

Purple

No calculators.

No pens: -10 pts.

No questions regarding content: -10 pts.

No late exams accepted. -100 pts.

No smart phones anywhere: -100 pts.

Tamara Knight

show all relevant work for full credit

1. Test the series for convergence or divergence. Name the test utilized. Support your conclusion. If convergent, find the sum whenever possible.

a.  $\sum_{n=0}^{\infty} \frac{2n^2}{n^2+1}$

Test:  $n^{\text{th}}$  term test  
for divergence

Apply test / reasoning:

$$\lim_{n \rightarrow \infty} \frac{2n^2}{n^2+1} = 2 \neq 0$$

series diverges

Converge / diverge:

Sum, if possible:

b.  $\sum_{n=0}^{\infty} (-1)^n \frac{4}{3^{n-1}} = \sum_{n=0}^{\infty} 12 \left(-\frac{1}{3}\right)^n$

Test: geometric

Apply test / reasoning:

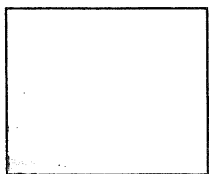
$$r = -\frac{1}{3} \quad | \quad 0 < \left| -\frac{1}{3} \right| < 1$$

converges

$$S = \frac{12}{1 - (-\frac{1}{3})} = \frac{12}{\frac{4}{3}} = 9$$

Converge / diverge:

Sum, if possible:



P

2. Find the center, radius, and the (open) interval of convergence for the power series. You need NOT test endpoints.

$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^n (x-5)^n}{3^n \cdot n} \quad u_n = \frac{(x-5)^n}{3^n \cdot n} \quad u_{n+1} = \frac{(x-5)^{n+1}}{3^{n+1} (n+1)}$$

Center:  $C = 5$

$$\lim_{n \rightarrow \infty} \left| \frac{(x-5)^{n+1}}{3^{n+1} (n+1)} \cdot \frac{3^n \cdot n}{(x-5)^n} \right|$$

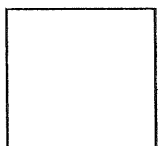
$$\lim_{n \rightarrow \infty} \left| \frac{(x-5)^{n+1}}{(x-5)^n} \cdot \frac{3^n}{3^{n+1}} \cdot \frac{n}{n+1} \right|$$

$$\left| \frac{x-5}{3} \right| \lim_{n \rightarrow \infty} \left| \frac{n}{n+1} \right| = \left| \frac{x-5}{3} \right| < 1$$

for convergence  $\left| \frac{x-5}{3} \right| < 1$

$$|x-5| < 3$$

Radius:  $R = 3$



(open) Interval of Convergence

$$(2, 8)$$

P

3. Find the center, radius, and the (open) interval of convergence for the power series,

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{n+1}}{(n+1)!} \quad \text{You need NOT test endpoints.}$$

Center:  $c = 0$

$$u_n = \frac{x^{n+1}}{(n+1)!}$$

$$u_{n+1} = \frac{x^{n+2}}{(n+2)!}$$

$$\lim_{n \rightarrow \infty} \left| \frac{x^{n+2}}{(n+2)!} \cdot \frac{(n+1)!}{x^{n+1}} \right|$$

$$\lim_{n \rightarrow \infty} \left| \frac{x^{n+2}}{x^{n+1}} \cdot \frac{(n+1)!}{(n+2)!} \right|$$

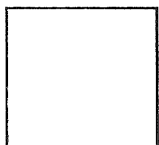
$$\lim_{n \rightarrow \infty} \left| \frac{x}{1} \cdot \frac{1}{n+2} \right| = |x| \lim_{n \rightarrow \infty} \left| \frac{1}{n+2} \right|$$

$$= |x| (0)$$

for convergence

$|x| (0) < 1$  true for  
all  $x$   $R = \infty$

Radius:  $R = \infty$



(open) Interval of Convergence  $(-\infty, \infty)$

p

4. Given the power series,  $f(x) = \sum_{n=1}^{\infty} \frac{(-1)^n (x-2)^n}{3^n \cdot n}$ , and the fact that the radius of convergence is  $R = 3$  to find the interval of convergence, including testing endpoints.

Open interval of convergence  
prior to testing endpoints:

$$(-1, 5)$$

TEST LEFT ENDPOINT:

$$\begin{aligned} x &= -1 \\ f(-1) &= \sum_{n=1}^{\infty} \frac{(-1)^n (-1-2)^n}{3^n \cdot n} \\ &= \sum_{n=1}^{\infty} \frac{(-1)^n (-3)^n}{3^n \cdot n} \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n} \\ &= \sum_{n=1}^{\infty} \frac{1}{n} \\ \text{div. p-series } p=1 \leq 1 \end{aligned}$$

TEST RIGHT ENDPOINT:

$$\begin{aligned} x &= 5 \\ f(5) &= \sum_{n=1}^{\infty} \frac{(-1)^n (5-2)^n}{3^n \cdot n} \\ &= \sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{3^n \cdot n} \\ &= \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \\ a_n &= \frac{1}{n} \quad a_{n+1} = \frac{1}{n+1} \\ 1) \lim_{n \rightarrow \infty} \frac{1}{n} &= 0 \\ 2) \frac{1}{n+1} &\leq \frac{1}{n} \\ \text{converges AST} \end{aligned}$$

FINAL interval of Convergence Radius:  
(after testing endpoints)

$$[-1, 5]$$

P

5. For the power series,  $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x-1)^{n+1}}{2^n \cdot (n+1)!}$  find AND simplify or rewrite  $n^{\text{th}}$  term for each of the following. You need NOT find interval of convergence, nor test endpoints.

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x-1)^{n+1}}{2^n \cdot (n+1)!} = (x-1) - \frac{1}{2 \cdot 2!} (x-1)^2 + \frac{1}{2^2 \cdot 3!} (x-1)^3 - \frac{1}{2^3 \cdot 4!} (x-1)^4 + \frac{1}{2^4 \cdot 5!} (x-1)^5 - \dots$$

$$\begin{aligned} \text{a. } f'(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)(x-1)^n}{2^n (n+1)!} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n (x-1)^n}{2^n \cdot n!} \end{aligned}$$

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$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x-1)^{n+1}}{2^n \cdot (n+1)!} = (x-1) - \frac{1}{2 \cdot 2!} (x-1)^2 + \frac{1}{2^2 \cdot 3!} (x-1)^3 - \frac{1}{2^3 \cdot 4!} (x-1)^4 + \frac{1}{2^4 \cdot 5!} (x-1)^5 - \dots$$

$$\begin{aligned} \text{b. } \int f(x) dx &= C + \sum_{n=0}^{\infty} \frac{(-1)^n (x-1)^{n+2}}{2^n (n+2)(n+1)!} \\ &= C + \sum_{n=0}^{\infty} \frac{(-1)^n (x-1)^{n+2}}{2^n \cdot (n+2)!} \end{aligned}$$



P

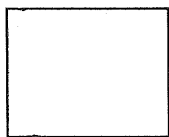
6. Use the definition of the Taylor Polynomial and Taylor Series to find the 5<sup>th</sup> Taylor Polynomial for  $f(x) = \sin 2x$  centered at  $c = \pi$ . Then use your polynomial to write the Taylor Series for  $f(x) = \sin 2x$  centered at  $c = \pi$ .

$$P_5(x) = a_0 + a_1(x-\pi) + a_2(x-\pi)^2 + a_3(x-\pi)^3 + a_4(x-\pi)^4 + a_5(x-\pi)^5$$

$n=0$	$f(x) = \sin 2x$	$f(\pi) = 0$	$a_0 = 0/0! = 0$
$n=1$	$f'(x) = 2\cos 2x$	$f'(\pi) = 2$	$a_1 = 2/1! = 2$
$n=2$	$f''(x) = -4\sin 2x$	$f''(\pi) = 0$	$a_2 = 0/2! = 0$
$n=3$	$f'''(x) = -8\cos 2x$	$f'''(\pi) = -8$	$a_3 = -8/3! = -4/3$
$n=4$	$f^{(4)}(x) = 16\sin 2x$	$f^{(4)}(\pi) = 0$	$a_4 = 0/4! = 0$
$n=5$	$f^{(5)}(x) = 32\cos 2x$	$f^{(5)}(\pi) = 32$	$a_5 = 32/5! = 4/15$

5th Taylor polynomial:  $P_5(x) = \frac{2}{1!}(x-\pi) - \frac{8}{3!}(x-\pi)^3 + \frac{32}{5!}(x-\pi)^5$

Taylor Series:  $P(x) = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} (x-\pi)^{2n+1}}{(2n+1)!}$



or

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2^{2n-1} (x-\pi)^{2n-1}}{(2n-1)!}$$

P

7. Use the Maclaurin Series for  $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$  to find AND simplify the series for

a.  $f(x) = e^{-2x^4}$

$$e^{-2x^4} = \sum_{n=0}^{\infty} \frac{(-2x^4)^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n 2^n x^{4n}}{n!}$$

b.  $g(x) = x^3 e^x$

$$x^3 e^x = x^3 \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{x^n x^3}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{x^{n+3}}{n!}$$

8. Suppose the interval of convergence for a particular power series  $f(x)$  is  $(-2, 1]$ .  
Answer the following, **TRUE OR FALSE**.

a. It is possible that the interval of convergence of  $f'(x)$  is  $(-2, 1)$ . True.

b. It is possible that the interval of convergence of  $f'(x)$  is  $[-2, 1]$ . False.

c. It is possible that the interval of convergence of  $\int f(x) dx$  is  $(-2, 1)$ . False.

d. It is possible that the interval of convergence of  $\int f(x) dx$  is  $[-2, 1]$ . True.

9. (4 points): In your opinion the greatest song (or recording) of all time (in any musical genre).

Answers vary.

There is no wrong answer.

GREEN

No calculators.

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1. Test the series for convergence or divergence. Name the test utilized. Support your conclusion. If convergent, find the sum whenever possible.

a.  $\sum_{n=0}^{\infty} \frac{2n^2}{n^2+1}$

Test:  $n^{\text{th}}$  term test  
for divergence

Apply test / reasoning :

$$\lim_{n \rightarrow \infty} \frac{2n^2}{n^2+1} = 2 \neq 0$$

series  
diverges

Converge / diverge :

Sum, if possible :

b.  $\sum_{n=0}^{\infty} (-1)^n \frac{4}{2^{n-1}} = \sum_{n=0}^{\infty} 8 \left(-\frac{1}{2}\right)^n$

Test: geometric

Apply test / reasoning :

$$r = -\frac{1}{2}$$

$$0 < \left| -\frac{1}{2} \right| < 1$$

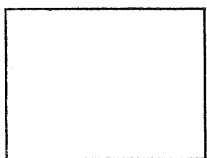
Converges

$$S = \frac{8}{1 - (-\frac{1}{2})} = \frac{8}{\frac{3}{2}}$$

Converge / diverge :

$$= 16/3$$

Sum, if possible :





G

2. Find the center, radius, and the (open) interval of convergence for the power series. You need NOT test endpoints.

$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^n (x-3)^n}{2^n \cdot n} \quad u_n = \frac{(x-3)^n}{2^n \cdot n} \quad u_{n+1} = \frac{(x-3)^{n+1}}{2^{n+1} (n+1)}$$

Center:  $c = 3$

$$\lim_{n \rightarrow \infty} \left| \frac{(x-3)^{n+1}}{2^{n+1} (n+1)} \cdot \frac{2^n \cdot n}{(x-3)^n} \right|$$

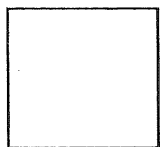
$$\lim_{n \rightarrow \infty} \left| \frac{(x-3)^{n+1}}{(x-3)^n} \cdot \frac{2^n}{2^{n+1}} \cdot \frac{n}{n+1} \right|$$

$$\left| \frac{x-3}{2} \right| \lim_{n \rightarrow \infty} \left| \frac{n}{n+1} \right| = \left| \frac{x-3}{2} \right| (1) = \left| \frac{x-3}{2} \right|$$

for convergence  $\left| \frac{x-3}{2} \right| < 1$

$$|x-3| < 2$$

Radius:  $R = 2$



(open) Interval of Convergence

$(1, 5)$

G

3. Find the center, radius, and the (open) interval of convergence for the power series,

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{n+1}}{(n+1)!} . \text{ You need } \underline{\text{NOT}} \text{ test endpoints.}$$

Center:  $c = 0$        $u_n = \frac{x^{n+1}}{(n+1)!}$        $u_{n+1} = \frac{x^{n+2}}{(n+2)!}$

$$\lim_{n \rightarrow \infty} \left| \frac{x^{n+2}}{(n+2)!} \cdot \frac{(n+1)!}{x^{n+1}} \right|$$

$$\lim_{n \rightarrow \infty} \left| \frac{x^{n+2}}{x^{n+1}} \cdot \frac{(n+1)!}{(n+2)!} \right|$$

$$|x| \lim_{n \rightarrow \infty} \left| \frac{1}{n+2} \right| = |x|(0)$$

for convergence  $|x|(0) < 1$   
true for all  $x$   
 $R = \infty$

Radius:  $R = \infty$



(open) Interval of Convergence  $(-\infty, \infty)$

G

4. Given the power series,  $f(x) = \sum_{n=1}^{\infty} \frac{(-1)^n (x-2)^n}{3^n \cdot n}$ , and the fact that the radius of convergence is  $R = 3$  to find the interval of convergence, including testing endpoints.

Open interval of convergence,  
prior to testing endpoints:

$$(-1, 5)$$

TEST LEFT ENDPOINT:

$$\begin{aligned} x &= -1 \\ f(-1) &= \sum_{n=1}^{\infty} \frac{(-1)^n (-1-2)^n}{3^n \cdot n} \\ &= \sum_{n=1}^{\infty} \frac{(-1)^n (-1)^n 3^n}{3^n \cdot n} \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n} \\ &= \sum_{n=1}^{\infty} \frac{1}{n} \end{aligned}$$

div. p-series  $p = 1 \leq 1$

TEST RIGHT ENDPOINT:

$$\begin{aligned} x &= 5 \\ f(5) &= \sum_{n=1}^{\infty} \frac{(-1)^n (5-2)^n}{3^n \cdot n} \\ &= \sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{3^n \cdot n} \\ &= \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \end{aligned}$$

$$a_n = \frac{1}{n} \quad a_{n+1} = \frac{1}{n+1}$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} = 0$$

$$\frac{1}{n+1} \leq \frac{1}{n} \quad \text{converges by AST}$$

FINAL interval of Convergence Radius:  
(after testing endpoints)

$$[-1, 5]$$

G

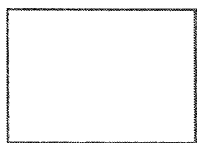
5. For the power series,  $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x-1)^{n+1}}{2^n \cdot (n+1)!}$  find AND simplify or rewrite  $n^{\text{th}}$  term for each of the following. You need NOT find interval of convergence, nor test endpoints.

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x-1)^{n+1}}{2^n \cdot (n+1)!} = (x-1) - \frac{1}{2 \cdot 2!} (x-1)^2 + \frac{1}{2^2 \cdot 3!} (x-1)^3 - \frac{1}{2^3 \cdot 4!} (x-1)^4 + \frac{1}{2^4 \cdot 5!} (x-1)^5 - \dots$$

$$\begin{aligned} \text{a. } f'(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n (n+1) (x-1)^n}{2^n \cdot (n+1)!} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n (x-1)^n}{2^n \cdot n!} \end{aligned}$$

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x-1)^{n+1}}{2^n \cdot (n+1)!} = (x-1) - \frac{1}{2 \cdot 2!} (x-1)^2 + \frac{1}{2^2 \cdot 3!} (x-1)^3 - \frac{1}{2^3 \cdot 4!} (x-1)^4 + \frac{1}{2^4 \cdot 5!} (x-1)^5 - \dots$$

$$\begin{aligned} \text{b. } \int f(x) dx &= C + \sum_{n=0}^{\infty} \frac{(-1)^n (x-1)^{n+2}}{2^n (n+2)(n+1)!} \\ &= C + \sum_{n=0}^{\infty} \frac{(-1)^n (x-1)^{n+2}}{2^n \cdot (n+2)!} \end{aligned}$$



G

6. Use the definition of the Taylor Polynomial and Taylor Series to find the 5<sup>th</sup> Taylor Polynomial for  $f(x) = \sin 2x$  centered at  $c = \pi$ . Then use your polynomial to write the Taylor Series for  $f(x) = \sin 2x$  centered at  $c = \pi$ .

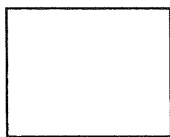
$$P_5(x) = a_0 + a_1(x-\pi) + a_2(x-\pi)^2 + a_3(x-\pi)^3 + a_4(x-\pi)^4 + a_5(x-\pi)^5$$

$n=0$	$f(x) = \sin 2x$	$f(\pi) = 0$	$a_0 = 0/0! = 0$
$n=1$	$f'(x) = 2\cos 2x$	$f'(\pi) = 2$	$a_1 = 2/1! = 2$
$n=2$	$f''(x) = -4\sin 2x$	$f''(\pi) = 0$	$a_2 = 0/2! = 0$
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$n=4$	$f^{(4)}(x) = 16\sin 2x$	$f^{(4)}(\pi) = 0$	$a_4 = 0/4! = 0$
$n=5$	$f^{(5)}(x) = 32\cos 2x$	$f^{(5)}(\pi) = 32$	$a_5 = 32/5! = 4/15$

5th Taylor polynomial:  $P_5(x) = 2(x-\pi) - \frac{8}{3!}(x-\pi)^3 + \frac{32}{5!}(x-\pi)^5$

Taylor Series:  $P(x) = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} (x-\pi)^{2n+1}}{(2n+1)!}$

or  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2^{2n-1} (x-\pi)^{2n-1}}{(2n-1)!}$



G

7. Use the Maclaurin Series for  $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$  to find AND simplify the series for

a.  $f(x) = e^{-4x^2}$

$$e^{-4x^2} = \sum_{n=0}^{\infty} \frac{(-4x^2)^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n 4^n x^{2n}}{n!}$$

b.  $g(x) = x^2 e^x$

$$x^2 e^x = x^2 \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{x^2 x^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{x^{n+2}}{n!}$$

8. Suppose the interval of convergence for a particular power series  $f(x)$  is  $(-2, 1]$ .

Answer the following, **TRUE OR FALSE**.

a. It is possible that the interval of convergence of  $f'(x)$  is  $(-2, 1)$ . True.

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d. It is possible that the interval of convergence of  $\int f(x) dx$  is  $[-2, 1]$ . True.

9. (4 points): In your opinion the greatest song (or recording) of all time (in any musical genre).

Answers vary.

There is no wrong answer.