The areas of math I am most interested in are number theory, combinatorics and harmonic analysis. I am often drawn to questions concerning the relationship between equidistribution and arithmetic structure. My favourite sort of problem has the following flavour: if an object is not governed by some sort of algebraic structure then it is best described by some sort of randomness. This type of problem is very common in number theory. One specific instance of it is the sum-product problem. Given a finite set of integers, say A, can it be the case that A is at one additively structured and multiplicatively structured? One measure of structure is the size of the set of sums

$$A + A = \{a + b : a, b \in A\}$$

and the set of products

$$A \cdot A = \{ab : a, b \in A\}.$$

If A were closed under addition then $|A + A| = |A|$ so that the set of sums would be much smaller than the number of possible sums which is about $|A|^2$. A similar statement is true about the set of products. Of course, there are no interesting finite subgroups of the integers, additive or multiplicative, but the sum set and the product set can still be quite small. A conjecture of Erdös and Szemerédi [ES] says that they cannot both be small. Specifically they conjecture that

$$\max\{|A + A|, |A \cdot A|\} \gg_\varepsilon |A|^{2-\varepsilon}$$

for any positive ε. Why should such a conjecture hold? Well sets A which have have a small sum set $A + A$ tend to look somewhat like an arithmetic progression while sets which have small product set $A \cdot A$ tend to look like geometric progressions which are rather different. So if a set A can have small sum set, and looks like an arithmetic progression, there appear to be no obvious multiplicative relations governing A and our best guess is that it looks multiplicatively random, which should mean that $|A \cdot A|$ is on the order of $|A|^2$. This principle appears in various forms in the work discussed below.

1 Past work

1.1 Hindman's Conjecture

The first question I became interested in is a weaker version of a conjecture of Hindman [HLS], and it asks the following. Suppose we partition the natural numbers $\mathbb{N}$

into finitely many sets $A_1, \ldots, A_r$, then do there exist numbers $(x, y) \neq (2, 2)$ such that the sum $x + y$ and the product xy both belong to the same part A_i for some $1 \leq i \leq k$? There is no heuristic reason for this to hold other than that there is no clear reason why it should not. This problem is widely considered to be difficult since the usual tools in Ramsay theory are not well suited to handling two binary operations. However, in a finite field one can ask an analogous question. I was able to show that if $A \subset \mathbb{F}_q$ has size at least $\sqrt{q}$ then A must contain the sum $x + y$ and product xy of two elements $x, y \in \mathbb{F}_q$. In fact, one can replace the sum with a general linear form $ax + by$ and the product with a general quadratic from $cx^2 + dxy + ey^2$ and the result will still hold as long as these forms satisfy a simple and necessary non-degeneracy condition. I also was able to construct a subset $A \subset \mathbb{F}_q$ of size on the order of $\log q$ which does not contain both xy and $x + y$ for any pair of elements $x, y \in \mathbb{F}_q$. These results were published in an article in Acta Arithmetica in 2013 [Ha].

1.2 Character sums over Bohr sets

Suppose we have a finite field $\mathbb{F}_p$ and a small subset Γ. A Bohr set $B(\Gamma, \varepsilon)$ consists of those elements $x \in \mathbb{F}_p$ for which $\|\frac{x\gamma}{p}\| \leq \varepsilon$ holds for each $\gamma \in \Gamma$, where $\|t\|$ is the distance from t to the nearest integer. Because this distance is roughly the same as the distance on the unit circle, Bohr sets are like approximate level sets of exponentials. What I aimed to show was that, while certain additive characters are roughly constant on a Bohr set, a multiplicative character will have to oscillate. I did this by proving non-trivial estimates for the multiplicative character sums

$$S(\chi) = \sum_{x \in B(\Gamma, \varepsilon)} \chi(x).$$

This estimate also had a nice application. A well-known result of Dirichlet says that if $\alpha_1, \ldots, \alpha_d$ are real numbers we can find an integer $q \leq Q$ for which $\|\alpha_i q\| \leq Q^{\frac{-1}{d}}$. Schmidt [Sch] investigated how good the approximation can be if we ask that q is a perfect square. Since multiplicative characters can be used to pick out the k'th powers in $\mathbb{F}_p$, I was able to deduce a recurrence theorem analogous to Schmidt's. Given elements $\alpha_1, \ldots, \alpha_d \in \mathbb{F}_p$ we can find a k'th power x^k for which

$$\|x^k \alpha_i\| \leq \frac{k^{\frac{1}{d}} \log p}{p^{\frac{1}{2d}}}.$$

1.3 Incidence geometry

Incidence geometry is a branch of combinatorics which is not at first related to number theory. However a beautiful observation of Elekes showed that it was a very useful tool for analysing sum-product type problems. Incidence geometry is by and large concerned with extremal geometric configurations. One of the crown jewels of the subject is the Szemerédi-Trotter theorem [ST]. It says the following. Suppose we have a collection of L lines and P points in the plane. An incidence is a pair (p, l) consisting of one of our points and one of our lines and such that p lies on l. The Szemerédi-Trotter theorem says that we can have at most $O(L^{\frac{2}{3}}P^{\frac{2}{3}} + L + P)$ incidences. It was first used by Elekes [E] to obtain a fairly strong sum-product theorem - namely that for any set of real numbers A, $\max\{|A + A|, |A \cdot A|\} \gg |A|^{\frac{5}{4}}$. The reason is in essence that sums and products are the operations used do define lines. If both the sum set and the product set of A were small, then we could create a family of points and lines which had too many incidences.

Now a simpler theorem than the Szemerédi-Trotter theorem is Beck's theorem. If we have a collection of P points in the plane, each pair of points determines a line, the one passing through them. As a function of P, how many lines are determined? Beck's theorem [B] states that under certain non-degeneracy hypotheses, almost P^2 lines arise in this fashion. Beck's theorem is a simple consequence of Szemerédi's theorem. However, if we replace our real points and lines by points and lines over a finite plane there is no incidence estimate which is as strong as the Szemerédi-Trotter theorem. Nonetheless, with A. Iosevich, B. Lund and O. Roche-Newton, we proved finite geometry variation of Beck's theorem. Suppose one has a collection of $\mathcal{P}$ of $P \gg q^{\frac{3}{2}}$ points in the plane $\mathbb{F}_q^2$ over the finite field with q elements. Then the number of lines which arise as a perpendicular bisector of two points in the collection is $\gg q^2$. The condition that we have such a large number of points rules out certain bad configurations, such as all points colinear and thus is in a sense necessary. As a consequence, we were able to show that if $P \gg q^{\frac{4}{3}}$ then there is a point $(x_0, y_0) \in \mathcal{P}$ such that the set of distances,

$$\{(x - x_0)^2 + (y - y_0)^2 : (x, y) \in \mathcal{P}\} \subset \mathbb{F}_q$$

contains $\gg q$ elements of $\mathbb{F}_q$.

2 Future work

2.1 Littlewood's problem on the Fourier Transform

There are a lot of ways to measure the additive structure of a set A of integers. One is to estimate the size of the sumset $A+A$ and another, closely related, is to estimate the number of solutions to $a+b=c+d$ with all variables in A. This latter count is the fourth moment of an exponential sum

$$|\{(a,b,c,d) \in A^4 : a+b=c+d\}| = \int_1^1 \left| \sum_{a \in A} e(a\theta) \right|^4 d\theta$$

where $e(x) = e^{2\pi i x}$. One way to guarantee that this fourth moment is large is to ask that the first moment is quite small and interpolate, owing to the fact that the second moment is always $|A|$. Littlewood conjectured that the minimum value of the first moment

$$\int_1^1 \left| \sum_{a \in A} e(a\theta) \right| d\theta$$

over all sets A of a given size was realized by an arithmetic progression. This was essentially established independently by McGehee, Pigno and Smith [MPS] and by Konyagin [K]. They showed that

$$\min_{\substack{A \subset \mathbb{Z} \\ |A|=n}} \left\{ \int_0^1 \left| \sum_{a \in A} e(a\theta) \right| d\theta \right\} \gg \log n$$

and $\log n$ is indeed the first moment when $A = \{1, \ldots, n\}$. What is yet to be proved is that in order for this bound to be sharp, A must closely resemble an arithmetic progression. In work in progress, I am investigating a weaker form of this question. Namely, suppose that A has some property that is very much unlike an arithmetic progression, then this lower bound can be improved. One property I am interested in is if A has multiplicative structure - which does not mesh well with additive structure in light of sum-product theory.

This problem also has some relation to character sums. Indeed, the Polya-Vinogradov method for estimating character sums uses the smallness of these exponential sums. However the method only allows for non-trivial estimates for sets larger than $\sqrt{p}$ where p is the modulus of the character. Extending estimates to smaller sets will be investigated in future work.

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