Barometric Price Leadership*

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Abstract

I study a dynamic Bertrand-duopoly model in which a firm initiates price changes while its competitor always matches the change with a lag. The firms produce a homogeneous product and are identical except for the information they possess about demand. The market size follows a Markov process and its realizations are observed by one of the firms but not the other. Price leadership patterns appear in equilibrium for a wide range of parameters. For high and fixed discount factors, price leadership allows firms to jointly approximate monopolistic profits in equilibrium as the the market size becomes more persistent.

JEL Classification: D43, L13.

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1 Introduction

Price leadership is an industry pricing pattern in which periods of uniform (across firms) and constant prices are disrupted by one firm, the leader, whose new price soon becomes the uniform price. Among the industries in which such patterns have been claimed to be observed are gasoline (Stigler, 1947), rayon (Markham, 1951), and airlines (Rotemberg and Saloner, 1988). Stigler (1947) and Markham (1951) suggested that such patterns could emerge without explicit collusion in situations in which the leader is better informed about industry demand conditions. Stigler labeled such price leadership *barometric price leadership* and went on to suggest that it should be more prevalent the more persistent are industry demand conditions (See Stigler (1947), p. 446).

Previous attempts to produce explicit models of price leadership following the Stigler-Markham ideas have interpreted price leadership as firms opting for sequential (Stackelberg) rather than simultaneous (Nash) pricing within each discrete period and before demand is realized (Cooper, 1997, Rotemberg and Saloner, 1990). There are two shortcomings of such models. First, a time series of prices generated by such models would not allow an observer to distinguish between sequential and simultaneous price setting models. Second, such models have nothing to say about the role of persistence of demand conditions. Moreover, having the firms *choose* between Stackelberg and Nash can be interpreted as explicit collusion, which is inconsistent with the above ideas of Stigler and Markham.

I develop a model of barometric price leadership that builds on the Stigler-Markham ideas. There is an infinite-horizon in discrete time and there are two firms; one is informed and the other uninformed. The market demand that the firms face can be either high or low and follows a symmetric persistent Markov process. The informed firm sees the market realization before setting its price; the uninformed firm never sees it. At each period, the two firms engage in price (Bertrand) competition; they set their prices simultaneously and are not able to engage in overt communication. If the two prices are different, then all sales go to the firm with the lower price; otherwise, they share sales equally. At the end of each period, each firm sees both prices, but only its own sales. In this model, price leadership is a sequence of prices in which the uninformed firm always matches the informed firm's previous price, while the informed firm sets different prices for different states.[1]

I show that price leadership is an equilibrium outcome for a broad set of parameters. The conditions can be summarized in terms of two parameters: the persistence of market demand and the common discount rate of the firms. The intuition is simple: as the market size becomes more persistent, the uninformed firm is able to infer more about tomorrow's state from today's state. Therefore, the informed firm's price becomes more informative as the market size becomes more persistent. These observations are consistent with Stigler's

[1] After formulating my model, it came to my attention that Escobar and Llanes (2015) were developing a similar idea. They present a model of general repeated games in which one player possess private information about her type. Their model can be applied to generate price leadership patterns if the costs of one firm are constant and known while the other firm's costs are private and evolve according to a persistent Markov process. Compared to Escobar and Llanes, our model not only delivers a different story but also allows for both payoffs to depend on the stochastic shock.

idea that prices in industries with price leaders are more persistent than those of industries without price leadership. Although the set of equilibria is large when firms are patient, price leadership is especially interesting because for high and fixed discount factors, price leadership in which joint profits approximate monopoly profits is a limiting equilibrium as persistence approaches perfect persistence.

The remainder of the paper is organized as follows. The next section contains the model. In Section 3, price leadership is defined and the results are presented. In Section 4, I discuss possible extensions and a potential application of the main idea in the model to a recent practice in the supermarket industry called *category management* (see Federal Trade Commission (FTC, 2001)).

2 Model

Consider a market with two firms, I and U, I stands for informed while U stands for uninformed. These firms interact in an infinite horizon game. At each period $t = 0, 1, 2, \ldots$, the firms compete in a homogeneous product Bertrand model. A state s^t, interpreted as the market size, is drawn at the beginning of each period from the set $\{s_l, s_h\}$ where $0 < s_l < s_h$. The state follows a Markov process with transition matrix

$$\begin{bmatrix} \Pr(s' = s_l | s = s_l) & \Pr(s' = s_h | s = s_l) \\ \Pr(s' = s_l | s = s_h) & \Pr(s' = s_h | s = s_h) \end{bmatrix} = \begin{bmatrix} 1-\phi & \phi \\ \phi & 1-\phi \end{bmatrix}$$

for some $\phi < \frac{1}{2}$ and initial distribution $\Pr(\cdot|s_h)$. The results will not depend on the selection of the initial distribution. We say that the state is persistent because the probability of the state changing, ϕ, is always less than one half.

While the Markov process is commonly known, only Firm I observes the realization of the state s^t.[2] After Firm I learns the state, the firms simultaneously set prices p_I^t and p_U^t from the support $[0, \overline{p}]$ where $\overline{p} > s_h$. For a given state s^t and prices p_I^t and p_U^t, the quantity demanded at period t is

$$\max\{s^t - \min\{p_1^t, p_2^t\}, 0\}.$$

If the firms set different prices, the firm with the lowest price gets the whole demand. Otherwise, firms share sales equually. Therefore, for prices p_I^t and p_U^t and state $s^t \in \{s_l, s_h\}$, firm i's stage payoff at period t is given by

$$u_i(p_i, p_j; s) = \begin{cases} \pi(p_i; s) & \text{if } p_i < p_j, j \neq i; \\ \frac{\pi(p_i;s)}{2} & \text{if } p_i = p_j, j \neq i; \\ 0 & \text{otherwise.} \end{cases}$$

where $\pi(p; s) = \max\{p(s-p), 0\}$. At the end of each period, firms observe both prices and their own quantity but are unable to observe their competitor's quantity. That is, at the of period t, firm i observes p_I^t, p_U^t and q_i^t.

[2] The model resembles a two-firm and two-state version of Kandori (1991). The information asymmetries is were the models differ. Firms are able to perfectly observe the state in Kandori's model.

The firms have a the common discount factor $\delta \in (0,1)$. Firm i's payoff from a sequence of prices $\{(p_I^t, p_U^t)\}_{t=0,1,\ldots}$ and a sequence of states $\{s^t\}_{t=0,1,\ldots}$ is

$$(1-\delta)\sum_{t=0}^{\infty} \delta^t u_i(p_I^t, p_U^t; s^t).$$

Irrespective of the state s^t, the only stage game Nash equilibrium prices are given by $p_I^t = p_U^t = 0$ and the unique stage Nash equilibrium payoffs are therefore zero. Similarly, at a period t in which the market size is $s^t = s_x$ for $x \in \{l, h\}$, a monopolist that knows s^t would set a price

$$p_x^M = \frac{s_x}{2}.$$

At each period t, when firms are about to set prices, they possess different information. The uninformed firm knows the sequence of prices and its own quantity up to period t, that is, its period t history is the sequence $h_U^t \equiv \{p_I^\tau, p_U^\tau, q_U^\tau\}_{\tau=0}^{t-1}$. Let $\mathcal{H}_U^t$ denote the set of all possible period t histories for the uninformed firm.

On the other hand, the informed firm knows the sequence of prices up to period t and the whole sequence of states including s^t. That is, by the time the informed firm sets p_I^t, it knows the history $h_I^t \equiv \{p_I^\tau, p_U^\tau, s^\tau\}_{\tau=0}^{t-1} \times s^t$. Denote the set of all period t histories for the informed firm as $\mathcal{H}_I^t$. Let $\mathcal{H}_j$ be the set all possible histories for firm j, or $\mathcal{H}_j = \bigcup_{t=0}^{\infty} \mathcal{H}_j^t$. A pure strategy for firm j is a mapping

$$P_j : \mathcal{H}_j \to [0, \overline{p}].$$

Note that the uninformed firm does not possess any private information. That is, for each history h_I^t that the informed firm observes, there is only one possible history h_U^t for the uninformed firm. The opposite is not true. Knowing both prices and its own quantity is not always enough for the uninformed firm to infer the market size. For example, the uninformed firm learns s^t if $p_U^t < p_I^t$ and $p_U^t < s^t$ because firm U knows that it captured the whole demand and therefore the quantity it sold was equal to $s^t - p_U^t$. On the contrary, firm U would not be able to learn s^t if $p_I^t < p_U^t$.

2.1 Monopolistic Profits

The maximum profits that can be attained in this environment are obtained by an informed monopolist, or a monopolist that at every period knows the state before setting its price. For that reason, we use monopolistic as the reference for high profits. In this subsection, we calculate those profits and establish necessary conditions for firms to be able jointly implement those profits.

Let $V^M(s_l)$ and $V^M(s_h)$ denote the expected payoff of a monopolist before it learns the realization of the current state given that the previous state was s_l and s_h respectively. Then, if the previous state was s_l: the current state is s_l with probability $(1-\phi)$ and the informed monopolist obtains a stage payoff of $\pi(p_l^M; s_l)$ and a continuation value $V^M(s_l)$; and, the

current state is s_h with probability ϕ in which case the monopolist obtains a stage payoff of $\pi(p_h^M; s_h)$ and a continuation value of $V^M(s_h)$. That is,

$$V^M(s_l) = (1-\delta)\big[(1-\phi)\pi(p_l^M; s_l) + \phi\pi(p_h^M; s_h)\big] + \delta\big[(1-\phi)V^M(s_l) + \phi V^M(s_h)\big]. \tag{1}$$

Similarly,

$$V^M(s_h) = (1-\delta)\big[\phi\pi(p_l^M; s_l) + (1-\phi)\pi(p_h^M; s_h)\big] + \delta\big[\phi V^M(s_l) + (1-\phi)V^M(s_h)\big]. \tag{2}$$

It is possible for firms to obtain joint profits equal to those of the informed monopolist but the restrictions on parameters are stringent. If firms are exactly implementing joint monopolistic profits, the uninformed firm is never setting a price below p_h^M because at every period the high demand state occurs with positive probability. Then, in such an equilibrium, the informed firm sets prices like a monopoly. Also, to be deterred from price cutting, the uninformed firm sometimes should sell in such an equilibrium. But at any period in which the uninformed firm is supposed to make a sale, the informed firm is tempted to pretend that the demand is low. For this deviation not to be profitable, we require that sharing the high-demand monopolistic profits is better that getting the whole high demand at the low price. That condition is established in the next proposition.

Proposition 1. *Firms can exactly implement joint profits equal to the informed monopolist profits in equilibrium only if and only if*

$$\frac{s_l}{s_h} \leq \frac{4-\sqrt{8}}{4} \approx 0.2929. \tag{3}$$

In the next section, a class of strategy profiles that generates price leadership patterns is introduced. Those strategies allow firms to obtain joint high profits even when the inequality (3) does not hold.

3 Price Leadership

Next, price leadership is formally defined as a class of strategy profiles that generate price leadership patterns.

Definition 1 (Price Leadership)**.** For any pair of prices, $\mathbf{p} = (p_l, p_h)$ with $p_l \neq p_h$, the price leadership pricing rules are defined as follow:

I. At any period $t \geq 0$, the informed firm sets a price p_I^t equal to p_l if the state is s_l and equal to p_h if the state is s_h;

U. the uninformed firm starts by setting the price p_h at $t = 0$ and after that always sets a price p_U^t that matches the informed firm's previous price, that is, $p_U^t = p_I^{t-1}$.

If a firm detects a deviation from the previous pricing rules in the past, then both firms set a price equal to 0 forever.

In this environment in which the market size is persistent, an strategy profile satisfying the previous definition will generate price leadership patterns. If the state changes, the informed firm changes its price accordingly and the uninformed firm follows in the next period.

Previous works have interpreted price leadership as firms opting for sequential (Stackelberg) rather than simultaneous (Nash) pricing within each discrete period before the demand is realized (Cooper, 1997, Deneckere and Kovenock, 1992, Mouraviev and Rey, 2011, Rotemberg and Saloner, 1990, Yano and Komatsubara, 2006, 2012). This interpretation differs from the price leadership introduced in this paper because the prices generated by those models do not distinguish between simultaneous and sequential pricing. At each period in those models, all firms have set the same prices by the time they start selling.

The price leadership strategy profile with prices $\mathbf{p}$ is simple in the sense that the period t's action only depends on the previous period actions for the uninformed firm and on the previous period actions and the current state for the informed firm. The remainder of this section shows that price leadership equilibria perform well in terms of joint profits despite its simplicity.

Because our interest lie in supporting price leadership outcomes in which firms attain high profits, we will restrain our attention to cases where $p_l < p_h$.

3.1 Payoffs from Price Leadership

The expected payoffs from the price leadership strategy profile are derived as follows. We start with the expected payoffs of the uninformed firm when both are playing according to the price leadership profile with prices $\mathbf{p} = (p_l, p_h)$ where $p_l < p_h$. In that case, Firm U's information is summarized by the informed firm previous price. Hence, let $V^U_{\mathbf{p}}(p)$ for $p \in \{p_l, p_h\}$ be the uninformed firm expected discounted payoff given that the informed firm previous price was equal to p. Then, provided that both firms are following the price leadership strategy profile,

- if the informed firm's previous price was p_l, the uninformed is setting the price p_l today. Also, it must be the case that the market size was s_l in the previous period so the state today is s_l with probability $(1-\phi)$ and s_h with probability ϕ. If the current state is s_l again, both firms set a price p_l and split the market today and the uninformed firm derives a continuation value of $V^U_{\mathbf{p}}(p_l)$. On the other hand, if the current state is s_h, the informed firm sets a price p_h and the uninformed gets the whole market today at a price p_l and a continuation value of $V^U_{\mathbf{p}}(p_h)$. That is,

$$V^U_{\mathbf{p}}(p_l) = (1-\delta)\left[(1-\phi)\frac{\pi(p_l;s_l)}{2} + \phi\pi(p_l;s_h)\right] + \delta\left[(1-\phi)\,V^U_{\mathbf{p}}(p_l) + \phi\,V^U_{\mathbf{p}}(p_h)\right] \quad (4)$$

- if the informed firm's previous price was p_h, the expected discounted payoff for the

uninformed firm is given by

$$V_{\mathbf{p}}^{U}(p_h) = (1-\delta)\left[(1-\phi)\frac{\pi(p_h;s_h)}{2}\right] + \delta\left[\phi V_{\mathbf{p}}^{U}(p_l) + (1-\phi)V_{\mathbf{p}}^{U}(p_h)\right]. \tag{5}$$

Similarly, assuming both firm are following the price leadership strategy profile, we will derive the expected payoffs for firm I. Remember that the informed firm knows the state by the time the prices are set. All the information that firm I needs to calculate its expected payoff is the previous and the current state. Let $V_{\mathbf{p}}^{I}(s,s')$ with $s,s' \in \{s_l, s_h\}$ be the informed firm's expected discounted payoff provided that the current state is s' and the previous state was s. Then,

- when both the previous and the current states are low, the informed firm knows that the uninformed is going to set a price equal to p_l because the previous state (and the informed firm previous price) was low. Hence, because the state today is also s_l, the informed firm also sets a price equal to p_l and both firms equally split the market today. The next state is s_l with probability $(1-\phi)$ in which case the continuation value of firm I is $V_{\mathbf{p}}^{I}(s_l,s_l)$. With probability ϕ, the next state is s_h in which case the informed gets a continuation payoff of $V_{\mathbf{p}}^{I}(s_l,s_h)$. Then,

$$V_{\mathbf{p}}^{I}(s_l,s_l) = (1-\delta)\frac{\pi(p_l;s_l)}{2} + \delta\left[(1-\phi)V_{\mathbf{p}}^{I}(s_l,s_l) + \phi V_{\mathbf{p}}^{I}(s_l,s_h)\right] \tag{6}$$

- when the previous state was low and the current state is high, the expected discounted payoff for the informed firm is

$$V_{\mathbf{p}}^{I}(s_l,s_h) = \delta\left[\phi V_{\mathbf{p}}^{I}(s_h,s_l) + (1-\phi)V_{\mathbf{p}}^{I}(s_h,s_h)\right] \tag{7}$$

- when the previous state was high and the current state is low, the expected discounted payoff for the informed firm is

$$V_{\mathbf{p}}^{I}(s_h,s_l) = (1-\delta)\pi(p_l;s_l) + \delta\left[(1-\phi)V_{\mathbf{p}}^{I}(s_l,s_l) + \phi V_{\mathbf{p}}^{I}(s_l,s_h)\right] \tag{8}$$

- when both the previous and the current states are high, the expected discounted payoff for the informed firm is

$$V_{\mathbf{p}}^{I}(s_h,s_h) = (1-\delta)\frac{\pi(p_h;s_h)}{2} + \delta\left[\phi V_{\mathbf{p}}^{I}(s_h,s_l) + (1-\phi)V_{\mathbf{p}}^{I}(s_h,s_h)\right]. \tag{9}$$

3.2 Price Leadership as an Equilibrium

Conditions for the price leadership strategy profile to be a PBE are derived in this section. First, we need to specify the beliefs on the equilibrium path. At any $t \in \mathbb{N}_0$, let $\mu^t(\cdot|h_U^t) \in \Delta(S)$ be the belief that the uninformed firm has about the state at period t given a history $h_U^t \in \mathcal{H}_U^t$. The initial belief $\mu^0(\cdot)$ is given by $\Pr(\cdot|s_h)$. If firms are following price leadership

with prices p_l and p_h with $p_l < p_h$, then μ^t only depends on the action that the informed firm took at period $t-1$. Then, given a history $h_U^t \in \mathcal{H}_U^t$, the uninformed firm beliefs about s^t are given by

$$\mu^t(\cdot|h_U^t) = \begin{cases} \Pr(\cdot|s^{t-1} = s_l) & \text{if } p_I^{t-1} = p_l \\ \Pr(\cdot|s^{t-1} = s_h) & \text{if } p_I^{t-1} = p_h \end{cases}$$

Next, we need to show that there are no profitable deviations from the price leadership profile. Firm can potentially make a profit by deviating to charge a lower price than the price prescribed by the strategy profile. All price cuts except one are immediately detected and therefore trigger a Nash reversal. Hence, price cuts that are immediately detected are not profitable for patient firms. The only price cut that is not detected occurs when both firms are supposed to set a price p_h but firm I deviates by setting a price p_l. In that case, the uninformed firm does not sell and cannot distinguish if there was a deviation or the state was s_l. That deviation is depicted in the next figure.

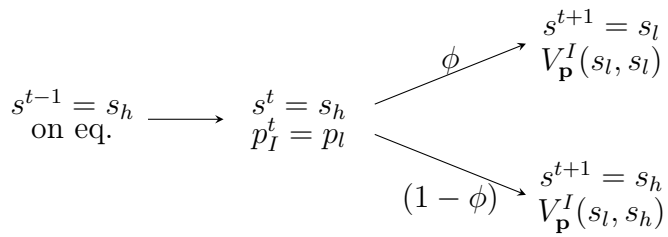

Figure 1: Undetected Deviation.

Figure 1 represents a situation in which firms have played according to the price leadership strategy profile up to period t and $s^{t-1} = s^t = s_h$. Because the price of the informed firm was p_h in the previous period, the uninformed sets a price p_h at period t. Then, firm I can deviate by setting a price p_l and getting the whole market instead instead of sharing the demand at price p_h. Hence, the informed firm's stage payoff from such deviation at period t is given by $\pi(p_l; s_h)$. The continuation values also change because the deviation leads to the uninformed firm setting a price p_l at period $t+1$. With probability ϕ the state at $t+1$ will change to s_l and in that situation the informed firm faces an identical problem as when on equilibrium the state is going from s_l to s_l therefore obtaining a expected discounted payoff of $V_{\mathbf{p}}^I(s_l, s_l)$. Similarly, with probability $1-\phi$ the state remains s_h at period $t+1$ and the informed firm obtains a discounted payoff of $V_{\mathbf{p}}^I(s_l, s_h)$.

Although high discount factors are enough to discourage all other price cuts in a price leadership profile, the firms being patient is not sufficient to discourage the informed firm from pretending the state is low when it is high. Conditions to ensure that the inform firm does not want to set the price p_l when it is supposed to set the price p_h are presented next.

3.3 Jointly approaching monopolistic profits

It is natural to start by deriving conditions that guarantee that price leadership with the monopolistic prices can be sustained as a PBE. Remember that an informed monopolist will

set a price $p_l^M = \frac{s_l}{2}$ whenever the market size is low and a price $p_h^M = \frac{s_h}{2}$ if the market size is high. The next proposition establishes a condition on s_l, s_h and ϕ, so that price leadership with monopolistic prices is a PBE for patient enough firms.

Proposition 2. *If the following inequality holds,*

$$\frac{s_l}{s_h} < \frac{2-\phi}{2+\phi} \tag{$\star$}$$

there exists $\overline{\delta} \in (0,1)$ *such that price leadership with monopolistic prices is a PBE for any* $\delta > \overline{\delta}$.

The condition ($\star$) is intuitive given that the inequality holds for low $\frac{s_l}{s_h}$ and low ϕ. The informed does not want to deviate and set a price p_l^M when both firms are supposed to set a price p_h^M and split the high demand because:

i. when $\frac{s_l}{s_h}$ is low, the monopolistic price for the low state, p_l^M, is low relative to p_h^M and that makes the option of deviating to set p_l^M less desirable.

ii. when ϕ is low, the demand is persistent, the market size is likely to stay high in the next period and deviating would lead to a low price by the uninformed firm.

Also, condition ($\star$) is consistent with Stigler's conjecture that "the prices of industries with price leaders are less flexible than those of industries without price leaders, despite the larger fluctuations of output of the former group."[3] Note that condition ($\star$) implies that price leadership is more likely to arise if ϕ is low, a condition that generates more persistent prices. In a similar fashion, price leadership is more likely to arise if s_l/s_h is not close to 1 which generates larger output fluctuations.

Moreover, there is a minimum $\overline{\delta}$ satisfying Proposition 2. In the next figure, we represent that minimum $\overline{\delta}$ as $\frac{s_l}{s_h}$ goes from 0 to $(2-\phi)/(2+\phi)$ for $\phi = 1/9$ and $\phi = 2/9$.

In Figure 2 we can observe that as $\frac{s_l}{s_h}$ approaches $(2-\phi)/(2+\phi)$ firms need to be more patient to support monopolistic prices in a price leadership equilibrium.

Similarly, in the next figure, we present the minimum $\overline{\delta}$ satisfying Proposition 2 as ϕ changes for two different values of $\frac{s_l}{s_h}$.

[3] In Stigler (1947), on p. 446.

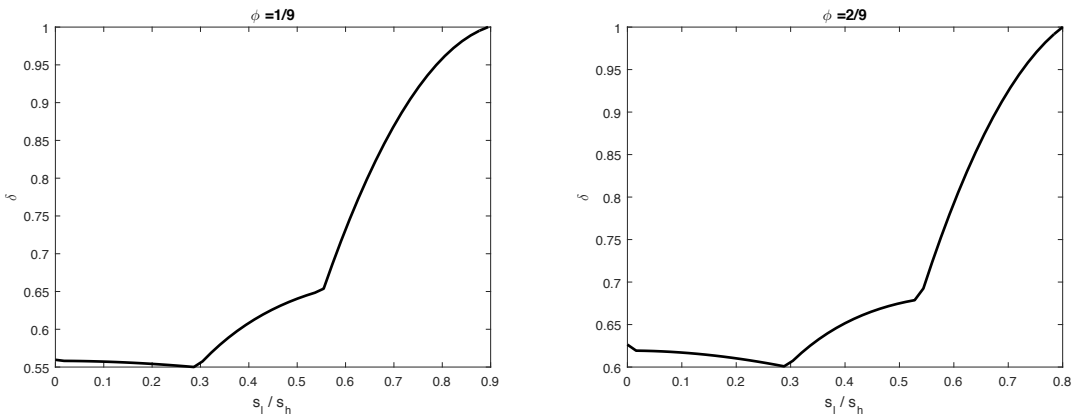

Figure 2: $\overline{\delta}$ as $\frac{s_l}{s_h}$ changes.

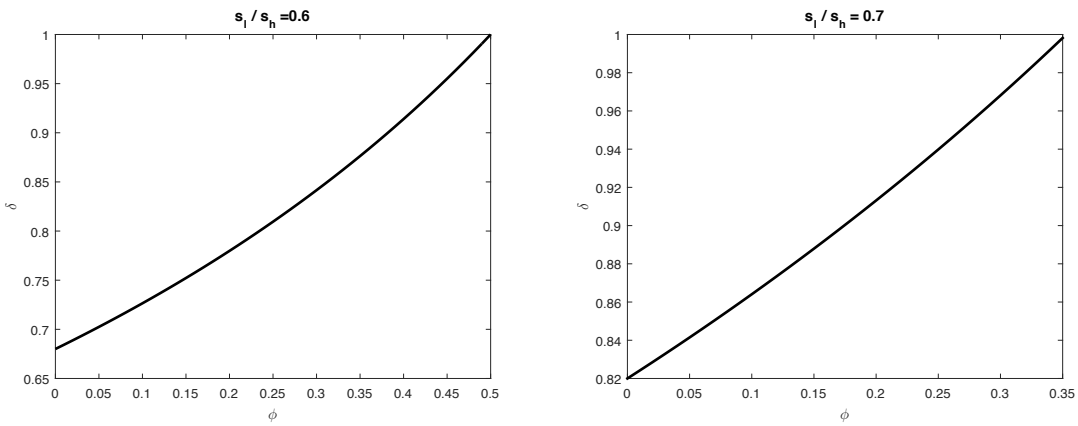

Figure 3: $\overline{\delta}$ as ϕ changes.

We can observe that as the market size becomes more persistent we can sustained monopolistic prices at a price leadership equilibrium with lower discount factors. Also, note that when $\frac{s_l}{s_h} = 0.6$, monopolistic prices can be sustained if firms are patient enough for any $\phi \in (0, 1/2)$.

As we previously showed in Proposition 1, whenever inequality (3) is not holding, the firms cannot exactly implement monopolistic joint profit. But Proposition 2 implies that for a wide range of parameters in which firms cannot exactly implement joint monopolistic profits, price leadership with monopolistic prices is still an equilibrium as can be seen in the next figure.

Looking at Figure 4, patient firms are only able to exactly obtain joint monopolistic profits when parameters are in the darker shaded area. But price leadership with monopolistic prices appears in equilibrium when firms are patient for parameters in both the dark and light shaded area.

If the firms are able to sustain price leadership with monopolistic prices, joint profits are below the monopolistic profits only when there was a change of state and the the uninformed

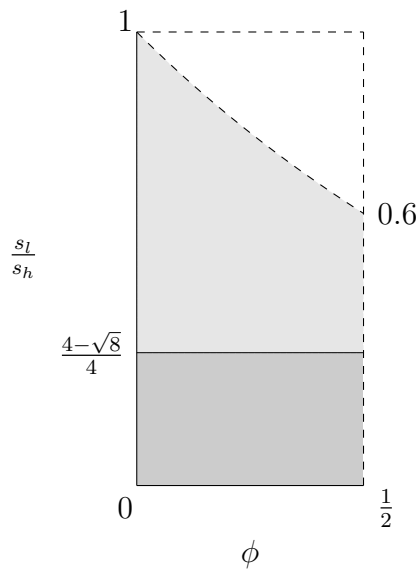

Figure 4: Results in the parameter space

firm is not able to adjust its price. Therefore, as the market size becomes more persistent the joint profits must approximate monopolistic profits. The next lemma contains a closed form solution for the difference between the informed monopolist expected profits and the joint expected profits from following price leadership with monopolistic prices.

We compare the joint profits derived from price leadership with monopolistic prices to those obtained by an informed monopolist in the next lemma. The comparison will consider ex-ante profits, that is, before the informed firm or the informed monopolist know the realization of the current state.

Lemma 1. *The difference in ex-ante discounted payoffs between the informed monopolist and the joint profits of firms following price leadership with monopolistic prices are*

$$V^M(s_l) - [V^U_{\mathbf{p}^M}(p^M_l) + V^I_{\mathbf{p}^M}(s_l)] = \left(\frac{1-\delta+\delta\phi}{1-\delta+2\delta\phi}\right)\phi\left(\frac{s_h-s_l}{2}\right)^2 \tag{10}$$

and

$$V^M(s_h) - [V^U_{\mathbf{p}^M}(p^M_h) + V^I_{\mathbf{p}^M}(s_h)] = \left(\frac{\delta\phi^2}{1-\delta+2\delta\phi}\right)\left(\frac{s_h-s_l}{2}\right)^2 \tag{11}$$

where $V^I_{\mathbf{p}^M}(s_l) = (1-\phi)V^I_{\mathbf{p}^M}(s_l,s_l)+\phi V^I_{\mathbf{p}^M}(s_l,s_h)$*, the ex-ante expected payoff for the informed firm provided that the previous state was low, and similarly* $V^I_{\mathbf{p}^M}(s_h) = \phi V^I_{\mathbf{p}^M}(s_h,s_l) + (1-\phi)V^I_{\mathbf{p}^M}(s_h,s_h)$ *is its expected value when the previous state was high.*

From Lemma 1, we conclude that the joint profits approach those of an informed monopolist as ϕ goes to 0. That is not necessarily the case when δ goes to 1 and everything else is fixed.

Proposition 3. *Fix s_l and s_h, then there exist a $\overline{\delta}_{s_l,s_h} \in (0,1)$ such that for any fixed $\delta > \overline{\delta}_{s_l,s_h}$,*

- *exists a $\overline{\phi} > 0$, such that for any $\phi < \overline{\phi}$, price leadership with monopolistic profits is a PBE.*
- *the ex-ante expected joint profits go to the monopolistic profits as ϕ goes to 0.*

It is important to point out that when ϕ goes to 0, a price leadership equilibrium approximates monopolistic profits for high and *fixed* discount factors. In contrast, it is unclear whether that would be the case for revision strategies, or strategy profiles that rely on a statistical test to enforce equilibrium play. For example, suppose we try to implement joint profits that are near monopolistic profits by asking the informed firm to always set a price p_h^M while the informed firm sets a price p_l^M when the state is s_l and a price p_h^M when the state is s_h. If the uninformed firm wants to test whether the informed firm is playing according to the specified profile, a reliable test would likely need to be longer and more complicated as ϕ becomes smaller and that would require a larger δ in order to implement the desired outcome.

3.4 Extending the Equilibrium Results

The next proposition shows that when price leadership with monopolistic prices is not an equilibrium, price leadership still is an equilibrium for other prices if firms are patient. As previously argued, if monopolistic prices cannot be sustained for patient firms, it must be the case that is too tempting for the informed firm to pretend that the demand is low when it is actually high. In one such case, the payoff derived from that deviation would be lower if firms decide to set a lower price p_l than the monopolistic one. The previous intuition leads to the next result.

Proposition 4. *If condition ($\star$) does not hold, there exists a price $\overline{p}_l$ with $0 < \overline{p}_l < p_l^M$, such that for any $p_l \in [0, \overline{p}_l)$, the price leadership strategy profile with prices p_l and p_h^M is a PBE provided that firms are patient enough.*

Therefore, price leadership can be sustained as an equilibrium for any s_l, s_h and $\phi < 1/2$ as long as firms are patient enough.

4 Extensions

Here I discuss extending the model to multiple states, to the case in which firms can only observe prices and a possible application of the model to a practice called *category management*.

Multiple states

The model can be extended to allow for more than two states. Now assume that at each period t, the market size s^t is drawn from the set $\{s_1, \ldots, s_n\}$ for some $n > 2$. Let $s_j < s_{j+1}$

for $j = 1, \ldots, n-1$. Again the market size follows Markov process with transition matrix $\mathbf{P}$ and initial distribution μ. Let $\mathbf{P}(s'|s)$ be the probability that the current state is s' given that the previous state was s. To show that the intuition behind the results holds for more than two states, we capture the idea that the market size is persistent by assuming that $\mathbf{P}(s_j|s_j) = 1 - \phi$ for $j = 1, \ldots, n$, and consider the case in which ϕ becomes small.

Now, let's consider price leadership with monopolistic prices, $p_j^M = s_j/2$, and derive the incentive constraints that capture the idea that the informed firm does not want to price according to the wrong state. The informed firm has an incentive to do so only when the current market size is greater or equal than the previous market size. Suppose that $s^{t-1} = s_j$ and $s^t \geq s_j$ for some $j > 1$, then at period t the uninformed firm would not want to set a price equal to p_{j-1}^M if

$$V_{\mathbf{p}^M}^I(s_j, s^t) \geq (1-\delta)\pi(p_{j-1}^M; s^t) + \delta \mathbb{E}_{s^{t+1}}\left[V_{\mathbf{p}^M}^I(s_{j-1}, s^{t+1})|s^t\right].$$

Compared to the two states case, extra assumptions on the state space are required for these incentive constraints to hold.

Considering the case with $n = 3$ is enough to get an idea. The incentive constraints that guarantee that the informed firm does not price according to the wrong state are,

- when the previous and the current state are s_j for $j = 2, 3$, the informed firm does not want to deviate and set a price p_{j-1}^M if

$$V_{\mathbf{p}^M}^I(s_j, s_j) \geq (1-\delta)\pi(p_{j-1}^M; s_j) + \delta \sum_{k=1}^{3} \mathbf{P}(s_k|s_j) V_{\mathbf{p}^M}^I(s_{j-1}, s_k). \tag{12}$$

- when the previous state was s_2 and the current state is s_3, the uninformed firm does not want to deviate and set a price equal to

$$V_{\mathbf{p}^M}^I(s_2, s_3) \geq (1-\delta)\pi(p_1^M; s_3) + \delta \sum_{k=1}^{3} \mathbf{P}(s_j|s_3) V_{\mathbf{p}^M}^I(s_1, s_k). \tag{13}$$

Again, values and incentive constraints are continuous on ϕ and taking limits when ϕ goes to 0, the incentive constraint (12) turns into

$$\frac{\pi(p_j^M; s_j)}{2} \geq (1-\delta)\pi(p_{j-1}^M; s_j) + \delta^2 \frac{\pi(p_j^M; s_j)}{2}$$

or

$$(1+\delta)\frac{\pi(p_j^M; s_j)}{2} > \pi(p_{j-1}^M; s_j).$$

Because $\pi(p_j^M; s_j) > \pi(p_{j-1}^M; s_j)$, the previous inequality always hold for high discount factors.

Similarly, taking limits when ϕ goes to 0, the incentive constraint (13) turns into

$$\delta \frac{\pi(p_3^M; s_3)}{2} > (1-\delta)\pi(p_1^M; s_3) + \delta^2 \frac{\pi(p_3^M; s_3)}{2}$$

or

$$\delta \frac{\pi(p_3^M; s_3)}{2} > \pi(p_1^M; s_3).$$

The previous inequality holds as long as,

$$\frac{s_1}{s_3} < 1 - \sqrt{1 - \frac{\delta}{2}}.$$

Consequently, to obtain an analogous result to Proposition 3 for the $n = 3$ case, we would require that

$$\frac{s_1}{s_3} < \frac{4 - \sqrt{8}}{4}.$$

But it is also important to notice that when the previous inequality does not hold, we can still sustain price leadership with prices $\mathbf{p} = (p_1, p_2^M, p_3^M)$ as a PBE for some $p_1 < p_1^M$ if the firms are patient and the market size is persistent.

Firms only observing prices

The fact that the uninformed firm is sometimes able to infer the state does not drive the results. This can be observed by assuming that the uninformed firm observes prices but does not observe quantities at all. The modified model allows to better understand the intuition behind the results. Consider the environment from Section 2 with firms only observing prices at the end of each period as the only difference. Therefore, when firms are setting prices at period t, the uninformed firm has only observed the sequence $\{p_I^\tau, p_U^\tau\}_{\tau=0}^{t-1}$ while the informed firm knows the sequence $\{p_I^\tau, p_U^\tau, s^\tau\}_{\tau=0}^{t-1} \times s^t$. Consequently, the uninformed firm is never able to infer s^t. In that case, when firms are following the price leadership strategy profile, the uninformed firm is not able to detect any deviation in which the informed firm sets a price p_l or p_h.

Compared to the case in which firms are able to observe their own quantities, there is one more incentive constrained that needs to be verified for the price leadership strategy profile to be a PBE: when the previous state was s_l and the current state is s_h, the informed firm does not want to set a price p_l. Expected payoff do not change and the expected payoff from following price leadership is the same as in equation (7),

$$V_{\mathbf{p}}^I(s_l, s_h) = \delta \left[\phi V_{\mathbf{p}}^I(s_h, s_l) + (1 - \phi) V_{\mathbf{p}}^I(s_h, s_h)\right].$$

Note that the informed firm does not sell in the current period when it follows price leadership. Instead, the informed firm can deviate to set p_l and share the market without being detected. But if the informed firm sets a price p_l today, the uninformed firm will also set a price p_l tomorrow. Therefore, the incentive constraint is given by,

$$V_{\mathbf{p}}^I(s_l, s_h) \geq (1 - \delta) \frac{\pi(p_l; s_h)}{2} + \delta \big[\phi V_{\mathbf{p}}^I(s_l, s_l) + (1 - \phi) V_{\mathbf{p}}^I(s_l, s_h)\big]. \tag{14}$$

Therefore, in a price leadership equilibrium with monopolistic prices, when the state goes from low to high, the informed firms does not want to deviate to set a price p_l because doing

so will imply another period of low prices tomorrow when a high price could be charge by following price leadership. The following lemma simplifies the previous incentive constraint for the case in which firms are using monopolistic prices.

Lemma 2. *When firms are following price leadership with monopolistic prices* $\mathbf{p}^M = (p_l^M, p_h^M)$*, the incentive constraint*(14) *holds if and only if*

$$\left[1+\delta\phi\right]\left(\frac{s_l}{s_h}\right)^2 - 2\left(\frac{s_l}{s_h}\right) + \delta(1-\phi) \geq 0. \tag{15}$$

The left-hand-side of the previous inequality is continuous on ϕ and δ. When $\delta \in (0,1)$ and $\phi \in (0, 1/2)$, the left-hand-side of inequality (15) is increasing on δ and decreasing on ϕ.

Hence, Proposition 3 holds without any extra assumptions because when ϕ goes to 0, the inequality (15) holds in the limit if $\delta \geq k(2-k)$ where $k = s_l/s_h$ and $k(2-k) < 1$ for $k \in [0,1)$.

Proposition (2) would hold if firms cannot observe quantities if condition ($\star$) is replaced by

$$\frac{s_l}{s_h} < \frac{1-\phi}{1+\phi}. \tag{16}$$

That is the case because (16) implies ($\star$) and, as δ goes to 1, the left-hand-side of inequality (15) goes to $(1-\phi) - 2k + (1+\phi)k^2$ where $k = \frac{s_l}{s_h}$ and the expression is positive as long as inequality (16) holds.

Category management

Category management refers to "an organizational approach in which the management of a retail establishment is broken down into categories of like products" (See FTC (2001), p. 47). A common way of managing a category is by assigning a "category captain," usually the category leading manufacturer, as the primary advisor for the category. In some instances, the category captains may advise the retailer on the prices for all category brands and the shelving of the category products (See American Antitrust Institute (2003), p. 4). Some have argued that the motive behind the practice is that all the parties benefit from the information and expertise that the captain possess, including information about demand, as a leading manufacturer (See p. 204 in Desrochers et al. (2003) or p. 46-47 in FTC (2001)).

The use of a category captain raises concern because a manufacturer that is selected as a category captain can use its position to increase profits at the expense of its competitors. However, the practice's prevalence suggests that captains do not abuse their positions. Still, it is unclear as to why manufacturers other than the category captains comply with the practice. The intuition behind price leadership in this model may provide an explanation: If the category captain has better information, then the competitors may benefit from delegating decisions to the informed firm. On the other hand, the captains allow their competitors to attain high profits because doing so is better than unleashing a retaliation or manufactures and retailers independently setting prices.

5 Concluding Remarks

In this model, I study an infinite-horizon duopoly model in which firms engage in Bertrand competition at each period. The market size follows a Markov process, and at each period the realization of the current state is only known by one firm. In that environment, I show the existence of equilibria that generates price leadership patterns for a wide set of parameters. Moreover, firms can derive joint profits that approach the monopolistic profits from this type of equilibria as the market size becomes more persistent. In such cases, given that overt communication is not feasible, the informed firm leads the uninformed firm towards joint profit maximization.

Moreover, compared to some previous models of barometric price leadership, we dispose of the assumption that firms have the option of pricing sequentially at each stage. We can understand pricing sequentially as a firm communicating their price intentions to competitors before making the price change, or as firms making non-immediately-effective price announcements as in the vitamins industry (Marshall et al., 2008). Our results suggest that firms can attain high profits through price leadership with no need for overt communication or price announcements.

Although our model differs from the "secret price cuts models"[4] because firms are able to observe prices, the informed firm can cut prices without being detected by pretending that the demand is low when it is actually high. Price leadership discourages this type of deviation with no need for on-equilibrium punishments because if the informed firm lowers its price the uninformed will match the low price in the next period.

This paper provides theoretical support for Stigler's barometric price leadership observations. The patterns can emerge in the presence of asymmetric information about persistent market conditions because the price of the better informed firm can be used to infer information about the unknown market conditions.

A Proofs

Proposition 1. Let us start with the "if" part. Assuming that the condition (3) is satisfied, we will propose an equilibrium that exactly achieves joint monopolistic profits. The proposed strategy profile is one in which the uninformed firm always set the price p_h^M while the informed sets the price p_l^M when the state is s_l and p_h^M when the market size is s_h. When a deviation is detected they trigger a stage Nash reversal. Let $\widetilde{V}^I(s)$ be the discounted payoff when the state is s for $s \in \{s_l, s_h\}$. If both firms are following the prescribed strategy profile, then

$$\widetilde{V}^I(s_l) = (1-\delta)\pi(p_l^M; s_l) + \delta\left[(1-\phi)\widetilde{V}^I(s_l) + \phi\widetilde{V}^I(s_h)\right]$$

and

$$\widetilde{V}^I(s_h) = (1-\delta)\frac{\pi(p_h^M; s_h)}{2} + \delta\left[\phi\widetilde{V}^I(s_l) + (1-\phi)\widetilde{V}^I(s_h)\right].$$

[4]See subsection 6.7.1 in Tirole (1988) for an example.

Detectable deviations trigger a Nash reversal and as a consequence are not profitable for patient firms. Then, we just need to verify that deviations that are not detected are not profitable. The only deviation that is not detected is one in which the informed firm plays as if the market size is s_l when it is actually s_h. Then, we require that

$$\widetilde{V}^I(s_h) \geq (1-\delta)\pi(p_l^M; s_h) + \delta\left[\phi\widetilde{V}^I(s_l) + (1-\phi)\widetilde{V}^I(s_h)\right]$$

or

$$(1-\delta)\frac{\pi(p_h^M; s_h)}{2} + \delta\left[\phi\widetilde{V}^I(s_l) + (1-\phi)\widetilde{V}^I(s_h)\right] \geq \\ (1-\delta)\pi(p_l^M; s_h) + \delta\left[\phi\widetilde{V}^I(s_l) + (1-\phi)\widetilde{V}^I(s_h)\right]$$

which holds as long as

$$\pi(p_h^M; s_h) \geq 2\pi(p_l^M; s_h).$$

Just plugging in the functions, we can see that the informed firm will not have incentives to lie about the state as long as,

$$\frac{s_h^2}{4} \geq s_l\left(s_h - \frac{s_l}{2}\right).$$

Finally, the previous inequality holds if,

$$2\left(\frac{s_l}{s_h}\right)^2 - 4\left(\frac{s_l}{s_h}\right) + 1 \geq 0,$$

or

$$\frac{s_l}{s_h} \leq \frac{4-\sqrt{8}}{4}.$$

We need to show the "only if" part. Note that condition (3) implies that a myopic informed firm will prefer to share the high demand at a price p_h^M to get the whole high demand at a price p_l^M. That is, $\pi(p_h^M; s_h)/2 \geq \pi(p_l^M; s_h)$.

Also, at each period, the probability that the market size is always positive, if the uninformed firm sets a price below p_h^M with positive probability the demand will be high and the lowest price in the industry will be below the monopolistic price which leads to profits below the monopolistic ones.

As a consequence, if firms are exactly achieving monopolistic profits jointly, it must be the case that the uninformed firm never sets a price below p_h^M and that the informed is always setting the monopolistic price according to the state of the demand. Also, the uninformed firm has to make enough sales otherwise it will be tempted to price cut the informed firm and here is were the problem lays. Every time that the demand is high and the uninformed is supposed to sell, the informed firm can set a price p_l^M and the uninformed firm will not be able to distinguish if this was in fact a deviation or the demand was low. That deviation would be profitable if the condition (3) is not satisfied.

Whenever this is the case, firms cannot exactly implement joint monopolistic profits. ■

Before getting into the proof of the propositions, we start by presenting close-form solutions for expected discounted payoffs when both firms are playing according to the price leadership strategy profile.

The following lemma will prove useful to calculate the values that firm derive from price leadership.

Lemma 3. *The solutions V_1 and V_2 to the system of equations*

$$V_1 = (1-\delta)\Pi_1 + \delta[(1-\phi)V_1 + \phi V_2] \tag{17}$$

and

$$V_1 = (1-\delta)\Pi_2 + \delta[\phi V_1 + (1-\phi)V_2] \tag{18}$$

with $\Pi_1, \Pi_2 \in \mathbb{R}_+$, $\delta \in (0,1)$ and $\phi \in (0, 1/2)$ are given by

$$V_1 = \left[\frac{1-\delta}{[1-\delta(1-\phi)]^2 - \delta^2\phi^2}\right]([1-\delta(1-\phi)]\Pi_1 + \delta\phi\,\Pi_2)$$
$$V_2 = \left[\frac{1-\delta}{[1-\delta(1-\phi)]^2 - \delta^2\phi^2}\right](\delta\phi\,\Pi_1 + [1-\delta(1-\phi)]\Pi_2)\,.$$

Proof. The system of equations (17) and (18) can be written as

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = (1-\delta)\begin{pmatrix} \Pi_1 \\ \Pi_2 \end{pmatrix} + \delta\begin{bmatrix} 1-\phi & \phi \\ \phi & 1-\phi \end{bmatrix}\begin{pmatrix} V_1 \\ V_2 \end{pmatrix}.$$

Then,

$$\begin{bmatrix} 1-\delta(1-\phi) & -\delta\phi \\ -\delta\phi & 1-\delta(1-\phi) \end{bmatrix}\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = (1-\delta)\begin{pmatrix} \Pi_1 \\ \Pi_2 \end{pmatrix},$$

or

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = (1-\delta)\begin{bmatrix} 1-\delta(1-\phi) & -\delta\phi \\ -\delta\phi & 1-\delta(1-\phi) \end{bmatrix}^{-1}\begin{pmatrix} \Pi_1 \\ \Pi_2 \end{pmatrix}.$$

By inverting the matrix in the right hand side of the equation,

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \left(\frac{1-\delta}{[1-\delta(1-\phi)]^2 - \delta^2\phi^2}\right)\begin{bmatrix} 1-\delta(1-\phi) & \delta\phi \\ \delta\phi & 1-\delta(1-\phi) \end{bmatrix}\begin{pmatrix} \Pi_1 \\ \Pi_2 \end{pmatrix}.$$

That completes the proof. ■

Using the lemma, we can see a close form solution for the uninformed firm values.

Corollary 1. *The uninformed firm discounted expected utilities $V^U(p_l)$ and $v^U(p_h)$ that solve equations (4) and (5) are given by*

$$V^U_{\mathbf{p}}(s_l) = \left[\frac{1-\delta}{[1-\delta(1-\phi)]^2 - \delta^2\phi^2}\right]\left([1-\delta(1-\phi)]\Pi^U_l + \delta\phi\,\Pi^U_h\right)$$
$$V^U_{\mathbf{p}}(s_h) = \left[\frac{1-\delta}{[1-\delta(1-\phi)]^2 - \delta^2\phi^2}\right]\left(\delta\phi\,\Pi^U_l + [1-\delta(1-\phi)]\Pi^U_h\right)$$

where $\Pi^U_l = (1-\phi)\frac{\pi(p_l;s_l)}{2} + \phi\pi(p_l;s_h)$ and $\Pi^U_h = (1-\phi)\frac{\pi(p_h;s_h)}{2}$.

In a similar fashion we can obtain the values for the informed firm.

Corollary 2. *The discounted expected values for the informed firm, V^I_{ll}, V^I_{lh}, V^I_{hl} and V^I_{hh} that are solutions to the system of equations (6 - 9) are given by*

$$V^I_{\mathbf{p}}(s_l, s_l) = \left[\frac{1-\delta}{c^2-\delta^2\phi^2}\right]\left[(c-\delta^2\phi^2)\Pi^I_{ll} + \delta\phi c\Pi^I_{lh} + \delta^2\phi^2\Pi^I_{hl} + \delta^2\phi(1-\phi)\Pi^I_{hh}\right] \tag{19}$$

$$V^I_{\mathbf{p}}(s_l, s_h) = \left[\frac{1-\delta}{c^2-\delta^2\phi^2}\right]\left[\delta^2\phi(1-\phi)\Pi^I_{ll} + c^2\Pi^I_{lh} + \delta\phi c\Pi^I_{hl} + \delta(1-\phi)c\Pi^I_{hh}\right] \tag{20}$$

$$V^I_{\mathbf{p}}(s_h, s_l) = \left[\frac{1-\delta}{c^2-\delta^2\phi^2}\right]\left[\delta(1-\phi)c\Pi^I_{ll} + \delta\phi c\Pi^I_{lh} + c^2\Pi^I_{hl} + \delta^2\phi(1-\phi)\Pi^I_{hh}\right] \tag{21}$$

$$V^I_{\mathbf{p}}(s_h, s_h) = \left[\frac{1-\delta}{c^2-\delta^2\phi^2}\right]\left[\delta^2\phi(1-\phi)\Pi^I_{ll} + \delta^2\phi^2\Pi^I_{lh} + \delta\phi c\Pi^I_{hl} + (c-\delta^2\phi^2)\Pi^I_{hh}\right] \tag{22}$$

where $c = [1-\delta(1-\phi)]$, $\Pi^I_{ll} = \frac{\pi(p_l;s_l)}{2}$, $\Pi^I_{lh} = 0$, $\Pi^I_{hl} = \pi(p_l;s_l)$ and $\Pi^I_{hh} = \frac{\pi(p_l;s_l)}{2}$.

Proof. Again, let

$$V^I_{\mathbf{p}}(s_l) \equiv (1-\phi)V^I_{\mathbf{p}}(s_l, s_l) + \phi V^I_{\mathbf{p}}(s_l, s_h)$$

and

$$V^I_{\mathbf{p}}(s_h) \equiv \phi V^I_{\mathbf{p}}(s_h, s_l) + (1-\phi)V^I_{\mathbf{p}}(s_h, s_h).$$

Then, plugging equations (6) to (9) in the previous two equations we obtain,

$$V^I_{\mathbf{p}}(s_l) = (1-\delta)\Big[(1-\phi)\Pi^I_{ll} + \phi\Pi^I_{lh}\Big] + \delta\Big[(1-\phi)V^I_{\mathbf{p}}(s_l) + \phi V^I_{\mathbf{p}}(s_h)\Big]$$

and

$$V^I_{\mathbf{p}}(s_h) = (1-\delta)\Big[\phi\Pi^I_{hl} + (1-\phi)\Pi^I_{hh}\Big] + \delta\Big[\phi V^I_{\mathbf{p}}(s_l) + (1-\phi)V^I_{\mathbf{p}}(s_h)\Big].$$

Therefore, following exactly the same steps as in lemma 3 we obtain

$$V^I_{\mathbf{p}}(s_l) = \left[\frac{1-\delta}{[1-\delta(1-\phi)]^2-\delta^2\phi^2}\right]\big([1-\delta(1-\phi)]\Pi^I_l + \delta\phi\Pi^I_h\big) \tag{23}$$

and

$$V^I_{\mathbf{p}}(s_h) = \left[\frac{1-\delta}{[1-\delta(1-\phi)]^2-\delta^2\phi^2}\right]\big(\delta\phi\Pi^I_l + [1-\delta(1-\phi)]\Pi^I_h\big) \tag{24}$$

where $\Pi^I_l = (1-\phi)\Pi^I_{ll} + \phi\Pi^I_{lh}$ and $\Pi^I_h = \phi\Pi^I_{hl} + (1-\phi)\Pi^I_{hh}$.

Note that equation (6) is equal to,

$$\begin{aligned}
V^I_{\mathbf{p}}(s_l, s_l) =& (1-\delta)\Pi^I_{ll} + \delta V^I_{\mathbf{p}}(s_l)\\
=& (1-\delta)\Pi^I_{ll} + \delta\left[\frac{1-\delta}{c^2-\delta^2\phi^2}\right]\big(c[(1-\phi)\Pi^I_{ll} + \phi\Pi^I_{lh}] + \delta\phi[\phi\Pi^I_{hl} + (1-\phi)\Pi^I_{hh}]\big)\\
=& \left[\frac{1-\delta}{c^2-\delta^2\phi^2}\right]\Big(\big[c^2-\delta^2\phi^2 + c(1-c)\big]\Pi^I_{ll} + \delta\phi c\,\Pi^I_{lh} + \delta^2\phi^2\Pi^I_{hl} + \delta^2\phi(1-\phi)\Pi^I_{hh}]\Big)\\
=& \left[\frac{1-\delta}{c^2-\delta^2\phi^2}\right]\big([c-\delta^2\phi^2]\Pi^I_{ll} + \delta\phi c\,\Pi^I_{lh} + \delta^2\phi^2\Pi^I_{hl} + \delta^2\phi(1-\phi)\Pi^I_{hh}]\big).
\end{aligned}$$

Similarly, equation (7)is equal to,

$$\begin{aligned}
V^I_{\mathbf{p}}(s_l, s_h) =&(1-\delta)\Pi^I_{lh} + \delta V^I_{\mathbf{p}}(s_h)\\
=&(1-\delta)\Pi^I_{lh} + \delta\left[\frac{1-\delta}{c^2-\delta^2\phi^2}\right]\left(\delta\phi[(1-\phi)\Pi^I_{ll} + \phi\Pi^I_{lh}] + c[\phi\Pi^I_{hl} + (1-\phi)\Pi^I_{hh}]\right)\\
=&\left[\frac{1-\delta}{c^2-\delta^2\phi^2}\right]\left(\delta^2\phi(1-\phi)\Pi^I_{ll} + [c^2-\delta^2\phi^2+\delta^2\phi^2]\Pi^I_{lh} + \delta\phi c\Pi^I_{hl} + \delta(1-\phi)c\Pi^I_{hh}\right)\\
=&\left[\frac{1-\delta}{c^2-\delta^2\phi^2}\right]\left(\delta^2\phi(1-\phi)\Pi^I_{ll} + c^2\Pi^I_{lh} + \delta\phi c\Pi^I_{hl} + \delta(1-\phi)c\Pi^I_{hh}\right)
\end{aligned}$$

The same argument be done with equations (8) and (9) to get $V^I_{\mathbf{p}}(s_h, s_l)$ and $V^I_{\mathbf{p}}(s_h, s_h)$. ■

Corollary 3. *The expected ex-ante discounted payoff for an informed monopolist, solutions to the system of equations (1-2), are given by*

$$\begin{aligned}
V^M(s_l) &= \left[\frac{1-\delta}{[1-\delta(1-\phi)]^2-\delta^2\phi^2}\right]\left([1-\delta(1-\phi)]\Pi^M_l + \delta\phi\,\Pi^M_h\right)\\
V^M(s_h) &= \left[\frac{1-\delta}{[1-\delta(1-\phi)]^2-\delta^2\phi^2}\right]\left(\delta\phi\,\Pi^M_l + [1-\delta(1-\phi)]\Pi^M_h\right)
\end{aligned}$$

where $\Pi^M_l = (1-\phi)\pi(p^M_l; s_l) + \phi\pi(p^M_h; s_h)$ *and* $\Pi^M_h = \phi\pi(p^M_l; s_l) + (1-\phi)\pi(p^M_h; s_h)$.

Finally, before getting into the proof of Proposition 2, we provide a list of all potentially profitable deviations from price leadership with monopolistic prices.

Deviations from price leadership with monopolistic prices.

- Lets start by analyzing the potential deviations by the uninformed firm.
  - If the informed firm previous price was p^M_l, the uninformed firm believes that the previous market size was s_l and that the current state is s_l with probability $(1-\phi)$ and s_h with probability ϕ. Then, the uninformed firm can deviate by
    - charging a slightly lower price than p^M_l. In that case, the uninformed firm gets the whole market and believes that with probability $(1-\phi)$ its stage payoff will be arbitrarily close to $\pi(p^M_l; s_l)$ and with probability ϕ its stage payoff will be arbitrarily close to $\pi(p^M_l; s_h)$. This deviation triggers a Nash reversal. Then, for that deviation not to be profitable we require that

$$V^U_{\mathbf{p}^M}(p^M_l) \geq (1-\delta)[(1-\phi)\pi(p^M_l; s_l) + \phi\pi(p^M_l; s_h)]. \tag{25}$$

    - charging a slightly lower price than p^M_h (and above p^M_l). In that case, the uninformed firm sells only if the current state is s_h, a scenario the uninformed firm believes to occur with probability ϕ and will lead to a stage payoff of $\pi(p^M_h; s_h)$. Again, this deviation triggers a Nash reversal and is not profitable as long as,

$$V^U_{\mathbf{p}^M}(p^M_l) \geq (1-\delta)\phi\pi(p^M_h; s_h). \tag{26}$$

- Similarly, if the informed firm previous price was p_h, the uninformed firm believes that the current state is s_l with probability ϕ and s_h with probability $(1-\phi)$ and can deviate by
  - charging a slightly lower price than p_l^M. This type of deviation is not profitable as long as,

$$V^U_{\mathbf{p}^M}(p_h^M) \geq (1-\delta)[\phi\pi(p_l^M; s_l) + (1-\phi)\pi(p_l^M; s_h)]. \tag{27}$$

  - charging a slightly lower price than ${}^M p_h$. This type of deviation is not profitable if

$$V^U_{\mathbf{p}^M}(p_h^M) \geq (1-\delta)(1-\phi)\pi(p_h^M; s_h). \tag{28}$$

- Informed firm.
  - The demand goes from s_l to s_l. Because firms are following the price leadership profile, the informed firm previous price was p_l^M implying that the uninformed firm current price is also p_l^M. In this case, the only potentially profitable deviation is for the informed firm is to charge a price slightly below p_l^M, a scenario in which the informed firm gets a stage payoff arbitrarily close to $\pi(p_l^M; s_l)$.

$$V^I_{\mathbf{p}^M}(s_l, s_l) \geq (1-\delta)\pi(p_l^M; s_l) \tag{29}$$

  - The demand goes from s_l to s_h. The informed firm knows that the the uninformed is setting a price equal to p_l^M so the only potentially profitable deviation is for the informed firm is to charge a price slightly below p_l^M and get a stage payoff close to $\pi(p_l^M; s_h)$.

$$V^I_{\mathbf{p}^M}(s_l, s_h) \geq (1-\delta)\pi(p_l^M; s_h) \tag{30}$$

  - The demand goes from s_h to s_l. In that case, the informed firm is obtaining the whole monopolistic profits in that period so there is no potential profitable deviation.
  - The demand goes from s_h to s_h. In that case, the uninformed sets a current price of p_h^M because the informed previous price was p_h^M. Then, there are two potential profitable deviations for the informed firm,
    - it can charge a price slightly below p_h^M (and above p_l^M) and get a stage payoff close to $\pi(p_h^M; s_h)$. Such a deviation is not profitable as long as

$$V^I_{\mathbf{p}^M}(s_h, s_h) \geq (1-\delta)\pi(p_h^M; s_h) \tag{31}$$

    - it can deviate by pretending the state is s_h by charging a price p_l^M as in Figure 1. The informed firm gets the whole market deriving a stage payoff of $\pi(p_l^M; s_h)$. The uninformed firm does not make a sale and therefore is not able to distinguish whether the market size was low or there was a deviation. Therefore, the next period the uninformed firm will set a price equal to p_l^M.

Also, next period market size is s_l with probability ϕ and the informed firm will face an identical problem as the case in which market size went from s_l to s_l. Similarly, next period market size is s_h with probability $(1-\phi)$ and the informed firm will face an identical problem to the case in which the market size went from s_l to s_h. Consequently, if price leadership with monopolistic prices is an equilibrium, it must be the case that

$$V^I_{\mathbf{p}^M}(s_h, s_h) \geq (1-\delta)\pi(p^M_l; s_h) + \delta[\phi V^I_{\mathbf{p}^M}(s_l, s_l) + (1-\phi)V^I_{\mathbf{p}^M}(s_l, s_h)] \quad (32)$$

The following lemma rewrites the incentive constraint (32) in a way that not only simplifies the proof of Proposition 2 but also provides some intuition over the necessary conditions for the informed firm not to want to pretend that the demand is low when it is actually high.

Lemma 4. *The incentive constraint* (32) *holds if and only if*

$$[1+\delta(1-\phi)]\pi(p^M_h; s_h) - 2\pi(p^M_l; s_h) + \delta\phi\pi(p^M_l; s_l) \geq 0. \quad (33)$$

Proof. We start with the incentive constraint (32),

$$V^I_{\mathbf{p}^M}(s_h, s_h) \quad \geq \quad (1-\delta)\pi(p^M_l; s_h) + \delta\left[\phi V^I_{\mathbf{p}^M}(s_l, s_l) + (1-\phi)V^I_{\mathbf{p}^M}(s_l, s_h)\right]$$

and if we plug equation (9),

$$(1-\delta)\frac{\pi(p^M_h; s_h)}{2} + \delta\left[\phi V^I_{\mathbf{p}^M}(s_h, s_l) + (1-\phi)V^I_{\mathbf{p}^M}(s_h, s_h)\right] \geq$$
$$(1-\delta)\pi(p^M_l; s_h) + \delta\left[\phi V^I_{\mathbf{p}^M}(s_l, s_l) + (1-\phi)V^I_{\mathbf{p}^M}(s_l, s_h)\right].$$

The previous inequality can be rewritten as,

$$(1-\delta)\left[\frac{\pi(p^M_h; s_h)}{2} - \pi(p_l; s_h)\right] \geq$$
$$\delta\left\{\phi\left[V^I_{\mathbf{p}^M}(s_l, s_l) - V^I_{\mathbf{p}^M}(s_h, s_l)\right] + (1-\phi)\left[V^I_{\mathbf{p}^M}(s_l, s_h) - V^I_{\mathbf{p}^M}(s_h, s_h)\right]\right\}$$

If we plug equations (6)-(9) in the right hand side of the previous inequality, we obtain

$$(1-\delta)\left[\frac{\pi(p^M_h; s_h)}{2} - \pi(p^M_l; s_h)\right] \quad \geq \quad -\delta(1-\delta)\left[\frac{\phi\pi(p^M_l; s_l) + (1-\phi)\pi(p^M_h; s_h)}{2}\right].$$

The last inequality can be written in the following way,

$$[1+\delta(1-\phi)]\pi(p^M_h; s_h) - 2\pi(p^M_l; s_h) + \delta\phi\pi(p^M_l; s_l) \geq 0$$

which completes the proof. ■

Proposition 2. Price leadership is a PBE as long as all the incentive constraints, (25)-(32), hold. Fix the prices to the respective monopolistic prices, that is, $p^M_l = s_l/2$ and $p^M_h = s_h/2$.

First note that the first seven incentive constraints, (25)-(31), correspond to price cutting deviations that are immediately detected. Therefore, any such deviation triggers a Nash reversal. As a result, any of those deviations is profitable if firms are patient enough because price leadership has a positive continuation value. Then, there exists a $\overline{\delta}_1 < 1$ such that the incentive constraints (25)-(31) hold for any $\delta \in (\overline{\delta}_1, 1)$.

Now, it remains to show that the incentive constraint (32) holds if firms are patient enough given the assumption ($\star$). To do so, we start by plugging the monopolistic prices in equation (33).

$$[1 + \delta(1 - \phi)]\pi(s_h/2; s_h) - 2\pi(s_l/2; s_h) + \delta\phi\pi(s_l/2; s_l) \geq 0.$$

Just plugging the profits,

$$[1 + \delta(1 - \phi)]\frac{s_h^2}{4} - s_l\left(s_h - \frac{s_l}{2}\right) + \delta\phi\frac{s_l^2}{4} \geq 0$$

or

$$[1 + \delta(1 - \phi)]\frac{s_h^2}{4} - s_l s_h + (2 + \delta\phi)\frac{s_l^2}{4} \geq 0.$$

Then, multiplying by $\frac{4}{s_h^2}$, we can conclude that (33) with monopolistic prices holds if and only if

$$[1 + \delta(1 - \phi)] - 4k + (2 + \delta\phi)k^2 \geq 0$$

where $k = \frac{s_l}{s_h}$.

Denote the left-hand side of the previous inequality as the function f, that is, $f(\delta, \phi, k) = [1+\delta(1-\phi)]-4k+(2+\delta\phi)k^2$. The function f is continuous and differentiable in all arguments. Moreover, f is increasing on δ because

$$\frac{\partial f}{\partial \delta}(\delta, \phi, k) = (1 - \phi) + \phi k^2 > 0$$

and decreasing on k whenever ($\star$) holds since

$$\frac{\partial f}{\partial k}(\delta, \phi, k) = 2(2 + \phi)k - 4 < 2(2 + \phi)\left(\frac{2 - \phi}{2 + \phi}\right) - 4 = -2\phi < 0.$$

Then, because f is decreasing on k when ($\star$) holds, for any k and ϕ satisfying ($\star$),

$$\begin{aligned} f(1, \phi, k) &> f\left(1, \phi, \frac{2 - \phi}{2 + \phi}\right) \\ &= (2 - \phi) - 4\left(\frac{2 - \phi}{2 + \phi}\right) + (2 + \phi)\left(\frac{2 - \phi}{2 + \phi}\right)^2 \\ &= \left(\frac{2 - \phi}{2 + \phi}\right)[(2 + \phi) - 4 + (2 - \phi)] \\ &= 0. \end{aligned}$$

Because f is continuous on δ, if $(\star)$ holds, there exists a $\overline{\delta}_2$ such that for any $\delta \in (\overline{\delta}_2, 1)$, $f(\delta; k, \epsilon) > 0$. Moreover, because f is increasing on δ,

$$\overline{\delta}_2 = \max\left\{0, -\left(\frac{1-4k+2k^2}{1-\phi+\phi k^2}\right)\right\}.$$

Therefore, for any $\delta \in (\overline{\delta}_2, 1)$, (32) holds if $(\star)$ holds.

As a result, if $(\star)$ holds and we let $\overline{\delta} = \max\{\overline{\delta}_1, \overline{\delta}_2\}$, for any $\delta \in (\overline{\delta}, 1)$, the incentive constraints (25)-(32) hold and therefore PL is a PBE. ■

Lemma 1. From equations (23) and (24),

$$V^I_{\mathbf{p}}(s_l) = (1-\phi)V^I_{\mathbf{p}}(s_l, s_l) + \phi V^I_{\mathbf{p}}(s_l, s_h) = \left[\frac{1-\delta}{[1-\delta(1-\phi)]^2 - \delta^2\phi^2}\right]\left([1-\delta(1-\phi)]\Pi^I_l + \delta\phi\Pi^I_h\right)$$

and

$$V^I_{\mathbf{p}}(s_h) = \phi V^I_{\mathbf{p}}(s_h, s_l) + (1-\phi)V^I_{\mathbf{p}}(s_h, s_h) = \left[\frac{1-\delta}{[1-\delta(1-\phi)]^2 - \delta^2\phi^2}\right]\left(\delta\phi\Pi^I_l + [1-\delta(1-\phi)]\Pi^I_h\right)$$

where $\Pi^I_l = (1-\phi)\frac{\pi(p_l;s_l)}{2}$ and $\Pi^I_h = \phi\pi(p_l; s_l) + (1-\phi)\frac{\pi(p_h;s_h)}{2}$.

From Corollary 1,

$$V^U_{\mathbf{p}}(s_l) = \left[\frac{1-\delta}{[1-\delta(1-\phi)]^2 - \delta^2\phi^2}\right]\left([1-\delta(1-\phi)]\Pi^U_l + \delta\phi\,\Pi^U_h\right)$$
$$V^U_{\mathbf{p}}(s_h) = \left[\frac{1-\delta}{[1-\delta(1-\phi)]^2 - \delta^2\phi^2}\right]\left(\delta\phi\,\Pi^U_l + [1-\delta(1-\phi)]\Pi^U_h\right)$$

where $\Pi^U_l = (1-\phi)\frac{\pi(p_l;s_l)}{2} + \phi\pi(p_l; s_h)$ and $\Pi^U_h = (1-\phi)\frac{\pi(p_h;s_h)}{2}$.

Then, letting $c = 1-\delta(1-\phi)$

$$\begin{aligned}
V^I_{\mathbf{p}}(s_l) + V^U_{\mathbf{p}}(p_l) &= \left[\frac{1-\delta}{c^2-\delta^2\phi^2}\right]\left(c\left[\Pi^I_l + \Pi^U_l\right] + \delta\phi\left[\Pi^I_h + \Pi^U_h\right]\right) \\
&= \left[\frac{1-\delta}{c^2-\delta^2\phi^2}\right]\left(c\left[\Pi^I_l + \Pi^U_l\right] + \delta\phi\left[\Pi^I_h + \Pi^U_h\right]\right) \\
&= \left[\frac{1-\delta}{c^2-\delta^2\phi^2}\right]\left(c\left[(1-\phi)\pi(p_l; s_l) + \phi\pi(p_l; s_h)\right] + \delta\phi\left[\phi\pi(p_l; s_l) + (1-\phi)\pi(p_h; s_h)\right]\right)
\end{aligned}$$

and

$$\begin{aligned}
V^I_{\mathbf{p}}(s_h) + V^U_{\mathbf{p}}(p_h) &= \left[\frac{1-\delta}{c^2-\delta^2\phi^2}\right]\left(\delta\phi\left[\Pi^I_l + \Pi^U_l\right] + c\left[\Pi^I_h + \Pi^U_h\right]\right) \\
&= \left[\frac{1-\delta}{c^2-\delta^2\phi^2}\right]\left(\delta\phi\left[\Pi^I_l + \Pi^U_l\right] + c\left[\Pi^I_h + \Pi^U_h\right]\right) \\
&= \left[\frac{1-\delta}{c^2-\delta^2\phi^2}\right]\left(\delta\phi\left[(1-\phi)\pi(p_l; s_l) + \phi\pi(p_l; s_h)\right] + c\left[\phi\pi(p_l; s_l) + (1-\phi)\pi(p_h; s_h)\right]\right)
\end{aligned}$$

From Corollary 3,

$$V^M(s_l) = \left[\frac{1-\delta}{c^2-\delta^2\phi^2}\right]\left(c\big[(1-\phi)\pi(p_l;s_l)+\phi\pi(p_h;s_h)\big]+\delta\phi\left[\phi\pi(p_l;s_l)+(1-\phi)\pi(p_h;s_h)\right]\right)$$
$$V^M(s_h) = \left[\frac{1-\delta}{c^2-\delta^2\phi^2}\right]\left(\delta\phi\left[(1-\phi)\pi(p_l;s_l)+\phi\pi(p_h;s_h)\right]+c\big[\phi\pi(p_l;s_l)+(1-\phi)\pi(p_h;s_h)\big]\right)$$

Therefore,

$$V^M(s_l) - \left(V^I_{\mathbf{p}}(s_l)+V^U_{\mathbf{p}}(p_l)\right) = \left[\frac{1-\delta}{c^2-\delta^2\phi^2}\right]c\phi\big[\pi(p_h;s_h)-\pi(p_l;s_h)\big]$$

and

$$V^M(s_h) - \left(V^I_{\mathbf{p}}(s_h)+V^U_{\mathbf{p}}(p_h)\right) = \left[\frac{1-\delta}{c^2-\delta^2\phi^2}\right]\delta\phi^2\big[\pi(p_h;s_h)-\pi(p_l;s_h)\big].$$

Also,

$$\left[\frac{1-\delta}{c^2-\delta^2\phi^2}\right] = \left[\frac{1-\delta}{[(1-\delta)+\delta\phi)]^2-\delta^2\phi^2}\right] = \left[\frac{1-\delta}{(1-\delta)^2+2\delta(1-\delta)\phi}\right] = \left[\frac{1}{1-\delta+2\delta\phi}\right]$$

and plugging the monopolistic prices,

$$\pi(p_h^M;s_h)-\pi(p_l^M;s_h) = \frac{s_h^2}{4}-\frac{s_l}{2}\left(s_h-\frac{s_l}{2}\right) = \frac{1}{4}\left(s_h^2-s_ls_h+s_l^2\right) = \frac{1}{4}\left(s_h-s_l\right)^2.$$

Finally,

$$V^M(s_l) - \left(V^I_{\mathbf{p}^M}(s_l)+V^U_{\mathbf{p}^M}(p_l)\right) = \frac{1}{4}\left[\frac{c\phi}{1-\delta+2\delta\phi}\right]\left(s_h-s_l\right)^2$$

and

$$V^M(s_h) - \left(V^I_{\mathbf{p}}(s_h)+V^U_{\mathbf{p}}(p_h)\right) = \frac{1}{4}\left[\frac{\delta\phi^2}{1-\delta+2\delta\phi}\right]\left(s_h-s_l\right)^2.$$

■

Proposition 3. Fix s_l and s_h and let $k=\frac{s_l}{s_h}$. We define

$$\overline{\delta}_{s_l,s_h} \equiv \max\left\{\frac{1}{2}\,,\ \frac{4k-2k^2}{1+4k-2k^2}\,,\ -2k^2+4k-1\right\}.$$

We will show that for any fixed $\delta > \overline{\delta}_{s_l,s_h}$, there exists a $\overline{\phi}\in(0,1/2)$, such that for any $\phi<\overline{\phi}$ price leadership with monopolistic prices is a PBE.

We fixed any $\delta > \overline{\delta}_{s_l,s_h}$. Now, given that we are fixing the discount factor, we must pay attention to those incentive constraints that correspond to deviations that trigger a Nash reversal. Therefore, we need to verify that all all the incentive constraints (25) to (32) are

satisfied when ϕ goes to 0. First note that,

$$\lim_{\phi\downarrow 0} V^U_{\mathbf{p}^M}(p_l) = \frac{\pi(p_l^M; s_l)}{2}$$
$$\lim_{\phi\downarrow 0} V^U_{\mathbf{p}^M}(p_h) = \frac{\pi(p_h^M; s_h)}{2}$$
$$\lim_{\phi\downarrow 0} V^I_{\mathbf{p}^M}(s_l, s_l) = \frac{\pi(p_l^M; s_l)}{2}$$
$$\lim_{\phi\downarrow 0} V^I_{\mathbf{p}^M}(s_l, s_h) = \delta\frac{\pi(p_h^M; s_h)}{2}$$
$$\lim_{\phi\downarrow 0} V^I_{\mathbf{p}^M}(s_h, s_l) = (1-\delta)\pi(p_l^M; s_l) + \delta\frac{\pi(p_l^M; s_l)}{2}$$
$$\lim_{\phi\downarrow 0} V^I_{\mathbf{p}^M}(s_h, s_h) = \frac{\pi(p_h^M; s_h)}{2}$$

Next, we show that as ϕ goes to 0, the incentive constraint (25) is satisfied as long as $\delta > 1/2$. Note that (25) is satisfied if,

$$V^U_{\mathbf{p}^M}(p_l) - (1-\delta)[(1-\phi)\pi(p_l^M; s_l) - \phi\pi(p_l^M; s_h)] \geq 0.$$

The left hand side of the previous inequality is continuous on ϕ for $\phi \in [0, 1/2]$. Also, when ϕ goes to 0, the left hand side goes to

$$\frac{\pi(p_l^M; s_l)}{2} - (1-\delta)\pi(p_l^M; s_h).$$

Therefore, the incentive constraint (25) holds when ϕ goes to 0 as long as $\delta > 1/2$. The same argument applies for incentive constraints (26) to (29), and (31).

We are left with verifying that the incentive constraints (30) and (32) hold. Let's look at (30), it holds if

$$V^I_{\mathbf{p}^M}(s_l, s_h) - (1-\delta)\pi(p_l^M; s_h) \geq 0.$$

When ϕ goes to 0, the left hand size of the previous inequality goes to

$$\delta\frac{\pi(p_h^M; s_h)}{2} - (1-\delta)\pi(p_l^M; s_h).$$

The previous expression can be written as

$$\delta\frac{s_h^2}{8} - (1-\delta)\frac{s_l s_h}{2} + (1-\delta)\frac{s_l^2}{4}$$

which is greater than 0 if

$$\delta > \frac{4k - 2k^2}{1 + 4k - 2k^2}.$$

Therefore, the incentive constraint (30) holds as ϕ goes to 0 as long as $\delta > \frac{4k-2k^2}{1+4k-2k^2}$.

Finally, using lemma 4, we know that the incentive constraint (32) holds as long as

$$[1+\delta(1-\phi)]\pi(p_h^M; s_h) - 2\pi(p_l^M; s_h) + \delta\phi\pi(p_l^M; s_l) \geq 0$$

and plugging the profits, that turns into,

$$[1+\delta(1-\phi)]\frac{s_h^2}{4} - s_l\left(s_h - \frac{s_l}{2}\right) + \delta\phi\frac{s_l}{4} \geq 0.$$

If we multiply the previous inequality by $\frac{4}{s_h^2}$, we would know that the inequality holds as

$$[1+\delta(1-\phi)] - 4k + 2k^2 + \delta\phi k^2 \geq 0.$$

Note that the left hand side of the previous inequality is continuous on ϕ. Also, as ϕ goes to 0, the left hand side goes to,

$$(1+\delta) - 4k + 2k^2.$$

Consequently, the incentive constrain (32) holds when ϕ goes to 0 as long as

$$\delta > 4k - 2k^2 - 1.$$

Therefore, for any $\delta > \overline{\delta}_{s_l,s_h}$, there exists a $\overline{\phi} > 0$, such that for any $\phi < \overline{r}$, price leadership with monopolistic prices is a PBE.

In the next figure, we can observe $\overline{\delta}_{s_l,s_h}$ as a function of s_l/s_h.

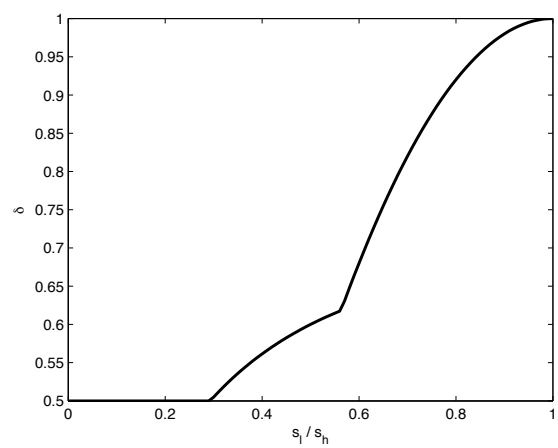

Figure 5: $\overline{\delta}_{s_l,s_h}$ as a function of s_l/s_h.

As a corollary of lemma 1, we can see that

$$\lim_{\phi\downarrow 0}\left[V^M(s_h) - (V^U_{\mathbf{p}^M}(s_h) + V^I_{\mathbf{p}^M}(p_h))\right] = 0.$$

This concludes the proof. ■

Proposition 4. The proof follows an argument similar to the one in the proof of Proposition 2. We need to show that no firm wants to price cut its competitor provided that both firms are following the price leadership profile with an arbitrary price $p_l < p_l^M$ for the low demand and the monopolistic price p_h^M for the high demand. As we have argued before the only price cut that is not immediately detected is when the both firms are supposed to set a price p_h^M and the informed firm deviates by setting a price p_l. Any other price cut would not be profitable for patient firms because it would be immediately detected and would trigger a Nash reversal.

Hence, we need to show that even when condition ($\star$) is not satisfied, there exists a $\overline{p_l} < p_l^M$ such that for any $p_l < \overline{p}_l$, a patient enough informed firm would not want to set a price p_l when both firms are supposed to set a price p_h^M. That is, for $p_l < \overline{p}_l$, exists a $\overline{\delta} < 1$ such that for any $\delta > \overline{\delta}$,

$$V_{\mathbf{p}}^I(s_h, s_h) \geq (1-\delta)\pi(p_l; s_h) + \delta\big[\phi V_{\mathbf{p}}^I(s_l, s_l) + (1-\phi)V_{\mathbf{p}}^I(s_l, s_h)\big].$$

Following exactly the same steps as in lemma (4), we can show that the previous inequality holds if

$$[1+\delta(1-\phi)]\pi(p_h^M; s_h) - 2\pi(p_l; s_h) + \delta\phi\pi(p_l; s_l) \geq 0.$$

When plugging $p_h^M = s_h/2$ in the previous inequality, we obtain

$$[1+\delta(1-\phi)]\frac{s_h^2}{4} - 2p_l(s_h - p_l) + \delta\phi p_l(s_l - p_l) \geq 0.$$

Define the function $f(p_l, \delta)$ as the left-hand side of the previous inequality. That is,

$$\begin{aligned} f(p_l, \delta) =& [1+\delta(1-\phi)]\frac{s_h^2}{4} - 2p_l(s_h - p_l) + \delta\phi p_l(s_l - p_l) \\ =& [1+\delta(1-\phi)]\frac{s_h^2}{4} - (2s_h - \delta\phi s_l)p_l + (2-\delta\phi)p_l^2. \end{aligned}$$

First, note that

$$f(0, \delta) = [1+\delta(1-\phi)]\frac{s_h}{2} > 0.$$

Hence, the informed firm would never want to deviate in such a way if $p_l = 0$. Therefore, irrespective of s_l, s_h and ϕ, price leadership where firms set a price equal to 0 for the low demand and a monopolistic price for the high demand is always an equilibrium for patient firms.

Moreover, f is a continuous and differentiable function and

$$\frac{\partial f}{\partial p_l}(p_l, \delta) = 2(2-\delta\phi)p_l - (2s_h - \delta\phi s_l).$$

Then, for any $0 \leq p_l < p_l^M$,

$$\frac{\partial f}{\partial p_l}(p_l, \delta) < (2-\delta\phi)s_l - (2s_h - \delta\phi s_l) = 2(s_l - s_h) < 0$$

and

$$\frac{\partial f}{\partial \delta}(p_l, \delta) = (1-\phi)\pi(p_h^M; s_h) + \delta\phi\pi(p_l; s_l) > 0.$$

Therefore, because of the continuity of f, as long as $f(p_l, 1) > 0$, it will exist a $\overline{\delta} < 1$ such that for any $\delta \in (\underline{\delta}, 1)$ price leadership with prices p_l and p_h^M is a PBE. Hence, because f is decreasing in p_l, there is a maximum $\overline{p}_l$ such that there exists exist a $\overline{\delta} < 1$ such that for any $\delta \in (\underline{\delta}, 1)$ price leadership with prices $p_l < \overline{p}_l$ and p_h^M is a PBE. That maximum $\overline{p}_l$ is the one that solves $f(\overline{p}_l, 1) = 0$ and it is well defined since $f(0, 1) > 0$ and $f(p_l^M, 1) < 0$.

Therefore, we have shown that we can always support a price leadership equilibrium if firms are patient. ■

Lemma 2. We start with the incentive constraint (14),

$$V^I_{\mathbf{p}^M}(s_l, s_h) \geq (1-\delta)\frac{\pi(p_l^M; s_h)}{2} + \delta\big[\phi V^I_{\mathbf{p}^M}(s_l, s_l) + (1-\phi)V^I_{\mathbf{p}^M}(s_l, s_h)\big].$$

and if we plug equation (7) in the left-hand-side of the previous inequality,

$$\delta\big[\phi V^I_{\mathbf{p}^M}(s_h, s_l) + (1-\phi)V^I_{\mathbf{p}^M}(s_h, s_h)\big] \geq (1-\delta)\frac{\pi(p_l^M; s_h)}{2} + \delta\big[\phi V^I_{\mathbf{p}^M}(s_l, s_l) + (1-\phi)V^I_{\mathbf{p}^M}(s_l, s_h)\big].$$

The previous inequality can be rewritten as,

$$\delta\big\{\phi\left[V^I_{\mathbf{p}^M}(s_h, s_l) - V^I_{\mathbf{p}^M}(s_l, s_l)\right] + (1-\phi)\left[V^I_{\mathbf{p}^M}(s_h, s_h) - V^I_{\mathbf{p}^M}(s_l, s_h)\right]\big\} \geq (1-\delta)\frac{\pi(p_l^M; s_h)}{2}.$$

If we plug equations (6)-(9) in the left hand side of the previous inequality, we obtain

$$\delta(1-\delta)\left[\frac{\phi\pi(p_l^M; s_l) + (1-\phi)\pi(p_h^M; s_h)}{2}\right] \quad \geq \quad (1-\delta)\frac{\pi(p_l^M; s_h)}{2}.$$

or

$$\delta(1-\phi)\pi(p_h^M; s_h) - \pi(p_l^M; s_h) + \delta\phi\pi(p_l^M; s_l) \geq 0.$$

Plugging the profits, the previous inequality turns into,

$$\delta(1-\phi)\frac{s_h^2}{4} - \frac{s_l s_h}{2} + \frac{s_l^2}{4} + \delta\phi\frac{s_l^2}{4} \geq 0$$

and multiplying by $\frac{4}{s_h^2}$,

$$\big[1+\delta\phi\big]\left(\frac{s_l}{s_h}\right)^2 - 2\left(\frac{s_l}{s_h}\right) + \delta(1-\phi) \geq 0.$$

■

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