Identification of solution concepts for semi-parametric discrete games with complete information

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Introduction
Objective

- Economic models often involve a description of the environment as well as a solution concept.
- We identify solution concepts with the implied behavior assumptions.
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- We identify solution concepts with the implied behavior assumptions
- Crucial for theory, for estimation, and for counterfactual analyses and policy implications
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- We identify solution concepts with the implied behavior assumptions
- Crucial for theory, for estimation, and for counterfactual analyses and policy implications

When is it possible to uncover from the data whether actual choices satisfy such assumptions?
Scope

- Discrete semi-parametric games with complete information
Scope

• Discrete semi-parametric games with complete information
• Solution concepts map characteristics of the environment into sets of admissible distributions over outcomes
Scope

- Discrete semi-parametric games with complete information
- Solution concepts map characteristics of the environment into sets of admissible distributions over outcomes
- For this talk only consider the entry game

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Overview of results

• Three difficulties
  → Unknown structural parameters
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  → Unobserved heterogeneity
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  → Unobserved heterogeneity
  → Multiple equilibria
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- Three difficulties
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    - Identification-at-infinity strategy
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A solution concept can rationalize observed data if and only if the underlying assumptions are satisfied almost surely
Overview of results

- Three difficulties
  - Unknown structural parameters
    - Identification-at-infinity strategy
  - Unobserved heterogeneity
  - Multiple equilibria
    - Exclusion restriction + completeness

A solution concept can rationalize observed data if and only if the underlying assumptions are satisfied almost surely.

Two solution concepts can be discriminated as long as they make disjoint predictions with positive probability.
Related literature

- **Point identification** — Björn & Vuong (1985), Bresnahan & Reis (1990, 1991), Bajari, Hong & Ryan (2010), Kline (2013)


- **Completeness & Exclusion restrictions** — Berry & Haile (2014), Newey & Powell (2003), Hall & Horowitz (2005)
Framework
Data generating process

- Random characteristics
  - Outcomes $y$, with finite support
  - Exogenous observables $x$
  - Exogenous unobservables $e$
Data generating process

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- Researcher knows the distribution of $e$ (in this talk) and $x$, and observes

  \[ y|x \sim \mu_0 \]
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- Behavior is characterized by distributions of play $h : E \times X \rightarrow \Delta(Y)$
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- Behavior is characterized by distributions of play $h : E \times X \to \Delta(Y)$
- $h$ is consistent with the data if
  \[ \mathbb{E} \left[ h(e, x) \bigg| x \right] = \mu_0(x) \quad \text{a.s.} \]
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  \[ E[h(e, x) | x] = E[h_0(e, x) | x] \quad \text{a.s.} \]
Structural model

- Structural index
  - Unknown structural parameter $\beta_0 \in B$
  - Unobserved structural index $u = u(\beta_0, e, x)$
Structural model

- **Structural index**
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- **Solution concepts**
  - A solution concept is a correspondence $q : B \times E \times X \Rightarrow \Delta(Y)$
**Structural model**

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- **Solution concepts**
  - A solution concept is a correspondence $q : B \times E \times X \Rightarrow \Delta(Y)$
  - $(\beta, h)$ jointly satisfy $q$ if and only if
    \[ h(e, x) \in q(\beta, e, x) \text{ a.s.} \]
  - $q$ is satisfied if it is satisfied by $(\beta_0, h_0)$
Entry game — variables and payoffs

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- Outcomes — $Y = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$
- Observables — $x = (x_1, x_2)$ with large support $X = \mathbb{R}^2$
- Unobservables — $e = (e_1, e_2)$ with $e|x \sim N(0, I)$
- Structural parameters — $\beta = (\beta_1, \beta_2, \beta_3) \in \mathbb{R}^3_{++}$
- Payoffs — $u = (u_1, u_2)$ with
  $$u_i(y) = (\beta_i x_i - e_i - \beta_3 y_{-i}) y_i$$
Entry game — solution concepts

\[ \beta_0 x_2 - \beta_0 x_1 = \delta_{(1,1)} \]

\[ \beta_0 x_1 - \beta_0 x_3 = \delta_{(1,0)} \]

\[ \beta_0 x_2 - \beta_0 x_3 = \delta_{(0,1)} \]

\[ \Delta(Y) \]

Level-2 rationality correspondence
Entry game — solution concepts

Nash equilibria correspondence
Entry game — distribution of play

Uniform randomization over NE
Identified set

- Assumptions as restrictions on the parameter space $\Theta \subseteq B \times H$
Identified set

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- $\theta = (\beta, h)$ belongs to the identified set under $\Theta$, if $\theta \in \Theta$ and $h$ is consistent with the observed data, i.e.,

$$E \left[ h(e, x) \middle| x \right] = \mu_0(x) \quad \text{a.s.}$$
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• A solution concept is consistent with the data under $\Theta$, if there exist some $\theta = (\beta, h)$ in the identified set that satisfies it, i.e.,

$$h(e, x) \in q(\beta, e, x) \quad \text{a.s.}$$
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**Theorem** (Beresteanu, Molchanov and Molinari, 2010) – A solution concept $q$ is consistent with the data if and only if there exists some $\beta$ such that

$$\mu_0(x) \in \mathbb{E}[q(\beta, e, x)|x] \quad \text{a.s.}$$
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$$
\mu_0(x) \in E[q(\beta, e, x)|x] \quad \text{a.s.}
$$
Identification
Identification

Distribution of play
Assumptions

- *Exclusion restriction (ER)*
  - $(e, x)$ only affect the distribution of play through $u$
  - There exists a function $\tilde{h}_0 : U \to \Delta(Y)$ such that
    \[
    h_0(e, x) = \tilde{h}_0(u) \quad \text{a.s.}
    \]
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• *Bounded completeness* (BC)
  → There is sufficient variation in the distribution of $u$ conditional on $x$
  → The family $\{F_{u|x}(u|x; \beta) \mid x \in X\}$ is boundedly complete
  → That is, for every measurable function $m : U \to [-1, 1]$
    
    $$\mathbb{E}[m(u)|x] = 0 \quad \text{a.s.} \quad \Rightarrow \quad m(u) = 0 \quad \text{a.s.}$$
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    \]
  - Satisfied if $v$ is a sufficient statistic for $u$, $F_{v|x}$ belongs to exponential family and $X$ contains an open set
Identification of the distribution of play

**Proposition** – If \((\beta, h)\) and \((\beta, h')\) belong to the identified set under (ER) and (BC), then

\[
h(e, x) = h'(e, x) \quad \text{a.s.}
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Identification of the distribution of play

**Proposition** – If $(\beta, h)$ and $(\beta, h')$ belong to the identified set under (ER) and (BC), then

$$h(e, x) = h'(e, x) \quad \text{a.s.}$$

- Sketch of proof:
  - If $h$ and $h'$ are consistent with the data then
    $$\mathbb{E}[h(e, x)|x] = \mu_0(x) = \mathbb{E}[h'(e, x)|x] \quad \text{a.s.}$$
  - Then, under (ER)
    $$\mathbb{E}[\tilde{h}(u) - \tilde{h}'(u)|x] = 0 \quad \text{a.s.}$$
  - Then, under (BC)
    $$h(e, x) = \tilde{h}(u) = \tilde{h}'(u) = h'(e, x) \quad \text{a.s.}|x}$$
Distribution of play in the entry game

- A sufficient statistic for $u$ given $\beta$ is
  \[ v = \begin{pmatrix} \beta_1 x_1 - e_1 \\ \beta_2 x_2 - e_2 \end{pmatrix} \]

- (BC) is satisfied because
  \[ v | x \sim N \left( \begin{pmatrix} \beta_1 x_1 \\ \beta_2 x_2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \]

- Hence, if (ER) is satisfied, then $h_0$ is identified up to $\beta$
Identification

Structural parameters
Point identification of payoff parameters

- **Identification at infinity (II)**
  - For each player $i$, there is a covariate that does not affect her payoffs
  - For large values of this covariate, $i$ faces a single agent decision problem and behaves rationally
  - These single decision problems are identified

**Proposition** – Under (II), the vector of structural parameters $\beta_0$ is point identified
Payoff parameters in the entry game

- Under level-2 rationality

  → Firm $i$ faces a single agent problem when $x_{-i} \to \pm \infty$

  → These single agent decision problems identify $\beta_i$ and $\beta_3$

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$x_2 \rightarrow -\infty$
Payoff parameters in the entry game

- Under level-2 rationality

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\end{array}
\]

\[ x_2 \rightarrow +\infty \]
Identification

Solution concepts
Main result

- $(BC') + (ER) \Rightarrow h_0$ is point identified up to $\beta_0$
- $(II) \Rightarrow \beta_0$ is point identified
- Hence, we can back out payoffs and behavior conditional on both $(e, x)$
Main result

- $(BC) + (ER) \Rightarrow h_0$ is point identified up to $\beta_0$
- $(II) \Rightarrow \beta_0$ is point identified
- Hence, we can back out payoffs and behavior conditional on both $(e, x)$

**Theorem** – Under $(BC)$, $(ER)$ and $(II)$, a solution concept $q$ is consistent with the data if and only if it is satisfied, i.e., there exists $\beta$ such that

$$\mu_0(x) \in \mathbb{E}\left[q(\beta, e, x)|x\right] \quad \text{a.s.}$$

if and only if

$$h_0(e, x) \in q(\beta_0, e, x) \quad \text{a.s.}$$
Competing solution concepts

• *True solution concept* (TS)
  → There exists some $q_0$ belonging to a known set $\Sigma$ which is satisfied
Competing solution concepts

• True solution concept (TS)
  → There exists some $q_0$ belonging to a known set $\Sigma$ which is satisfied

• Disjoint predictions (DP)
  → For every $q, q' \in \Sigma$ there exists some $Z \subseteq E \times X$ realized with positive probability and such that

$$q(\beta_0, e, x) \cap q'(\beta_0, e, x) = \emptyset \quad \text{a.s. in } Z$$
Competing solution concepts

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\]

**Theorem** – Under (BC), (ER), (II), (TS) and (DP), the true solution concept is point identified
Identification

Applications and limitations
Solution concepts

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- Are firms choices in equilibrium?
Solution concepts

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- Are firms choices in equilibrium?
- For which markets are firms choices in equilibrium?
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- Are firms choices in equilibrium?
- For which markets are firms choices in equilibrium?
- Which equilibria are more likely to be selected?
- Are entry choices simultaneous or sequential?
### Policy implications

- **Policymaker’s objective — decrease monopolies**

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- Policymaker’s objective — decrease monopolies
- Policy instrument — duopoly subsidy $\delta$
Policy implications

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- Policymaker’s objective — decrease monopolies
- Policy instrument — duopoly subsidy $\delta$
- Assumption — Nash equilibrium in pure strategies (PNE)
\[
\begin{align*}
\beta_0 x_1 - \beta_0 x_2 &= 0 \\
\beta_0 x_2 - \beta_0 x_3 &= 0 \\
\beta_1 x_1 - \beta_0 x_3 &= 0 \\
\beta_2 x_1 - \beta_0 x_3 &= 0
\end{align*}
\]

\textit{PNE correspondence}
Incidence of monopolies under PNE
Under PNE, the policy always reduces incidence of monopolies
Firm 1 monopoly 
or firm 2 monopoly

PNE correspondence
True behavior = 2nd level rationality + strategic-ambiguity aversion
Under true behavior — no monopolies in the “multiplicity” region
Policy benefit is always smaller and could even increase monopolies!
Information structure

• Our (ER) cannot be satisfied if error terms $e_i$ are private information
Information structure

- Our (ER) cannot be satisfied if error terms $e_i$ are private information.
- For the entry game, assuming equilibrium play, we can still distinguish between private and common information.

\[ \Pr(0,1), \Pr(1,0), \Pr(1,1) \]

$M_{\text{NE}}(t)$, $m_{\text{BNE}}(t)$
Closing remarks
Takeaway

• With sufficiently rich data and parametric assumptions we can
  → Identify the distribution of play and solution concepts
Takeaway

- With sufficiently rich data and parametric assumptions we can
  - Identify the distribution of play and solution concepts
  - Test commonly used assumptions using field data
  - Policy implications that are robust to misspecification of behavior assumptions
Thank you for your attention!

paper available at brunosalcedo.com

contact me at bruno@psu.edu