b \cdot ln(x), \quad y = a + b \cdot log(x)$$

Examples of *(basic)* exponential and logarithmic models:



Law of Exponential Growth and Decay

If y is a positive quantity whose rate of change with respect to time is proportional to the quantity present at any time t, then y is of the form

$$y = Ce^{kt}$$

where C is the **initial value** and k is the **constant of proportionality.** **Exponential growth** is indicated by $k > 0$ and **exponential decay** by $k < 0$.