Balanced Contributions and Fairness in Exchange Economies

Gustavo Gudiño

gustavo@psu.edu

October 27, 2014

Abstract

This paper presents a new way to fairly distribute welfare gains derived from trade in exchange economies. The imposed fairness condition is based on the balanced marginal contributions condition satisfied by the Shapley value in transferable utility games. The solution is defined for cardinal finite exchange economies. Then, we point out that ordinal solutions can be created by mapping finite ordinal exchange economies onto cardinal economies and then applying the original solution. Two ordinal solutions that inherit the properties of the original solution for cardinal economies are proposed.

1 Introduction

Trade in exchange economies leads to welfare gains, yet one of the lingering questions in economics is how to fairly distribute these gains. Our purpose is to propose a new definition of fairness to evaluate allocations in exchange economies with individual endowments. The proposed solution takes into consideration that initial endowments and trade possibilities among subgroups of the economy lead to agents possessing different rights.

Multiple conceptions of fairness have been suggested throughout the years. The concepts of envy-freeness (Foley, 1967) and egalitarian equivalence (Pazner and Schmeidler, 1978) propose ways to fairly distribute a social endowment. However, when considering an economy with *individual* endowments, fairness is not the only property that an allocation should satisfy. At a minimum, we should require allocations to be individually rational. Schmeidler and Vind (1972) extend the envy-free notion to economies with individual endowments by considering envy-free trades. In a work similar to ours, Pérez-Castrillo and Wettstein (2006) propose a way to extend egalitarian equivalence to the case of economies with individual endowments.

Any notion of fairness implies interpersonal comparisons. Pérez-Castrillo and Wettstein (2006) propose a solution, the *ordinal Shapley value*, which defines interpersonal comparisons

in terms of a reference bundle. This is reminiscent of the way the Nash bargaining solution can be considered a social welfare function through implicit interpersonal comparisons based on the status quo point. Pérez-Castrillo and Wettstein (2006) measure agents' trading contributions to the welfare of others in terms of the reference bundle. Then, as a fairness condition, they ask for these contributions to be balanced, that is, an agent contributes to the other agents in the economy in the same amount that the others contribute to her. The ordinal Shapley value receives its name because balance contributions is a property that characterizes the Shapley value in transferable utility games (Myerson, 1980).

We follow an approach similar to Pérez-Castrillo and Wettstein (2006) by also directly defining the balance contributions property in exchange economies but by interpreting contributions in a different manner. The solution is initially defined for finite cardinal exchange economies, and interpersonal comparisons are made in terms of utility gains. An agent's welfare gain when going from consuming one bundle to another is interpreted as the change in utility. Then, an agent's contribution to another agent is the utility gain that the latter obtains by having the former in the economy. Our solution maps each cardinal economy onto the set of efficient allocations that balance contributions. By defining a solution entirely based on properties that characterize the Shapley value, we also extend it to a non-transferable utility environment. Our approach differs from those of the non-transferable utility values in Harsanyi (1963), Shapley (1969) and Maschler and Owen (1992) because those three values associate each non-transferable utility game with transferable utility games to then use their Shapley value to select a distribution for the original environment.

The solution always satisfies uniqueness in utility levels, two kinds of monotonicity in endowments, symmetry in contributions and equal treatment of equals. Also, under minimal assumptions, the solution is non-empty. Although the solution is not invariant to affine transformation of utility functions, ordinal solutions can be created by mapping ordinal economies into cardinal economies whenever possible and then applying our solution. We present two such ordinal solutions that inherit all the properties of the cardinal solution. The first ordinal solution measures contributions in terms of a common reference bundle, similar to ordinal Shapley value and reminiscent of egalitarian equivalence. Following the guidelines of previous works, the second ordinal solution measures changes by means of compensating variations.[1]

The next section describes the environment and the solution. Section 3 presents the solution's properties. Finally, Section 4 discusses the cardinality of the solution.

2 The Environment and the Solution

In Section 2 and 3 we work in the space of **finite cardinal exchange economies**. A cardinal exchange economy consists of l divisible goods and a nonempty set of agents $N = \{1, \ldots, n\}$ where each agent $i \in N$ is represented by her utility function representation $u_i : \mathbb{R}^l_+ \to \mathbb{R}$ and her initial endowment $\omega_i \in \mathbb{R}^l_+$. Altogether, a cardinal exchange economy is given by

[1] See Chipman and Moore (1980) for more on using compensating variations as measures of welfare change.

$(u_i, \omega_i)_{i \in N}$.

For $(u_i, \omega_i)_{i \in N}$, an **allocation** is a vector $x = (x_i)_{i \in N} \in \mathbb{R}^{n \times l}_+$ which assigns the bundle $x_i \in \mathbb{R}^l_+$ to agent i. An allocation x is **feasible** if $\sum_{i \in N} x_i \leq \sum_{i \in N} \omega_i$. An allocation x is **Pareto efficient** if it is feasible and there is no other feasible allocation y such that $u_i(y_i) \geq u_i(x_i)$ for all $i \in N$ and $u_j(y_j) > u_j(x_j)$ for some $j \in N$. A utility function u is **monotone** if for $x, x' \in \mathbb{R}^l_+$, $x' \gg x$ implies $u(x') > u(x)$.[2] For notational simplicity, it will be assumed that for each $i \in N$, $u_i(0) = 0$.

A **solution** will be a mapping that assigns a set of feasible allocations to each economy $(u_i, \omega_i)_{i \in N}$. A solution is said to be efficient if every economy is mapped into a set of efficient allocations.

2.1 The solution

Next, we define a new solution, μ, that is based on the notion of balanced contributions satisfied by the Shapley value in TU-games.[3] Contributions capture the welfare gains derived from trade in terms of utility gains. We require contributions to be balanced as the fairness condition.

Definition 1 Given $(u_i, \omega_i)_{i \in N}$, the solution μ is inductively defined as follows,

a. For an economy with just one agent (u_1, ω_1), the solution μ is just the endowment, that is,
$$\mu(u_1, \omega_1) = \{\omega_1\}.$$

b. For an economy with $n \geq 2$, assume that the solution has been defined for each economy with $N' \subsetneq N$.Then, an allocation x is in the solution if it is Pareto efficient and there exists a $y^{-k} \in \mu((u_j, \omega_j)_{j \in N \setminus \{k\}})$ for each $k \in N$ such that,
$$\sum_{j \neq i} \left[u_j(x_j) - u_j(y_j^{-i}) \right] = \sum_{j \neq i} \left[u_i(x_i) - u_i(y_i^{-j}) \right] \qquad \forall\, i \in N. \qquad (\star)$$

For any allocation in the solution x, we will say that the allocations y^{-j} for $j \in N$ **support** x in the solution if they are the allocations in the solution for the economies with $n-1$ agents satisfying $(\star)$.

Note that if agent j was not in the economy, then the solution would assign a bundle y_i^{-j} to agent i. By adding agent j to the economy, the solution assigns the bundle x_i to agent i. Then, player j's contribution to agent i is defined as i's utility gain of having agent j, that is, $u_i(x_i) - u_i(y_i^{-j})$. Condition $(\star)$ requires contributions to be balanced as in the Myerson's characterization of the Shapley value.

Next, we provide an example of how the solution works for an economy with three agents.

[2]For two vectors of the same dimension $a = (a_1, \ldots, a_m)$ and $b = (b_1, \ldots, b_m)$, we will write $a \gg b$ if and only if $a_n > b_n$ for $n = 1, \ldots, m$.

[3]Characterization due to Myerson (1980). Appendix A contains the formal characterization.

Example. Consider the economy with two commodities and three agents represented by $u_1(x) = u_2(x) = x_1^{1/2}x_2^{1/2}$ and $u_3(x) = x_1 + x_2$ and $\omega_1 = (9,1)$, $\omega_2 = (1,9)$, and $\omega_3 = (5,5)$. Lets compute the solution for the economy $(u_i, \omega_i)_{i\in\{1,2,3\}}$. So the solution has to be computed inductively. For the economies with just one agent the solutions are just their endowments. The next table contains both allocations and utility levels for one agent subeconomies.

Set of agents	x_1	$u_1(x_1)$	x_2	$u_2(x_2)$	x_3	$u_3(x_3)$
$\{1\}$	$(9,1)$	3	-	-	-	-
$\{2\}$	-	-	$(1,9)$	3	-	-
$\{3\}$	-	-	-	-	$(5,5)$	10

Now lets consider economies with two agents. Considering the economy with agent 1 and 2, $(u_i, \omega_i)_{i\in\{1,2\}}$. The allocation $y^{-3} = (y_1^{-3}, y_2^{-3})$ is in the solution if it is efficient and

$$u_1(y_1^{-3}) - u_1(\omega_1) = u_2(y_2^{-3}) - u_2(\omega_2).$$

The following table contains the unique allocations in the solution.

Set of agents	x_1	$u_1(x_1)$	x_2	$u_2(x_2)$	x_3	$u_3(x_3)$
$\{1,2\}$	$(5,5)$	5	(5,5)	5	-	-
$\{1,3\}$	$(\frac{13}{3}, \frac{13}{3})$	$\frac{13}{3}$	-	-	$(\frac{29}{3}, \frac{5}{3})$	$\frac{34}{3}$
$\{2,3\}$	-	-	$(\frac{13}{3}, \frac{13}{3})$	$\frac{13}{3}$	$(\frac{5}{3}, \frac{29}{3})$	$\frac{34}{3}$

Finally, lets consider the three agents case. Then, when agent 3 is not in the economy the solution assigns the allocation $y^{-3} = (y_1^{-3}, y_2^{-3}) = ((5,5),(5,5))$. Similarly, for the other cases. Then, given a bundle x_1, the contribution of agent 3 to agent 1 will be given by $u_1(x_1) - u_1(y_1^{-3})$ and contributions for agent 1 will be balanced if

$$[u_1(x_1) - u_1(y_1^{-2})] + [u_1(x_1) - u_1(y_1^{-3})] = [u_2(x_2) - u_2(y_2^{-1})] + [u_3(x_3) - u_3(y_3^{-1})].$$

Then, an allocation is in the solution if it is efficient and balances contributions for every agent. The following table contains the only allocation in the solution and the corresponding utility levels for each of the agents. The corresponding contributions are

Set of agents	x_1	$u_1(x_1)$	x_2	$u_2(x_2)$	x_3	$u_3(x_3)$
$\{1,2,3\}$	$(\frac{71}{15}, \frac{71}{15})$	$\frac{71}{15}$	$(\frac{71}{15}, \frac{71}{15})$	$\frac{71}{15}$	$(\frac{83}{15}, \frac{83}{15})$	$\frac{166}{15}$

$$\begin{bmatrix} - & u_1(x_1) - u_1(y_2^{-1}) & u_1(x_1) - u_1(y_3^{-1}) \\ u_2(x_2) - u_2(y_2^{-1}) & - & u_2(x_2) - u_2(y_2^{-3}) \\ u_3(x_3) - u_3(y_3^{-1}) & u_3(x_3) - u_3(y_3^{-2}) & - \end{bmatrix} = \begin{bmatrix} - & \frac{6}{15} & -\frac{4}{15} \\ \frac{6}{15} & - & -\frac{4}{15} \\ -\frac{4}{15} & -\frac{4}{15} & - \end{bmatrix}$$

Note that agent 3 contribution's are both negative. Consequently, agents 1 and 2 are worse off with the inclusion of agent 3 and the solution is not in the core of the economy. Agents 3 contributions to the others are negative because the solution considers the case in which agent 1 (resp. 2) leaves the economy and in that situation, agent 2 (resp. 1) will benefit from having agent 3 in the economy.

3 Properties of the solution

3.1 Uniqueness of utility levels

We say that the solution induces unique utility levels (U) if each agent is indifferent between the bundles assigned to him at any two allocations in the solution. Formally, the utility levels induced by the solution μ are unique if

(U) For any cardinal economy $(u_k, \omega_k)_{k\in N}$ and for any two allocations in its solution, $\{x, x'\} \subset \mu((u_k, \omega_k)_{k\in N})$,

$$u_i(x_i) = u_i(x'_i) \qquad \forall\, i \in N.$$

Proposition 1 *The solution satisfies (U).*

3.2 Symmetry

Note that our solution requires that for any $i \in N$, the sum of all others' contribution to i are equal to the sum of i's contributions to all the others. Furthermore, the balancedness condition holds in a stronger sense that we name symmetry. The solution μ satisfies symmetry if for any two pair of agents, each one is contributing to the other in the same amount. Formally, the solution satisfies symmetry if,

(S) For any economy $(u_k, \omega_k)_{k\in N}$ with $n \geq 2$, and for any $x \in \mu((u_k, \omega_k)_{k\in N})$ supported by the allocations $y^{-i} \in \mu((u_j, \omega_j)_{j\neq i})$ for each $i \in N$, the following equation holds for any pair $i, j \in N$ with $i \neq j$,

$$u_i(x_i) - u_i(y_i^{-j}) = u_j(x_j) - u_j(y_j^{-i}).$$

Proposition 2 *The solution satisfies (S).*

3.3 Non-emptyness

In the following claim, conditions for the solution μ to be non-empty are stated.

Proposition 3 *Consider $(u_i, \omega_i)_{i\in N}$ with $1 \leq n < \infty$ and $\omega_i \gg 0$ for each $i \in N$. Moreover, assume that u_i is continuous and monotone for each $i \in N$, then the solution $\mu((u_i, \omega_i)_{i\in N})$ satisfies,*

(NE) $\mu((u_i, \omega_i)_{i\in N}) \neq \varnothing$.

3.4 Individual Rationality

The solution satisfies individual rationality meaning that it never makes an agent worse than how she would be on her own. Formally, an allocation x is individually rational if $x_i \succeq_i \omega_i$ for all $i \in N$. The solution is individually rational if every economy is mapped into a set of individually rational allocations.

Proposition 4 *The solution μ is individually rational.*

3.5 Monotonicity in initial endowments

Monotonicity in initial endowments can be understood in at least two ways: $(M.1)$ if there are two agents with the same utility function but one of them initially has more resources than the other then the former is at least as good as the latter at any allocation in the solution; or, $(M.2)$ if an agent's initial endowment increases, then the solution cannot make her worse off. Formally, the solution μ satisfies $(M.1)$ and $(M.2)$ if

(M.1) For any economy $(u_i, \omega_i)_{i \in N}$ and any two agents $j, k \in N$ with $u_j = u_k$ and $\omega_j \geq \omega_k$,

$$u_j(x_j) \geq u_k(x_k)$$

for any $x \in \psi((u_i, \omega_i)_{i \in N})$.

(M.2) For any two economies, $(u_i, \omega_i)_{i \in N}$ and $(u_i, \omega_i')_{i \in N}$, with the same set of agents N such that there exist one and only one agent k such that $\omega_k \geq \omega_k'$ and $\omega_j = \omega_j'$ for each $j \neq k$. Then,

$$u_k(x_k) \geq u_k(x_k')$$

for any $x \in \psi((u_i, \omega_i)_{i \in N})$ and any $x' \in \psi((u_i, \omega_i')_{i \in N})$.

The solution μ satisfies both.

Proposition 5 *The solution μ satisfies (M.1).*

Proposition 6 *The solution μ satisfies (M.2).*

3.6 Equal Treatment of Equals

Another property that our solution satisfies is the equal treatment of equals. In any economy with two agents that are represented by the the same utility function representation and have the same endowment, the solution always make them indifferent. Formally,

(ETE) For any $(u_i, \omega_i)_{i \in N}$ and $x \in \mu((u_i, \omega_i)_{i \in N})$, if there are two agents $j, k \in N$ such that $u_j = u_k$ and $\omega_j = \omega_k$, then $u_j(x_j) = u_j(x_k)$.

Proposition 7 *The solution μ satisfies (ETE).*

Note that the equal treatment of equals is a property that the non-transferable utility values do not satisfy.[4]

4 Discussion of the Solution's Cardinality

There are reasons to be concerned about the fact that the solution is not unique up to affine transformations of the utility function representations; an example of this is if we put ourselves in a situation in which agents' preferences must be inferred from choices. But by defining the solution for the space of cardinal exchange economies we obtain some versatility. The solution can easily be adapted in at least two ways to obtain an ordinal solution that continues to satisfy all the properties from the previous section.

Lets now consider **finite ordinal exchange economies**. An ordinal exchange economy consists of l divisible goods and a set of agents $N = \{1, \ldots, n\}$ where each agent $i \in N$ is described by her binary preference relation $\succeq_i$ on $\mathbb{R}^l_+$ and her initial endowment $\omega_i \in \mathbb{R}^l_+$. Altogether, an ordinal exchange economy is a tuple $(\succeq_i, \omega_i)_{i \in N}$.

Whenever possible, we can define a solution for ordinal economies that first map them into cardinal economies to then apply the solution μ. Note that a sufficient requirement for all the properties to hold is that each preference profile is mapped into a unique profile of utility function that represent those preferences in a way that any two agents with the same preferences are represented by the same utility function.

4.1 A reference bundle approach

The balanced contributions allocations admit a reference bundle interpretation similar to the one in the ordinal Shapley value in the following way. As long as a preference relation $\succeq$ is regular, continuous and monotone, we can fix an arbitrary bundle $a \in \mathbb{R}^l_{++}$ and represent the preferences through the continuous and monotone utility function $u^a_i : \mathbb{R}^l_+ \to \mathbb{R}$ where $u^a_i(x)$ is the unique real such that $x \sim_i u^a_i(x) \cdot a$ for any $x \in \mathbb{R}^l_+$.

So for any ordinal economy $(\succeq_i, \omega_i)_{i \in N}$ if preferences are regular, continuous and monotone, we can fix a common reference bundle $a \in \mathbb{R}^l_{++}$, and let the well-defined solution μ^a for the economy $(\succeq_i, \omega_i)_{i \in N}$ be the set of allocations

$$\mu^a((\succeq_i, \omega_i)_{i \in N}) = \mu((u^a_i, \omega_i)_{i \in N}).$$

The solution μ^a measures the welfare gain an agent gets when going from a bundle x to a bundle y is the proportion of the reference bundle that the agent needs to be added to $u^a_i(x) \cdot a$ to make her indifferent to y. Provided that the same reference bundle is used for each agent, μ^a possess all the properties from the previous section plus ordinality but the suggested outcome now depends on the selection of the reference bundle.

[4] Yannelis (1985) provides an example of an exchange economy in which the Shapley non-transferable utility value (Shapley, 1969) fails to treat equal equally.

4.2 An expenditure approach

We can also try to use the compensating variation as a measure of welfare change to tackle the cardinality issue. To do so, we will use the fact that if a preference relation $\succeq$ is regular, continuous and monotone, then the expenditure function $e_i(p, \cdot) : \mathbb{R}^l_+ \to \mathbb{R}$ is a utility function representation for any price $p \in \mathbb{R}^l_{++}$.[5]

Consider an ordinal exchange economy $(\succeq_i, \omega_i)_{i \in N}$. If preferences are regular, continuous and monotone, the expenditure function, we can fix a strictly positive set of prices $p \in \mathbb{R}^l_+$ and using the expenditure function as a utility representation for every agent, we can define the solution, μ^p, for the ordinal economy $(\succeq_i, \omega_i)_{i \in N}$ as the set of allocations

$$\mu^p((\succeq_i \omega_i)_{i \in N}) = \mu((e_i(p, \cdot), \omega_i)_{i \in N}).$$

The solution μ^p satisfies all the properties of the solution plus ordinality. Note that the symmetry of μ^p is equivalent as i's compensating variation for having j in the economy being equal to j's compensating variation of having i in the economy. Also, using expenditures functions to represent agents' preferences guarantees that any two agents with the same preferences are represented by the same utility function irrespective of the price.[6]

A The Shapley value

Consider a TU-game (N, v) where N is the set of agents and $v : 2^N \to \mathbb{R}^{|N|}$ is the characteristic function. A **value** is a function f that maps every game (N, v) to a vector in $\mathbb{R}^{|N|}$ such that $\sum_{i \in N} f_i(N, v) = v(N)$. The next theorem presents a characterization of the Shapley value (Shapley, 1953) due to Myerson (1980).

Theorem 8 *A value ξ is the Shapley value if and only if it satisfies*

$$\sum_{i \in N \setminus \{j\}} (\xi_i(N, v) - \xi_i(N \setminus \{j\}, v)) = \sum_{i \in N \setminus \{j\}} (\xi_j(N, v) - \xi_j(N \setminus \{i\}, v))$$

for all (N, v) with $|N| \geq 2$ and all $j \in N$.

For a given value ξ, the difference $\xi_i(N, u) - \xi_i(N \setminus \{j\}, u)$ is commonly known as the marginal contribution of player j to the value of player i. Therefore, Theorem 8 states that the Shapley value is the only one that balances contributions in the sense that for each agent, the sum of her marginal contributions to others add up to sum of others' marginal contributions to her.

[5] The expenditure function $e_i(p, \cdot)$ is defined as $e_i(p, x^0) = \min\{\, p \cdot x \,|x \in \mathbb{R}^l_+ \quad \text{and} \quad x \succeq_i x^0\}$.

[6] Appendix B contains an example of an economy in which the almost all the allocations in the contract curve can be sustained in the cardinal solution by adequately choosing utility function representations, but the unique allocation in the solution with the expenditure approach does not change when the prices change.

B Proofs

Proposition 1 and 2. The proof proceeds by induction on the number of agents.

When we are considering an economy with just one agent, the solution is just the endowment so the utility levels are unique.

An economy with two agents.
Consider an economy $((u_1, \omega_1), (u_2, \omega_2))$.

(S) Let $x \in \mu((u_1, \omega_1), (u_2, \omega_2))$. Then, by definition,

$$u_1(x_1) - u_1(\omega_1) = u_2(x_2) - u_2(\omega_2).$$

Then, contributions are balanced one to one.

(U) Suppose towards a contradiction that there are $x, x' \in \mu((u_1, \omega_1), (u_2, \omega_2))$ such that $u_1(x_1) > u_1(x'_1)$. But then, since x and x' are in the solution,

$$u_1(x_1) - u_1(\omega_1) = u_2(x_2) - u_2(\omega_2)$$

and

$$u_1(x'_1) - u_1(\omega_1) = u_2(x'_2) - u_2(\omega_2).$$

But then, $u_1(x_1) > u_1(x'_1)$ and the last two equalities imply that $u_2(x_2) > u_2(x'_2)$. But this implies that x Pareto dominates x' and therefore contradicts x' being in the solution since it is not efficient.

An economy with more than two agents.
Consider an economy $(u_i, \omega_i)_{i \in N}$ with $n > 2$. Lets assume that the solution satisfies uniqueness in utility levels (U) and contributibutions are balanced one to one (S) for all economies with less than n agents. For each agent $i \in N$, let $u^{-i} = (u_j^{-i})_{j \in N \setminus \{i\}}$ be the utility levels induced by the solution for the economy $(u_j, \omega_j)_{j \in N \setminus \{i\}}$.

For notational simplicity, for any allocation x in $(u_i, \omega_i)_{i \in N}$ and each pair of agents i and j with $i \neq j$ calculate the contribution of agent j to agent i, $c_i^{-j}(x)$, as

$$c_i^{-j}(x) = u_i(x_i) - u_i^{-j}.$$

Now lets define the function $C_i : \mathbb{R}_+^{n \times l} \to \mathbb{R}$ as

$$C_i(x) = \sum_{j \neq i} c_j^{-i}(x) - \sum_{j \neq i} c_i^{-j}(x).$$

$C_i(x)$ can be seen as the net contribution of agent i. If $C_i(x) < 0$, then the sum of the contributions that agent i is receiving is larger than the sum of the contributions that i is giving to the others. An arithmetic result that will be used throughtout proofs is the fact that $\sum_{i \in N} C_i(x) = 0$ for any allocation $x \in \mathbb{R}_+^l$.

An intermediate result that will be helpful is presented in the following Lemma.

Lemma 9 *Under the induction hypotheses, for any pair of agents $i, j \in N$ with $i \neq j$ and for any allocation x in the economy with n agents, the following equation holds,*

$$C_j(x) - C_i(x) = n\,(c_i^{-j}(x) - c_j^{-i}(x)).$$

The proof of Lemma 9 can be found at the end of this appendix. Lemma 9 will be used to show that properties (U) and (S) are satisfied by the solution in the economy with n agents.

(S) Suppose that $x \in \mu((u_k, \omega_k)_{k \in N})$. Then, by definition $C^i(x) = 0$ for each $i \in N$. Then, using Lemma 9, for any pair of agents i and j,

$$n\,(c_i^{-j}(x) - c_j^{-i}(x)) = C_j(x) - C_i(x) = 0$$

implying

$$c_i^{-j}(x) = c_j^{-i}(x)$$

or equivalently,

$$u_i(x_i) - u_i^{-j} = u_j(x_j) - u_j^{-i}.$$

Therefore, contributions are balanced one to one.

(U) Suppose towards a contradiction that x and x' are in the solution $\mu((u_i, \omega_i)_{i \in N})$ but $u_i(x_i) > u_i(x_i')$ for some agent i. Lemma 9 implies that for any $j \in N \setminus \{i\}$,

$$u_i(x_i) - u_i^{-j} = u_j(x_j) - u_j^{-i}$$

and

$$u_i(x_i') - u_i^{-j} = u_j(x_j') - u_j^{-i}.$$

Using $u_i(x_i) - u_i^{-j} > u_i(x_i') - u_i^{-j}$ and the last two equalities, it must be true that for any agent $j \neq i$

$$u_j(x_j) - u_j^{-i} > u_j(x_j') - u_j^{-i}.$$

But then x would Pareto dominate x' and therefore contradict x' being in the solution $\mu((u_k, \omega_k)_{k \in N})$. Hence, the utility levels must be unique.

□

Proposition 3. The proof proceeds by induction on the number of agents.

An economy with just one agent.

For an economy with just one agent, the endowment is in the solution. Therefore, the solution is non-empty.

Economies with more than one agent.

Consider an economy $(u_k, \omega_k)_{k\in N}$ with $n \geq 2$. Again, for each $j \in N$, let u^{-j} be the unique utility levels induced by the solution for the economy $(u_i, \omega_i)_{i\in N\setminus\{j\}}$. Also,

$$c_i^{-j}(x) = u_i(x_i) - u_i^{-j}$$

and

$$C_i(x) = \sum_{j\neq i} c_j^{-i}(x) - \sum_{j\neq i} c_i^{-j}(x).$$

The next lemma contains a result that will be used to show the existence of an allocation in the solution.

Lemma 10 *Let $(u_i, \omega_i)_{i\in N}$ be such that u_i is continuous and mononote and $\sum_{i\in N} \omega_i \gg 0$. Fix $y^{-j} \in \mu((u_i, \omega_i)_{i\neq j})$ for each $j \in N$. Then, there is an efficient allocation x^* in $(u_i, \omega_i)_{i\in N}$ satisfying*

$$u_i(x_i^*) > 0 \quad \textit{implies} \quad C_i(x^*) \geq 0 \qquad \textit{for each } i \in N. \tag{$\Diamond$}$$

The proof of Lemma 10 can be found at the end of this appendix.

Using Lemma 10, let x^* be the efficient allocation satisfying ($\Diamond$). To show that x is in the solution $\mu((u_k, \omega_k)_{k\in N})$ is enough to show that it cannot be the case that $u_i(x_i^*) = 0$ for some $i \in N$. That is true since then $u_i(x_i^*) > 0$ for each $i \in N$ and therefore $C_i(x^*) \geq 0$ for each $i \in N$ and $\sum_{j\in N} C_j(x^*) = 0$.

Suppose towards a contradiction that $u_i(x_i^*) = 0 < u_i(\omega_i)$ for some agent $i \in N$. Then, x^* being efficient implies that there is an agent $j \in N$ with $u_j(x_j^*) > u_j(y_j^{-i})$ otherwise (ω_i, y^{-i}) would Pareto dominate x^*. But then $c_j^{-i} = u_j(x_j^*) - u_j(y_j^{-i}) > 0$ and $c_i^{-j}(x^*) = -u_i(y_i^{-j}) \leq 0$. Then, using Lemma 9

$$C_i(x^*) - C_j(x^*) = n[c_j^{-i}(x*) - c_i^{-j}(x^*)] > 0.$$

This implies that $C_i(x^*) > C_j(x^*) \geq 0$. But then, $\sum_{k\in N} C_k(x^*) > 0$ which is a contradiction. Therefore, $u_i(x_i^*) > 0$ for each $i \in N$. Then, x^* is an efficient allocation satisfying $C_i(x^*) = 0$ for each $i \in N$ and is therefore in the solution. Hence, the solution is non-empty. □

Proposition 4. The proof proceeds by induction in the number of agents.

Economies with one agent.
For economies with just one agent the solution set is the endowment. Therefore the solution is individually rational.

Economies with more than one agent.
For an economy $(u_i, \omega_i)_{i\in N}$ with $n \geq 2$, assume that the solution is individually rational for all economies with less than n agents.

Suppose towards a contradiction that there exists $x^* \in \mu((u_i, \omega_i)_{i\in N})$ such that $u_j(x_j^*) < u_j(\omega_j)$ for some $j \in N$.

Then, since x^* is efficient, there must be an agent $k \in N$ with $u_k(x_k^*) > u_k(y_k^{-j})$ otherwise (ω_j, y^{-j}) would Pareto dominate x^*. But this implies

$$u_k(x_k^*) - u_k(y_k^{-j}) > 0 > u_j(x_j^*) - u_j(y_j^{-k})$$

contradicting symmetry (S). Therefore, the solution μ must be individually rational. □

Proposition 5. Consider an economy with two agents with the same utility function $u = u_1 = u_2$ but with $\omega_1 \geq \omega_2$. Then, if x is in the solution it must be the case that

$$u(x_1) = u(x_1) - u(\omega_1) + u(\omega_1) = u(x_2) - u(\omega_2) + u(\omega_1) \geq u(x_2).$$

Now, lets consider an economy with $n \geq 3$ agents. Assume that the solution satisfies M.1 for economies with $n-1$ agents.

Note that the economy with set of agents $N \setminus \{2\}$ has at least as many resources than the economy with set of agents $N \setminus \{1\}$. Then, there must be an agent $j \in N$ with $u_j(y_j^{-2}) \geq u_j(y_j^{-1})$ or equivalently $u_j(x_j) - u_j(y_j^{-1}) \geq u_j(x_j) - u_j(y_j^{-2})$. But then, using the fact that conributions are symmetric,

$$u(x_1) - u(y_1^{-j}) \geq u(x_2) - u(y_2^{-j})$$

and since $u(y_1^{-j}) \geq u(y_2^{-j})$ by induction hyothesis,

$$u(x_1) \geq u(x_2).$$

Therefore, the solution satisfies condition M.1. □

Proposition 6. Consider the case for economies with just two agents. Without loss of generality let $(u_i, \omega_i)_{i \in \{1,2\}}$ and $(u_i, \widetilde{\omega}_i)_{i \in \{1,2\}}$ differ only in $\omega_1 \geq \widetilde{\omega}_1$. Suppose towards a contradiction that there are $x \in \mu((u_i, \omega_i)_{i \in \{1,2\}})$ and $x' \in \mu((u_i, \widetilde{\omega}_i)_{i \in \{1,2\}})$ with $u_1(x_1) < u_1(\widetilde{x}_1)$. Then,

$$u_2(x_2) - u_2(\omega_2) = u_1(x_1) - u_1(\omega_1) < u_1(\widetilde{x}_1) - u_(\widetilde{\omega}_1) = u_2(\widetilde{x}_2) - u_2(\omega_2)$$

which implies $u_2(x_2) < u_2(\widetilde{x}_2)$. But x is feasible in $(u_i, \omega_i)_{i \in \{1,2\}}$ and Pareto dominates x. This is a contradiction since x is in the solution.

Consider a set of agents N with $n \geq 3$ and two economies $(u_i, \omega_i)_{i \in N}$ and $(u_i, \widetilde{\omega}_i)_{i \in N}$ differ only in $\omega_1 \geq \widetilde{\omega}_1$. Assume that the solution satisfies condition M.2 for economies with $n-1$ agents. Let $x \in \mu((u_i, \omega_i)_{i \in N})$ and $\widetilde{x} \in \mu((u_i, \widetilde{\omega}_i)_{i \in N})$. Suppose towards a contradiction that $u_1(x_1) < u_1(\widetilde{x}_1)$. Then, there must be an agent $j \in N$ with $u_j(x_j) > u_j(\widetilde{x}_j)$ or equivalently

$$u_j(x_j) - u_j(y_j^{-1}) > u_j(\widetilde{x}_j) - u_j(\widetilde{y}_j^{-1}).$$

The last inequality comes from the fact that $(u_i, \omega_i)_{i \neq 1} = (u_i, \widetilde{\omega}_i)_{i \neq 1}$ and the utility levels attained in the solution are unique. But using symmetry,

$$u_1(x_1) - u_1(y_1^{-j}) > u_1(\widetilde{x}_1) - u_1(\widetilde{y}_1^{-j})$$

but then

$$u_1(\widetilde{y_1}^{-j}) > u_1(y_1^{-j})$$

which is a contradiction to the induction hypothesis.

Therefore, the solution μ satisfies M.2. □

Proposition 7. Let $(u_i, \omega_i)_{i \in N}$ be such that there are two agents $j, k \in N$ such that $u_j = u_k$ and $\omega_j = \omega_k$. Then, the two $n-1$ agents economies that result from removing j and k are identical. Property (U) guarantees that for any $y^{-j} \in \mu((u_i, \omega_i)_{i \in N \setminus \{j\}})$ and $y^{-k} \in \mu((u_i, \omega_i)_{i \in N \setminus \{k\}})$,

$$u_j(y_k^{-j}) = u_j(y_j^{-k}).$$

Consider any $x \in \mu((u_i, \omega_i)_{i \in N})$ with $(y^{-i})_{i \in N}$ being the allocations supporting x in the solution, we know from property (S) that

$$u_j(x_j) - u_j(y_j^{-k}) = u_j(x_k) - u_j(y_k^{-j}).$$

Combining the last two equations we obtain the desired equality,

$$u_j(x_j) = u_j(x_k).$$

□

Lemma 9. Let x be an allocation in the economy with set of agents N. For any $j \in N$, by the induction hypotheses, let $(u_i^{-j})_{i \neq j}$ be the unique utility levels attained in the solution for the economy with set of agents $N \setminus \{j\}$. That is, for any $y^{-j} \in \mu((u_i, \omega_i)_{i \in N \setminus \{j\}})$, $u_i^{-j} \equiv u_i(y_i^{-j})$. Moreover, for any three distinct agents, let $u_i^{-j,k}$ be the unique utility level that i attains in the solution for the economy with set of agents $N \setminus \{j, k\}$. That is, for any $y^{-j,k} \in \mu((u_i, \omega_i)_{i \in N \setminus \{j,k\}})$, $u_i^{-j,k} \equiv u_i(y_i^{-j,k})$.

Without loss of generality, we will show that the equation holds for agent 1 and agent 2. Then, note that for any $k \neq 1, 2$ the following holds

$$u_1(x_1) - u_1^{-2} + u_1^{-2} - u_1^{-2,k} = u_1(x_1) - u_1^{-k} + u_1^{-k} - u_1^{-2,k} \tag{1}$$

$$u_2(x_2) - u_2^{-k} + u_2^{-k} - u_2^{-1,k} = u_2(x_2) - u_2^{-1} + u_2^{-1} - u_2^{-1,k} \tag{2}$$

$$u_k(x_k) - u_k^{-1} + u_k^{-1} - u_k^{-1,2} = u_k(x_k) - u_k^{-2} + u_k^{-2} - u_k^{-1,2}. \tag{3}$$

But note that by the induction hypothesis, in the economies with $n-1$ agents, (S) implies,

$$u_1^{-2} - u_1^{-2,k} = u_k^{-2} - u_k^{-1,2},$$

$$u_2^{-k} - u_2^{-1,k} = u_1^{-k} - u_1^{-2,k},$$

and

$$u_k^{-1} - u_k^{-1,2} = u_2^{-1} - u_2^{-1,k}.$$

But then, adding equations (1), (2) and (3),

$$\begin{aligned}\left[u_1(x_1) - u_1^{-2}\right] + \left[u_2(x_2) - u_2^{-k}\right] + \left[u_k(x_k) - u_k^{-1}\right] \\ = \left[u_1(x_1) - u_1^{-k}\right] + \left[u_2(x_2) - u_2^{-1}\right] + \left[u_k(x_k^*) - u_k^{-2}\right],\end{aligned}$$

or equivalently,

$$\begin{aligned}\left[u_1(x_1) - u_1^{-2}\right] - \left[u_2(x_2) - u_2^{-1}\right] = \\ = \left[u_1(x_1) - u_1^{-k}\right] - \left[u_k(x_k) - u_k^{-1}\right] + \\ + \left[u_k(x_k) - u_k^{-2}\right] - \left[u_2(x_2) - u_2^{-k}\right].\end{aligned}$$

Writing the last equation in terms of contributions,

$$(c_1^{-2}(x) - c_2^{-1}(x)) = (c_1^{-k}(x) - c_k^{-1}(x)) + (c_k^{-2}(x) - c_2^{-k}(x)).$$

Then, adding up for all $k \neq 1, 2$,

$$(n-2)(c_1^{-2}(x) - c_2^{-1}(x)) = \sum_{k\geq 3}(c_1^{-k}(x) - c_k^{-1}(x)) + \sum_{k\geq 3}(c_k^{-2}(x) - c_2^{-k}(x))$$

and adding $2(c_1^{-2}(x^*) - c_2^{-1}(x^*))$ on both sides of the equation,

$$n(c_1^{-2}(x) - c_2^{-1}(x)) = \sum_{k\neq 1}(c_1^{-k}(x) - c_k^{-1}(x)) + \sum_{k\neq 2}(c_k^{-2}(x) - c_2^{-k}(x)).$$

But $-C_1(x) = \sum_{k\neq 1}(c_1^{-k}(x) - c_k^{-1}(x))$ and $C_2(x) = \sum_{k\neq 2}(c_k^{-2}(x) - c_2^{-k}(x))$, hence

$$C_2(x^*) - C_1(x^*) = n(c_1^{-2}(x^*) - c_2^{-1}(x^*)).$$

The argument can be repeated for any other pair of agents. □

Lemma 10. Then, we will construct a mapping γ with a fixed point and this fixed point that will imply ($\Diamond$).

- *Construction of the mapping γ.*
  The domain will be the set

  $$U^P = \{u \in \mathbb{R}^n | \text{exists } x \text{ Pareto efficient such that } u_i = u_i(x_i) \text{ for each } i \in N\}.$$

  Note that since preferences are regular, continuous and monotone and $\sum_{i\in N}\omega_i \gg 0$, the set U^P is homeomorphic to the $(n-1)$-unit simplex.[7]

[7] Proposition 4.6.1 in Mas-Colell (1990).

For each $u \in U^P$ and for each for each pair of agents i and j with $i \neq j$, calculate the contribution of agent j to agent i, c_i^{-j}, as

$$c_i^{-j}(u) = u_i - u_i(y_i^{-j}).$$

Now lets define the function $C_i : \mathbb{R}^n \to \mathbb{R}$ as

$$C_i(u) = \sum_{j \neq i} c_j^{-i}(u) - \sum_{j \neq i} c_i^{-j}(u).$$

$C_i(u)$ can be understood as the net contribution of agent i. If $C_i(u) < 0$, then the sum of the contributions that agent i is receiving is larger than the sum of the contributions that i is giving to the others. Let $C_i^-(u) = \min\{0, C_i(u)\}$. Then, the function $p : U^P \to \mathbb{R}^n$ will punish all the agents with $C_i^- < 0$ by assigning them the utility level $p_i(u) < u_i$ while leaving the others the same. The utility levels $p_i(u)$ are defined as follows

$$p_i(u) = \left(1 + \frac{C_i^-(u)}{\max_{j \in N}\{|C_j(u)|\} + 1}\right) u_i.$$

Given that $p(u)$ may not be efficient and therefore not in U^P, the function f will map $p(u)$ into $f(u) = (f_i(u))_{i \in N} \in U^P$ in the following manner,

$$f_i(u) = p_i(u) + \delta(u) \qquad \forall i \in N,$$

where $\delta(u)$ is the unique real number such that $f(u)$ is efficient.[8] Note that the function f is clearly continuous since each contribution is a continuous function of u. [9]

- *f has a fixed point.*
  Since U^P is homeomorphic to the $(n-1)$-unit simplex and f is a continuous map from U^P into itself, there exists a $u^* \in U^P$ such that $u^* = f(u^*)$.

- *$u^* = f(u^*)$ implies that $f(u^*) = p(u^*)$.*
  This is true because if $u^* \neq p(u^*)$ then $u^* = f(u^*) \gg p(u^*)$. But this is a contradiction since $\sum_i C_i(u^*) = 0$ implies that $C_j^- = 0$ for some $j \in N$ and therefore for this j, $u_j^* = p_j(u^*) < f_j(u^*) = u_j^*$.

Then, $u^* = p(u^*)$ only if $u_i^* C_i^-(u^*) = 0$ for each $i \in N$. This implies that if $u_i^* > 0$ then $C_i(u^*) \geq 0$. □

Lemma 11 *For any feasible level of utility, exists a unique $\delta \in \mathbb{R}$ such that $u + \delta(1, \cdot, 1)$ is efficient.*

[8] Lemma 11 guarantees that δ is in fact well-defined.

[9] *Suppose towards a contradiction that f is not continuous.* Exists a sequence $u_n \to u$ and $f(u_n) \nrightarrow f(u)$. Since U^P is compact, there exists a subsequence u_{n_k} with $f(u_{n_k}) \to f^* \neq f(u)$. But, we know that $u_{n_k} \to u$ implies $p(u_{n_k}) \to p(u)$ and then $\delta(u) \neq \lim \delta(u_{n_k}) = f^* - p(u)$. But if $\delta(u) < f^* - p(u)$ the it is a contradiction to $f(u)$ being efficient. Also, $\delta(u) > f^* - p(u)$ contradicts $f(u_{n_k})$ being efficient for large k. Therefore, f must be continuous.

Proof. Let $U = \{u \in \mathbb{R}^{|N|} | \text{exists } x \text{ feasible allocation with } u_i = u_i(x_i) \text{ for } i \in N\}$. Consider the set

$$D = \{d \in \mathbb{R} \,|\, (u_i + d)_{i \in N} \in U\}.$$

Note that $0 \in D$. Also, D is bounded from above. Let $\delta = \sup D$. Since U is compact, $\delta \in D$. We want to show that $(u_i + \delta)_{i \in N}$ is efficient. Suppose not, then exists a feasible utility profile u' with $u'_i > u_i + \delta$ for each i. But then, letting $\epsilon = \min_{i \in N}\{u'_i - (u_i + \delta)\}$. But then, $(u_i + \delta + \epsilon)_{i \in N} \leq (u'_i)_{i \in N} \in U$. But this implies that $(u_i + \delta + \epsilon)_{i \in N} \in U$ contradicting $\delta = \sup D$. □

Example

Consider the economy $((u_1, \omega_1), (u_2, \omega_2))$ where

$$u_1(x_{11}, x_{12}) = q x_{11} x_{12} \quad \text{and} \quad \omega_1 = (3, 1)$$

$$u_2(x_{21}, x_{22}) = x_{21} x_{22} \quad \text{and} \quad \omega_2 = (1, 3).$$

Again, an allocation x is in the solution if it is efficient and

$$q\,(x_{11} x_{12} - 3) = (x_{21} x_{22} - 3).$$

The following table shows the unique allocations in the solution for different values of q.

q	x_{11}	x_{12}	x_{21}	x_{22}
1	2	2	2	2
2	1.9161	1.9161	2.0839	2.0839
10	1.7861	1.7861	2.2139	2.2139
100	1.7381	1.7381	2.2619	2.2619

As q grows, a small change in agent 1's bundle leads to a large change in her utility level. Then, agent 1's utility level in the solution goes to the endowment utility level as q goes to infinity.

Now, lets fix arbitrary and strictly positive prices $p \in \mathbb{R}^l_{++}$ and let the expenditure functions represent the agents preferences. In that case, the only allocation in the solution is each agent obtaining the bundle $(2, 2)$ irrespective of the price. This happens because for any $p \in \mathbb{R}^l_{++}$,

$$e_i(p, x^0) = 2\left(p_1 p_2 x_1^0 x_2^0\right)^{\frac{1}{2}} \qquad i = 1, 2$$

and therefore, the contributions are balanced at the allocation $((x_{11}, x_{12}), (x_{21}, x_{22}))$ if

$$2(p_1 p_2)^{\frac{1}{2}} \left[x_{11}^{\frac{1}{2}} x_{12}^{\frac{1}{2}} - \sqrt{3}\right] = 2(p_1 p_2)^{\frac{1}{2}} \left[x_{21}^{\frac{1}{2}} x_{22}^{\frac{1}{2}} - \sqrt{3}\right].$$

It can be seen that the the allocations satisfying the previous equation do not depend on the prices.

References

Chipman, J. S. and Moore, J. C. (1980). Compensating variation, consumer's surplus, and welfare. *The American Economic Review*, 70(5):933–949.

Foley, D. K. (1967). Resource allocation and the public sector. *Yale Economic Essays*, 7:45–98.

Harsanyi, J. C. (1963). A simplified bargaining model for the n-person cooperative game. *International Economic Review*, 4(2):194–220.

Mas-Colell, A. (1990). *The theory of general economic equilibrium: A differentiable approach*, volume 9. Cambridge University Press.

Maschler, M. and Owen, G. (1992). The consistent shapley value for games without side payments. In Selten, R., editor, *Rational Interaction*, pages 5–12. Springer Berlin Heidelberg.

Myerson, R. (1980). Conference structures and fair allocation rules. *International Journal of Game Theory*, 9:169–182.

Pazner, E. A. and Schmeidler, D. (1978). Egalitarian equivalent allocations: A new concept of economic equity. *The Quarterly Journal of Economics*, 92(4):671–687.

Pérez-Castrillo, D. and Wettstein, D. (2006). A Shapley value for economic environments. *Journal of Economic theory*, 12 (1):296–308.

Schmeidler, D. and Vind, K. (1972). Fair net trades. *Econometrica*, 40(4):637–642.

Shapley, L. (1953). A value for n-person games. In Kuhn, H. and Tucker, A., editors, *Contributions to the Theory of Games*, pages 307–317. Princeton University Press, Princeton.

Shapley, L. (1969). Utility comparison and the theory of games. *La Decision*, 171:251–263.

Yannelis, N. (1985). Value and fairness. In Aliprantis, C., Burkinshaw, O., and Rothman, N., editors, *Advances in Equilibrium Theory*, volume 244 of *Lecture Notes in Economics and Mathematical Systems*, pages 205–235. Springer Berlin Heidelberg.