If needed $\sin 2u = 2 \sin u \cos u$

1. Find the indefinite integral:

$$\int \frac{1}{\sqrt{x^2 + 4}} \, dx$$

$$\int \frac{2 \sec^2 \theta \, d\theta}{2 \sec \theta}$$

$$\int \sec \theta \, d\theta$$

$$= \ln | \sec \theta + \tan \theta | + C$$

$$= \ln \left| \frac{\sqrt{x^2 + 4}}{2} + \frac{x}{2} \right| + C$$

2. Find the indefinite integral:

$$\int \frac{x^2}{\sqrt{4-x^2}} \, dx$$

$$\int \frac{(2 \sin \theta)^2(2 \cos \theta) \, d\theta}{2 \cos \theta}$$

$$4 \int \sin^2 \theta \, d\theta$$

$$= 4 \int \left( \frac{1 - \cos 2\theta}{2} \right) \, d\theta$$

$$= 2 \int (1 - \cos 2\theta) \, d\theta$$

$$= 2 \left[ \theta - \frac{1}{2} \sin 2\theta \right] + C$$

$$= 2 \left[ \theta - \frac{1}{2} (2 \sin \theta \cos \theta) \right] + C$$

$$= 2 \left[ \arcsin \left( \frac{x}{2} \right) - \left( \frac{x}{2} \cdot \frac{2-x^2}{2} \right) \right] + C$$
3. Find the partial fraction decomposition:

\[
\frac{x^2 + x + 2}{x^2(x-2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2}
\]

\[
x^4 + x + 2 = Ax(x-2) + Bx(x-2) + Cx^2
\]

\[
x = 0 \quad 2 = -2B \quad x = 2 \quad 8 = 4C \quad x = 1 \quad 4 = -A - B + C
\]

\[
-1 = B \\
-1 = B
\]

\[
z = C \\
z = C
\]

P.F.D. \( \frac{-3 - \frac{1}{x} + \frac{2}{x-2}}{x^2} \)

4. Find the partial fraction decomposition:

\[
\frac{4x^2 - 5x + 5}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}
\]

\[
4x^2 - 5x + 5 = A(x^2+1) + (Bx+C)(x-1)
\]

\[
\text{if } x = 1 \quad 4 = 2A \\
2 = A
\]

\[
* \text{ if expand } \quad 4x^2 - 5x + 5 = Ax^2 + A + Bx^2 + Cx - Bx - C
\]

\[
= (A+B)x^2 + (C-B)x + (A-C)
\]

\[
A + B = 4 \quad \text{ if } A = 2 \quad B = 2 \]

\[
C - B = -5 \quad \text{ if } A = 2 \quad -C = -3
\]

\[
A - C = 5 \quad \quad C = 3
\]

P.F.D. \( \frac{2 + 2x - 3}{x-1} \)

\[
\frac{2 + 2x - 3}{x^2 + 1}
\]

* or \( 1 \neq x = 0 \) and other values
5. Find the limit, if it exists.

\[ \lim_{x \to 0} \left( \frac{e^{2x} - \cos x - 2x}{x^2} \right) = \lim_{x \to 0} \frac{e^0 \cos 0 - 0}{0} = \frac{0}{0} \]

L'Hopital's

\[ \lim_{x \to 0} \frac{2e^{2x} \sin x - 2}{2x} = \frac{2e^0 \sin 0 - 2}{0} = \frac{0}{0} \]

L'Hopital's

\[ \lim_{x \to 0} \frac{4e^{2x} + \cos x}{2} = \frac{4e^0 + \cos 0}{2} = \frac{4}{2} = 2 \]

6. Find the limit, if it exists.

\[ \lim_{x \to 0^+} (x \ln x) = 0 \cdot (-\infty) \]

\[ \lim_{x \to 0^+} \frac{\ln x}{1/x} = -\infty \]

L'Hopital's

\[ \lim_{x \to 0^+} \frac{1/x}{-1/x^2} = \lim_{x \to 0^+} \frac{\frac{1}{x^2}}{(-x)} \]

\[ = \lim_{x \to 0^+} (-x) = 0 \]
7. Determine if the improper integral converges or diverges. If convergent, find its value.

\[ \int_{2}^{4} \frac{1}{\sqrt{16-x^2}} \, dx = \lim_{b \to 4^-} \left( \int_{2}^{b} \frac{1}{\sqrt{16-x^2}} \, dx \right) \]

\[ = \lim_{b \to 4^-} \left[ (\arcsin x/4) \right]_{2}^{b} \]

\[ = \arcsin 1/4 - \arcsin 1/2 \]

\[ = \arcsin 1 - \arcsin 1/2 = \pi/2 - \pi/6 = \pi/3 \]

converges

8. Determine if the improper integral converges or diverges. If convergent, find its value.

\[ \int_{-\infty}^{0} \frac{e^x}{1+e^x} \, dx = \lim_{a \to -\infty} \int_{a}^{0} \frac{e^x}{1+e^x} \, dx \]

\[ = \lim_{a \to -\infty} \left[ \ln (1+e^x) \right]_{a}^{0} \]

\[ = \ln (1+e^0) - \ln (1+e^a) \]

\[ = \ln 2 - \ln (1+e^{-a}) \]

\[ = \ln 2 - \ln 1 = \ln 2 \]

converges
Bonus. If finite, find the volume of the solid generated by rotating the region bounded by the graphs of $y = e^{-2x}$, and the $x$-axis for $x \geq 0$ about the $x$-axis. If the volume is not finite, support your conclusion.

\[
\text{DISK} \quad V = \pi \int_a^b R^2 \, dx
\]

\[
V = \pi \int_0^\infty \left(e^{-2x}\right)^2 \, dx
\]

\[
= \pi \int_0^\infty e^{-4x} \, dx
\]

\[
= \pi \lim_{b \to \infty} \int_0^b e^{-4x} \, dx
\]

\[
= \pi \lim_{b \to \infty} \left[-\frac{1}{4} e^{-4x}\right]_0^b
\]

\[
= \pi \lim_{b \to \infty} \left(-\frac{1}{4} e^{-4b} + \frac{1}{4} e^0\right)
\]

\[
= \pi \lim_{b \to \infty} \left(-\frac{1}{4} e^{-4b} + \frac{1}{4}\right)
\]

\[
= \pi \left(0 + \frac{1}{4}\right) = \frac{\pi}{4} \text{ units}^3
\]

\[
= \frac{\pi}{4} \text{ units}^3
\]
If needed $\sin 2\theta = 2 \sin \theta \cos \theta$

1. Find the indefinite integral:

$$\int \frac{1}{x^2 \sqrt{x^2 - 4}} \, dx$$

$$\int \frac{\sec \theta \tan \theta \, d\theta}{(2 \sec \theta)^2 (2 \tan \theta)}$$

$$\frac{1}{4} \int \frac{1}{\sec \theta} \, d\theta$$

$$\frac{1}{4} \int \cos \theta \, d\theta = \frac{1}{4} \sin \theta + C$$

$$= \frac{1}{4} \left( \sqrt{x^2 - 4} \right) + C$$

2. Find the indefinite integral:

$$\int \frac{x^2}{\sqrt{4 - x^2}} \, dx$$

$$\int \frac{(2 \sin \theta)^2 (2 \cos \theta)}{2 \cos \theta} \, d\theta$$

$$4 \int \sin^2 \theta \, d\theta$$

$$4 \int \left( 1 - \cos^2 \theta \right) \, d\theta$$

$$2 \int \left( 1 - \cos \theta \right) \, d\theta$$

$$2 \int \left( \frac{\theta - \frac{1}{2} \sin 2 \theta \right) + C$$

$$2 \left[ \theta - \frac{1}{2} \left( 2 \sin \theta \cos \theta \right) \right] + C$$

$$2 \left[ \arcsin \frac{x}{2} - \left( \frac{x}{2} \cdot \frac{\sqrt{4-x^2}}{2} \right) \right] + C$$
3. Find the partial fraction decomposition: \[
\frac{x^2 - 3x - 2}{x(x-2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2}
\]

\[x^2 - 3x - 2 = Ax(x-2) + B(x-2) + Cx^2\]

If \(x = 0\) \(-2 = -2B\)
If \(x = 2\) \(-4 = 4C\)
\(1 = B\)
\(-1 = C\)

\(x = 1\)
\(-4 = -A - B + C\)
\(-4 = -A - 1 - 1\)
\(-2 = -A\)
\(2 = A\)

q.f.d.

\[
\begin{align*}
\frac{2}{x} + \frac{1}{x^2} - \frac{1}{x-2}
\end{align*}
\]

4. Find the partial fraction decomposition: \[
\frac{4x^2 - 3x + 3}{(x-1)(x^2 + 1)} = \frac{A}{x-1} + \frac{Bx + C}{x^2 + 1}
\]

\[4x^2 - 3x + 3 = A(x^2 + 1) + (Bx + C)(x-1)\]

If \(x = 1\) \(4 = 2A\)
\(2 = A\)

*If expand

\[4x^2 - 3x + 3 = Ax^2 + A + Bx^2 + Cx - Bx - C\]

\[= (A + B)x^2 + (C - B)x + (A - C)\]

\[A + B = 4\]
If \(A = 2\) \(B = 2\)
\(C - B = -3\)
If \(A = 2\) \(-B = -3\)
\[A - C = 3\]
If \(A = 2\) \(-C = 1\)

P.F.D.

\[
\frac{2}{x-1} + \frac{2x - 1}{x^2 + 1}
\]
5. Find the limit, if it exists.

\[
\lim_{x \to 0} \left( \frac{e^{2x} - \cos x - 2x}{x^2} \right) = \frac{e^0 - \cos 0 - 0}{0} = 0
\]

L'Hopital's

\[
\lim_{x \to 0} \frac{2e^x + \sin x - 2}{2x} = \frac{2e^0 + \sin 0 - 2}{0} = 0
\]

L'Hopital's

\[
\lim_{x \to 0} \frac{4e^{2x} + \cos x}{2} = \frac{4e^0 + \cos 0}{2} = \frac{5}{2}
\]

6. Find the limit, if it exists.

\[
\lim_{x \to 0^+} (x \ln x) = 0 (-\infty)
\]

L'Hopital's

\[
\lim_{x \to 0^+} \frac{\ln x}{\frac{1}{x}} = -\infty
\]

L'Hopital's

\[
\lim_{x \to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \infty
\]

L'Hopital's

\[
\lim_{x \to 0^+} \left( \frac{1}{x} \cdot (-x^2) \right) = \lim_{x \to 0^+} \left( -x \right) = 0
\]

\[
\lim_{x \to 0^+} (-x) = 0
\]
7. Determine if the improper integral converges or diverges. If convergent find its value.

\[ \int_{1}^{2} \frac{1}{\sqrt{4-x^2}} \, dx = \lim_{b \to 2^-} \int_{b}^{2} \frac{1}{\sqrt{4-x^2}} \, dx \]

Let \( u = x \) and \( du = dx \)

\[ = \lim_{b \to 2^-} \left[ \arcsin \frac{x}{2} \right]_{b}^{2} \]

\[ = \lim_{b \to 2^-} \arcsin \frac{2}{2} - \arcsin \frac{b}{2} \]

\[ = \arcsin 1 - \arcsin \frac{b}{2} \]

\[ = \frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3} \]

Converges

8. Determine if the improper integral converges or diverges. If convergent, find its value.

\[ \int_{0}^{\infty} \frac{e^{-x}}{1+e^{-x}} \, dx \]

Let \( w = 1 + e^{-x} \) and \( dw = -e^{-x} \, dx \)

\[ = \lim_{b \to \infty} \left[ -\ln |1+e^{-x}| \right]_{0}^{b} \]

\[ = \lim_{b \to \infty} \left( -\ln |1+e^{-b}| - \ln |1+e^{0}| \right) \]

\[ = -\ln |1+e^{\infty}| + \ln 2 \]

\[ = -\ln 1 + \ln 2 = \ln 2 \text{ converges} \]
Bonus. If finite, find the volume of the solid generated by rotating the region bounded by the graphs of $y = e^{-2x}$, and the $x$-axis for $x \geq 0$ about the $x$-axis. If the volume is not finite, support your conclusion.

Using the disk method, the volume $V$ is given by:

$$V = \pi \int_a^b R^2 \, dx$$

where $R = e^{-2x}$. Thus,

$$V = \pi \int_0^\infty (e^{-2x})^2 \, dx$$

$$= \pi \int_0^\infty e^{-4x} \, dx$$

$$= \pi \left[ -\frac{1}{4} e^{-4x} \right]_0^\infty$$

$$= \pi \lim_{b \to \infty} \left( -\frac{1}{4} e^{-4x} \right)_{x=0}^{x=b}$$

$$= \pi \lim_{b \to \infty} \left( -\frac{1}{4} e^{-4b} + \frac{1}{4} \right)$$

$$= \pi \left( 0 + \frac{1}{4} \right)$$

$$= \frac{\pi}{4} \text{ units}^3$$

Thus, the volume converges to a finite value.