Extensions and selections of maps with decomposable values

by

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Abstract. Let $X$ be a separable metric space and $E$ a Banach space. Let $\mu$ be a nonatomic probability measure on a measurable space $T$, and let $L^1 = L^1(T; E)$ be the Banach space of $\mu$-integrable functions $u: T \to E$. A subset $K$ of $L^1$ is decomposable if, for any $\mu$-measurable set $A \subseteq T$ and all $u, v \in K$, one has $u \cdot \chi_A + v \cdot \chi_{T\setminus A} \in K$. Using the property of decomposability as a substitute for convexity, the analogues of three theorems by Dugundji, Cellina and Michael are proved.

1) A continuous map $f$ from a closed set $Y \subseteq X$ into a decomposable subset $K$ of $L^1$ can be continuously extended to a map $	ilde{f}: X \to K$.

2) An upper semicontinuous multivalued map $F: X \to 2^{L^1}$ with decomposable values has a continuous $\varepsilon$-approximate selection, for any $\varepsilon > 0$.

3) A lower semicontinuous multifunction $G: X \to 2^{L^1}$ with closed decomposable values admits a continuous selection.

The compactness assumption on $X$, which appears in previous papers, is here never used. From 1) it follows that, if $L^1(T; E)$ is separable, then any closed decomposable subset $K \subseteq L^1$ is a retract of the whole space, hence it has the compact fixed point property.

1. Introduction. Consider a measure space $(T, \mathcal{F}, \mu)$, where $\mathcal{F}$ is a $\sigma$-algebra of subsets of $T$ and $\mu$ is a nonatomic probability measure on $\mathcal{F}$. If $E$ is a Banach space, let $L^1(T; E)$ be the Banach space of all functions $u: T \to E$ which are Bochner $\mu$-integrable [17]. According to [10], a subset $K \subseteq L^1(T; E)$ is decomposable if, for every measurable set $A \in \mathcal{F}$,

$$u \cdot \chi_A + v \cdot \chi_{T\setminus A} \in K \quad \forall u, v \in K.$$  

In several cases, the property of decomposability is a good substitute for convexity [15]. Three classical theorems, which make use of a convexity assumption, will be considered here.

Theorem I (Dugundji [6, p. 188]). Let $A$ be a closed subset of a metric space $X$ and let $K$ be a convex subset of a Banach space $Z$. Then every continuous map $f: A \to K$ has a continuous extension $\tilde{f}: X \to K$.

Theorem II (Cellina [2, p. 84]). Let $X$ be a metric space and $Z$ a Banach space. Let $F: X \to 2^Z$ be an upper semicontinuous map with convex values. Then, for every $\varepsilon > 0$, $F$ admits a continuous $\varepsilon$-approximate selection,
i.e. a continuous function \( f_\varepsilon : X \to Z \) such that

\[
\text{graph}(f_\varepsilon) \subseteq B(\text{graph}(F), \varepsilon).
\]

Here \( B(V, \varepsilon) \) denotes the \( \varepsilon \)-neighborhood of a set \( V \).

**Theorem III** (Michael [13]). Let \( X \) be a paracompact topological space and \( Z \) a Banach space. Then any lower semicontinuous multivalued map \( F : X \to 2^Z \) with closed convex values admits a continuous selection.

Aim of the present paper is to establish the analogues of the above results when \( Z \) is the Banach space \( L^1(T; E) \) and, in the assumptions, convexity is replaced by decomposability.

The first result concerning the existence of a continuous selection, for a continuous multifunction with decomposable but not necessarily convex values, is due to Antosiewicz and Cellina [1]. Their selection theorem yields the existence of solutions for the differential inclusion \( x \in F(t, x) \) with Hausdorff-continuous right-hand side, by means of a classical fixed point argument. These results were extended to the lower semicontinuous case by Bressan [3] and Łojasiewicz [12]. More recently, Fryszkowski stated a general selection theorem for lower semicontinuous maps with decomposable values [7]. Approximate continuous selections for upper semicontinuous maps with decomposable values were constructed in [5]. We remark that the proofs of Theorems I–III rely on the paracompactness of the space \( X \). In the decomposable case, however, all known results require that the space \( X \) be compact. This unnatural assumption is motivated only by a technical difficulty, which will be removed in the present paper.

In the analysis of decomposable sets, instead of taking convex combinations, one can continuously interpolate between different points following a well-established procedure. Consider an increasing family, \( \{A_\lambda ; \lambda \in [0, 1]\} \), of measurable subsets of \( T \) with the property that \( \mu(A_\lambda) = \lambda \cdot \mu(T) \) for every \( \lambda \).

The existence of such a family is proved in [9], Lemma 4. Let \( u_1, \ldots, u_p \) be elements of a decomposable set \( K \subseteq L^1(T; E) \), and let \( \lambda_1, \ldots, \lambda_p \) be nonnegative numbers which add up to 1. Setting \( \eta_0 = 0, \eta_i = \lambda_1 + \ldots + \lambda_i \) (\( i = 1, \ldots, p \)), a combination of the \( u_i \) with the \( \lambda_i \) as parameters is given by

\[
\Gamma(u, \lambda) = \sum_{i=1}^{p} u_i \cdot \chi_{A_{\eta_i}} \setminus A_{\eta_{i-1}}.
\]

As in the case of convex combinations, the right-hand side of (1.2) lies inside \( K \) and varies continuously with each \( u_i \) and \( \lambda_i \). Moreover, \( \Gamma(u, \lambda) = u_i \) whenever \( \lambda_i = 1 \). Together with these analogies there is, however, a major difference. In Banach spaces, the metric and the algebraic structures are linked together by the fact that balls are convex. On the other hand, balls in \( L^1(T; E) \) are not decomposable. The failure of this basic property is a primary source of technical difficulties. If \( \bar{u} \in L^1 \) and \( \|u_i - \bar{u}\| \leq \varepsilon \) for all
\( i \in \{1, \ldots, p\} \), without additional assumptions on the sets \( A_\lambda \) the only available estimate for (1.2) is

\[
\| \Gamma(u, \lambda) - \bar{u} \| \leq pq.
\]

This bound can be improved if the sets \( A_\lambda \) are more carefully chosen. In [7] the author defines the measures \( \mu_i \) by setting

\[
\mu_i(A) = \int_A \| u_i - \bar{u} \|_E \, d\mu \quad (A \in \mathcal{F}).
\]

By Lyapunov's Convexity Theorem [9], one can then choose a family of sets \( A_\lambda \) satisfying the additional conditions

\[
\mu_i(A_\lambda) = \lambda \mu_i(T), \quad (\lambda \in [0, 1], \ i = 1, \ldots, p).
\]

If these special sets are used in (1.2), the stronger estimate \( \| \Gamma(u, \lambda) - \bar{u} \| \leq q \) holds. So far, this technique has been applied only in the case of finitely many functions \( u_i \). Indeed, Lyapunov's theorem does not hold for an infinite family of measures \( \mu_i \) [11]. In order to extend the results given in [5] and [7] from the compact to the paracompact case, one has to construct continuous combinations for an infinite family of functions \( u_i \), taking advantage of the fact that only a finite number of \( u_i \) enter in a combination at any given time.

To do this, our key technical tool is Lemma 1 in § 4. It contains an extension of Lyapunov's theorem, valid for a countable set of measures, which is precisely fit for our purpose. Using this lemma, we can prove the analogues of Theorems I–III for the decomposable case, in a quite general setting. Indeed, a separability assumption is the only additional requirement. The statements of our main results are collected in § 3.

An interesting consequence of the extension theorem is that, in a separable space \( L^1(T; E) \), any closed decomposable subset \( K \) is a retract of the whole space. Therefore, \( K \) has the compact fixed point property [16, p. 33]. This provides a further generalization of the fixed point theorems of Cellina [4] and Fryszkowski [8], which holds for \( L^1 \) spaces over any abstract measure space \( (T, \mathcal{F}, \mu) \).

2. Notation and basic definitions. Throughout this paper, \( (T, \mathcal{F}, \mu) \) denotes a measure space, where \( \mathcal{F} \) is a \( \sigma \)-algebra of subsets of \( T \) and \( \mu \) is a nonatomic probability measure on \( \mathcal{F} \). Given a \( \mu \)-integrable function \( f: T \to \mathbb{R} \), we write \( f \cdot \mu \) for the measure having density \( f \) w.r.t. \( \mu \). We denote by \( \sigma \{ A_\lambda ; \lambda \in A \} \) the \( \sigma \)-algebra generated by a family of measurable sets \( A_\lambda \in \mathcal{F} \).

If \( E \) is a Banach space with norm \( \| \cdot \|_E \), \( L^1(T; E) \) denotes the Banach space of Bochner \( \mu \)-integrable functions \( u: T \to E \) [17, p. 132], with norm \( \| u \|_1 = \int_T \| u \|_E \, d\mu \). Given two metric spaces \( X, Y \) with distances \( d_X, d_Y \) respectively, the distance on their product is \( d_{X \times Y} = d_X + d_Y \). The open \( \epsilon \)-neighborhood of a set \( S \subseteq X \) is \( B(S, \epsilon) = \{ x \in X : d(x, S) < \epsilon \} \). The diameter of \( S \) is
diam(S) = sup \{d(x, x'); x, x' \in S\}. The set-theoretic difference between two sets \(A, B\) is written \(A \setminus B\); their symmetric difference is \(A \triangle B = (A \setminus B) \cup (B \setminus A)\). 

\# denotes the cardinality of the set \(A\), while \(\chi_A\) is the characteristic function of \(A\).

Following [10], we now introduce the main concept discussed in this paper.

**Definition.** A set \(K \subseteq L^1(T; E)\) is decomposable if 

\[ u \cdot \chi_A + v \cdot \chi_{T \setminus A} \in K \quad \text{whenever} \quad u, v \in K, \quad A \in \mathcal{F}. \]

The collection of all nonempty decomposable subsets of \(L^1(T; E)\) is denoted by \(D(L^1(T; E))\). For any set \(H \subseteq L^1(T; E)\), the decomposable hull of \(H\) is 

\[ \text{dec}[H] = \bigcap \{K \in D(L^1(T; E)); \ H \subseteq K\}. \]

Clearly, \(\text{dec}[H]\) represents the smallest decomposable set which contains \(H\).

At last, we recall two basic properties of multivalued mappings [2]. Let \(X, Y\) be metric spaces. A multifunction \(F: X \to 2^Y\) is lower semicontinuous (l.s.c.) iff the set \(\{x \in X; F(x) \subseteq C\}\) is closed for every closed set \(C \subseteq Y\). A map \(F: X \to 2^Y\) is Hausdorff-upper semicontinuous (H-u.s.c.) iff, for every \(x_0 \in X\) and every \(\varepsilon > 0\), there exists a neighborhood \(V\) of \(x_0\) such that \(F(x) \subseteq B(F(x_0), \varepsilon)\) for all \(x \in V\).

3. **Statement of the main results.** Our first result is the counterpart of Dugundji's extension theorem, for maps taking values in a separable \(L^1\) space, with the convex hull replaced by the decomposable hull.

**Theorem 1.** Let \(A\) be a closed subset of a metric space \(X\). If either \(X\) or \(L^1(T; E)\) is separable, then every continuous map \(f: A \to L^1(T; E)\) has a continuous extension \(\tilde{f}: X \to L^1(T; E)\) such that \(\tilde{f}(X) \subseteq \text{dec}[f(A)]\).

**Corollary 1.** If \(L^1(T; E)\) is separable, then every closed decomposable subset \(K \subseteq L^1(T; E)\) is a retract of the whole space.

Following [16, p. 33], we say that a topological space \(K\) has the compact fixed point property if every continuous map \(f: K \to K\) with relatively compact image has a fixed point. Theorem 1 yields a general fixed point theorem, which is valid for \(L^1\) spaces over any abstract measure space \((T, \mathcal{F}, \mu)\) with a nonatomic probability measure \(\mu\).

**Corollary 2.** Every closed decomposable set \(K \subseteq L^1(T; E)\) has the compact fixed point property.

Indeed, if \(L^1(T; E)\) is separable, then Corollary 2 is an immediate consequence of Corollary 1. To cover the case where \(L^1(T; E)\) is not separable, let \(f: K \to K\) be a continuous map whose image is relatively compact, and let \(X\) be the closure of the convex hull of \(f(K)\). Since \(X\) is compact, it is obviously separable. Using Theorem 1, extend the identity map \(i\) on \(X \cap K\) to a continuous map \(\tilde{i}: X \to K\). The composition \(f \circ \tilde{i}\) maps \(X\).
into $X \cap K$. By Schauder's theorem, it has a fixed point $\bar{x} \in X \cap K$, which is then a fixed point of $f$.

The next two theorems are concerned with multivalued maps having decomposable values. They provide the analogues of the selection theorems by Cellina and Michael, respectively.

**Theorem 2.** Let $X$ be a metric space and let $F: X \to D\left(L^1(T; E)\right)$ be a H-u.s.c. multifunction with decomposable values. If either $X$ or $L^1(T; E)$ is separable, then for every $\varepsilon > 0$ there exists a continuous map $f_\varepsilon: X \to L^1(T; E)$ such that

$$\text{graph}(f_\varepsilon) \subseteq B(\text{graph}(F), \varepsilon).$$

Moreover, $f_\varepsilon(X) \subseteq \text{dec}[F(X)]$.

**Theorem 3.** Let $X$ be a separable metric space, and let $F: X \to D\left(L^1(T; E)\right)$ be a l.s.c. multifunction with closed decomposable values. Then $F$ has a continuous selection.

**Remark.** For simplicity, the above results are stated in terms of a probability measure $\mu$, but they all can be easily extended to the case where $\mu$ is any nonatomic, nonnegative, bounded measure on $(T, \mathcal{F})$. For this purpose, it suffices to consider the probability measure $\tilde{\mu} = [\mu(T)]^{-1} \cdot \mu$, which is equivalent to $\mu$.

4. Three technical lemmas.

**Lemma 1.** Let $(T, \mathcal{F}, \mu)$ be a measure space with a $\sigma$-algebra $\mathcal{F}$ of subsets of $T$ and a nonatomic probability measure $\mu$ on $\mathcal{F}$. Let $(g_n)_{n \geq 0}$ be a sequence of nonnegative functions in $L^1(T; \mathbb{R})$ with $g_0 \equiv 1$. Then there exists a map $\Phi: \mathbb{R}^+ \times [0, 1] \to \mathcal{F}$ with the following properties:

(a) $\Phi(\tau, \lambda_1) \subseteq \Phi(\tau, \lambda_2)$ if $\lambda_1 \leq \lambda_2$,
(b) $\int_{\Phi(\tau_1, \lambda_1) \triangle \Phi(\tau_2, \lambda_2)} |\lambda_1 - \lambda_2| + 2|\tau_1 - \tau_2|,
(c) $\int_{\alpha(\tau, \lambda)} g_n \, d\mu = \lambda \int_{\tau} g_n \, d\mu \quad \forall n \leq \tau$,
for all $\lambda, \lambda_1, \lambda_2 \in [0, 1], \tau, \tau_1, \tau_2 \geq 0$.

**Proof.** The lemma will be proved first in a special case, assuming that

$$\int_{\tau} g_n \, d\mu = 1 \quad \forall n \geq 0.$$

By induction on $n$, we shall define a sequence of families of measurable sets $\{A^*_\lambda; \lambda \in [0, 1]\}, n \geq 0$, and a decreasing sequence of $\sigma$-algebras $\mathcal{F}^n \subseteq \mathcal{F}$ with the following properties:

(i) $\mu(A^*_\lambda) = \lambda$,
(ii) $A^*_\lambda \in \mathcal{F}^{n-1}$,
(iii) $A^*_\lambda \subseteq A^*_\lambda$ whenever $\lambda_1 \leq \lambda_2$,
(iv) $\mathcal{F}^n = \sigma\{A^*_\lambda; \lambda \in [0, 1]\}$,
(v) $\mu(A) = \int_A g_i \, d\mu$ whenever $A \in \mathcal{F}^n, i \leq n$. 

To do this, using Lyapunov’s theorem [9], construct a family of sets \( \{A^\lambda_n\} \) such that (i) and (iii) hold for \( n = 0 \), and let \( \mathcal{F}^0 \) be the \( \sigma \)-algebra generated by the sets \( A^\lambda_n \). Let now \( A^\lambda_n \) be defined for all \( \lambda \in [0, 1] \) and all \( m \leq n - 1 \) so that the properties (i)-(v) hold. Apply Lyapunov’s theorem to the two nonatomic measures \( \mu_n = g_n \cdot \mu \), on the measurable space \( (T, \mathcal{F}^{n-1}) \). This yields a family of sets \( \{A^\lambda_n; \lambda \in [0, 1]\} \) such that (i), (ii) and (iii) hold for \( n \). Define \( \mathcal{F}^n \) by (iv). We then have

\[
\mu(A) = \int_A g_n \, d\mu \quad (A \in \mathcal{F}^n),
\]

because the equality holds whenever \( A = A^\lambda_n \) for some \( \lambda \), and the family of sets \( \{A^\lambda_n; \lambda \in [0, 1]\} \) is increasing and, by definition, generates \( \mathcal{F}^n \). If \( i < n \), then \( A \in \mathcal{F}^n \) implies \( A \in \mathcal{F}^i \supseteq \mathcal{F}^n \), hence (v) is a consequence of the inductive hypothesis.

We now define the sets \( \Phi(\tau, \lambda) \) as follows.

If \( \tau \) is an integer, \( \Phi(\tau, \lambda) = A^\lambda_\tau \).

If \( \tau = n + \delta \) with \( n \) integer, \( 0 < \delta < 1 \), we consider two cases: when \( \lambda < \delta \), we set \( \Phi(\tau, \lambda) = A^\lambda_{n+1} \); when \( \lambda > \delta \) we set \( \Phi(\tau, \lambda) = A^\lambda_{n+1} \cup A^\lambda_n \), where \( \xi \) is the smallest number in \( [0, 1] \) for which the equality \( \mu(A^\lambda_{n+1} \cup A^\lambda_n) = \lambda \) holds.

Notice that for any \( n \), the function

\[
\xi \to \psi(\xi) = \mu(A^\lambda_{n+1} \cup A^\lambda_n)
\]

is Lipschitz-continuous and nondecreasing, with \( \psi(0) = \delta \), \( \psi(1) = 1 \). In the case \( \delta < \lambda \leq 1 \), the set \( \{\xi \in [0, 1]; \psi(\xi) = \lambda\} \) is nonempty, closed and connected, hence it contains a minimal element. The map \( \Phi \) is thus well defined.

The verification of (a) is elementary. By construction, we also have

\[
(4.2) \quad \mu(\Phi(\tau, \lambda)) = \lambda \quad \forall \tau \geq 0, \lambda \in [0, 1].
\]

Observe that on \( \mathcal{F}^n \) the measures \( g_1 \cdot \mu, \ldots, g_n \cdot \mu \) all coincide with \( \mu = g_0 \cdot \mu \), because of (v). Since \( \Phi(\tau, \lambda) \in \mathcal{F}^n \) whenever \( \tau \geq n \), (4.2) implies (c). To prove (b) notice that (a) and (4.2) together yield

\[
\mu(\Phi(\tau, \lambda_1) \triangle \Phi(\tau, \lambda_2)) = |\lambda_1 - \lambda_2| \quad \forall \tau, \lambda_1, \lambda_2.
\]

Therefore, to establish (b), it suffices to prove the inequality

\[
(4.3) \quad \mu(\Phi(\tau_1, \lambda) \triangle \Phi(\tau_2, \lambda)) \leq 2|\tau_1 - \tau_2|.
\]

Moreover, we can assume that \( \tau_1 < \tau_2 \) and that \( \tau_1, \tau_2 \) both belong to the same interval \([n, n+1]\). For \( i = 1, 2 \), set \( \delta_i = \tau_i - n \) and, if \( \lambda > \delta_i \), let \( \Phi(\tau_i, \lambda) = A^{\lambda+1}_{\delta_i} \cup A^\lambda_{\delta_i} \). Three cases must be considered.

1) If \( \lambda \leq \delta_1 < \delta_2 \), then \( \Phi(\tau_1, \lambda) = \Phi(\tau_2, \lambda) = A^{\lambda+1}_\lambda \) and (4.3) holds trivially.
2) If \( \delta_1 \leq \lambda \leq \delta_2 \), then
\[
\mu(\Phi(\tau_1, \lambda) \triangle \Phi(\tau_2, \lambda)) \leq \mu((A^{\delta_1}_{\delta_1} \cup A^{\delta_2}_{\delta_1}) \triangle A^{\delta_2}_{\delta_2}) + \mu(A^{\delta_2}_{\delta_1} \setminus A^{\delta_2}_{\delta_2})
\]
\[
= (\lambda - \delta_1) + (\delta_2 - \delta_1) \leq 2(\delta_2 - \delta_1) = 2|\tau_1 - \tau_2|.
\]

3) If \( \delta_1 < \delta_2 \leq \lambda \), observe that \( A^{\delta_1}_{\delta_1} \subseteq A^{\delta_2}_{\delta_2} \) and \( A^{\delta_1}_{\delta_2} \supseteq A^{\delta_2}_{\delta_2} \). Using these relations, we obtain
\[
\mu(\Phi(\tau_2, \lambda) \setminus \Phi(\tau_1, \lambda)) + \mu(\Phi(\tau_1, \lambda) \setminus \Phi(\tau_2, \lambda))
\]
\[
\leq \mu(A^{\delta_2}_{\delta_1} \setminus A^{\delta_2}_{\delta_1}) + \mu(\Phi(\tau_1, \lambda) \setminus (A^{\delta_1}_{\delta_1} \cup A^{\delta_2}_{\delta_2}))
\]
\[
= (\delta_2 - \delta_1) + \mu(\Phi(\tau_1, \lambda) \setminus (A^{\delta_1}_{\delta_1} \cup A^{\delta_2}_{\delta_2}))
\]
\[
\leq (\delta_2 - \delta_1) + \lambda - [\lambda - (\delta_2 - \delta_1)] = 2|\tau_1 - \tau_2|.
\]

Here the last inequality is deduced from the inclusion
\[
(A^{\delta_2}_{\delta_1} \setminus A^{\delta_2}_{\delta_2}) \setminus (A^{\delta_1}_{\delta_1} \setminus A^{\delta_1}_{\delta_2}) \subseteq (A^{\delta_1}_{\delta_1} \setminus A^{\delta_2}_{\delta_1}).
\]

The above estimates complete the proof of Lemma 1 under the additional assumption (4.1).

To treat the general case, for each \( n \geq 0 \), set \( \tilde{g}_n \equiv 1 \) if \( g_n = 0 \) \( \mu \)-almost everywhere; otherwise define \( \tilde{g}_n = [\int g_n d\mu]^{-1} \cdot g_n \). If \( \{\Phi(\tau, \lambda)\} \) is a family of sets which satisfy (a)–(c) for the sequence \( (\tilde{g}_n) \), one can easily check that these same sets satisfy (a)–(c) for the sequence \( (g_n) \) as well.

**Lemma 2.** Let \( X \) be a separable metric space, and let \( \varphi_n : X \to L^1(T; R) \), \( h_n : X \to [0, 1] \) \((n \geq 1)\) be two sequences of continuous functions, with \( \varphi_n(x)(t) \geq 0 , \forall x \in X , \forall t \in T \), and such that \( \{\text{supp}(h_n) ; n \geq 1\} \) is a locally finite (closed) covering of \( X \). Then, for every \( \varepsilon > 0 \) and every continuous strictly positive function \( l : X \to R^+ \), there exist a continuous function \( \tau : X \to R^+ \) and a map \( \Phi : R^+ 	imes [0, 1] \to \mathcal{F} \) which satisfy conditions (a), (b) in Lemma 1 together with
\[
(c) \quad \text{For all } x \in X , \lambda \in [0, 1] \text{ and } n \geq 1 \text{, if } h_n(x) = 1 \text{ then}
\]
\[
| \int_{\sigma(x,t)} \varphi_n(x) d\mu - \lambda \int_T \varphi_n(x) d\mu | < \varepsilon/(4l(x)).
\]

**Proof.** Let \( \varepsilon > 0 \) and \( l \) be given. For every \( x \in X \), choose an open neighborhood \( U_x \) of \( x \) which intersects the supports of finitely many functions \( h_n \), so that the set of indices \( I_x = \{n ; U_x \cap \text{supp}(h_n) \neq \emptyset\} \) is finite. Set \( \psi_n(x) = h_n(x) \varphi_n(x) \in L^1(T; R) \) and define
\[
V_x = \{x' \in U_x ; \|\psi_n(x') - \psi_n(x)\|_1 < \varepsilon/(8l(x)) \forall n \in I_x\}.
\]

The family \( \{V_x ; x \in X\} \) is an open covering of the paracompact separable space \( X \). Hence, there exists a sequence of functions \( k_m : X \to [0, 1] \) such that the family \( \{\text{supp}(k_m) ; m \geq 1\} \) is a countable nbd-finite refinement of \( \{V_x\} \) and
the sets \( W_m = \{ x \in X ; k_m(x) = 1 \} \) still cover \( X \). For all \( m \geq 1 \), select \( x_m \) such that \( W_m \subseteq V_{x_m} \). Define the sequence \( (g_j)_{j \geq 0} \) in \( L^1(T;R) \) by setting: \( g_j = \psi_n(x_m) \) if \( j = 2^m 3^n \) for some integers \( m, n \geq 1 \); \( g_j = 1 \) otherwise. Moreover, set

\[
\tau(x) = \sum_{m,n \geq 1} k_m(x) h_n(x) 2^m 3^n.
\]

The function \( \tau \) is continuous, because the summation in (4.5) is locally finite. Using Lemma 1, construct a map \( \Phi \) which satisfies (a)-(c) for the sequence \( (g_j)_{j \geq 0} \). We claim that (c') holds as well.

To see this, fix \( x \in X \), \( n \geq 1 \), and \( \lambda \in [0, 1] \). For some index \( m \), \( x \in W_m \). If \( h_n(x) = 1 \), then

\[
\left| \int_{\Phi(x)} \varphi_n(x) d\mu - \lambda \int \varphi_n(x) d\mu \right|
\leq \int_{\Phi(x)} |\psi_n(x) - \psi_n(x_m)| d\mu + \int_{\Phi(x)} |\psi_n(x_m) - \psi_n(x) - \lambda \int \psi_n(x_m) d\mu| d\mu
\]

\[+ \lambda \int |\psi_n(x_m) - \psi_n(x)| d\mu\]

\[\leq 2||\psi_n(x) - \psi_n(x_m)||_1 + \int_{\Phi(x)} g_{2^m 3^n} d\mu - \lambda \int g_{2^m 3^n} d\mu|.
\]

By (4.4), since \( x \in V_{x_m} \), the first term of this last expression is less than \( 2/(4l(x)) \), while the second term vanishes because \( \tau(x) \geq 2^m 3^n \), by (4.5).

**Lemma 3.** Let \( X \) be a paracompact topological space. For every \( x \in X \), let \( U_x \) be an open neighborhood of \( x \) and let \( M(x) \) be an integer number. Then there exists a continuous function \( \tau : X \to R \) such that \( \tau(x) \geq \min \{ M(x); \ x \in U_x \} \) for every \( x \in X \).

**Proof.** Let \( \{ V_i; i \in I \} \) be an open nbhd-finite refinement of the covering \( \{ U_x \} \), and let \( \{ p_i(\cdot); i \in I \} \) be a continuous partition of unity subordinate to \( \{ V_i \} \). For each \( i \), select a point \( x_i \) such that \( V_i \subseteq U_{x_i} \). Define \( \tau(x) = \sum_{i \in I} p_i(x) M(x_i) \). Clearly, \( \tau \) is continuous. Moreover,

\[\tau(x) \geq \min \{ M(x_i); p_i(x) \neq 0 \} \geq \min \{ M(x_i); x \in U_{x_i} \}\]

\[\geq \min \{ M(x); x \in U_x \} \].

**5. Proof of Theorem 1.** We assume first that \( L^1(T;E) \) is separable. For each \( x \in X \setminus A \), take an open ball \( B(x, r_x) \) with radius \( r_x < \frac{1}{2} d(x, A) \). The family \( \{ B(x, r_x); x \in X \setminus A \} \) is an open covering of the paracompact space \( X \setminus A \), hence it admits an open nbhd-finite refinement \( \{ V_i; i \in I \} \). Here \( I \) is a possibly uncountable set of indices. For each \( i \), choose two points \( x_i \in V_i \) and \( y_i \in A \) such that \( d(x_i, y_i) < 2d(x_i, A) \). Using the separability assumption, select a countable subset \( D = \{ u_n; n \geq 1 \} \) of \( f(A) \) which is dense in \( f(A) \).
Define the sequence \( (g_k)_{k \geq 0} \) in \( L^1(T; R) \) by setting
\[
g_k(t) = \begin{cases} 
\|u_m(t) - u_n(t)\|_E & \text{whenever } k = 2^m 3^n \text{ for some } m, n \geq 1; \\
1 & \text{otherwise.}
\end{cases}
\]
Applying Lemma 1 to this sequence, we obtain a family \( \{\Phi(\tau, \lambda)\} \) of measurable subsets of \( T \) with the properties (a)-(c). For each \( i \in I \), choose \( u_{v(i)} \in D \) such that \( \|u_{v(i)} - f(y_i)\| < d(x_i, y_i) \). Let \( \{p_i(\cdot); i \in I\} \) be a continuous partition of unity subordinate to the covering \( \{V_i\} \). For every \( n \geq 1 \), define the open set \( W_n = \bigcup \{V_i; v(i) = n\} \) and let \( q_n(x) = \sum_{v(i) = n} p_i(x) \). Clearly, \( \{q_n(\cdot); n \geq 1\} \) is a continuous partition of unity subordinate to the locally finite open covering \( \{W_n\} \). Construct a sequence of continuous functions \( (h_n)_{n \geq 1} \) such that \( h_n \equiv 1 \) on \( \text{supp}(q_n) \) and \( \text{supp}(h_n) \subseteq W_n \). For every \( x \in X \setminus A \), define \( \lambda_n(x) = \sum_{m \leq n} q_m(x) \), \( n \geq 0 \), and consider the function
\[
\tau(x) = \sum_{m, n \geq 1} h_m(x) h_n(x) 2^m 3^n.
\]
Notice that \( \tau \) is continuous on \( X \setminus A \) and that
\[
(5.1) \quad \tau(x) \geq 2^m 3^n \quad \forall x \in \text{supp}(q_m) \cap \text{supp}(q_n).
\]
We can now extend the map \( f \) to the whole space \( X \) by setting
\[
\tilde{f}(x) = \begin{cases} 
f(x) & \text{if } x \in A, \\
\sum_{n \geq 1} u_n \cdot \chi_n(x) & \text{if } x \in X \setminus A,
\end{cases}
\]
where
\[
\chi_n(x) = \chi_{\Phi(\tau(x), \lambda_n(x)) \setminus \Phi(\tau(x), \lambda_n - 1(x))}.
\]
It is clear that \( \tilde{f} \) maps \( X \) into \( \text{dec} [f(A)] \). Moreover, \( \tilde{f} \) is continuous on \( X \setminus A \), because the functions \( \tau(\cdot) \) and \( \lambda_n(\cdot) \) \( (n \geq 0) \) are continuous, the characteristic function of the set \( \Phi(\tau, \lambda) \) varies continuously in \( L^1(T; R) \) w.r.t. the parameters \( \tau \) and \( \lambda \), and because the summation defining \( \tilde{f} \) is locally finite.
To prove that \( \tilde{f} \) is continuous on \( A \), let \( a \in A \) and \( \varepsilon > 0 \) be given. Choose \( \delta > 0 \) such that \( \delta < \varepsilon/12 \) and \( \|f(y) - f(a)\|_1 < \varepsilon/2 \) whenever \( y \in A \), \( d(y, a) < 12\delta \). If \( d(x, a) < \delta \) and \( x \in V_i \) for some \( i \in I \), then \( \text{diam}(V_i) < 2\delta \), \( d(x_i, A) < 3\delta \) and \( d(x_i, y_i) < 6\delta \). Therefore, \( p_i(x) \neq 0 \) implies that \( d(y_i, a) < 9\delta, \|f(y_i) - f(a)\|_1 < \varepsilon/2 \) and \( \|u_{v(i)} - f(a)\|_1 < \varepsilon \). From the last inequality, it follows that
\[
(5.2) \quad \|u_n - f(a)\|_1 < \varepsilon \quad \forall n \text{ such that } q_n(x) \neq 0.
\]
For any \( x \in X \setminus A \) with \( d(x, a) < \delta \), fix an integer \( j \) for which \( q_j(x) \neq 0 \). Using (5.1), (5.2) and the property (c) of the sets \( \Phi(\tau, \lambda) \), we obtain the estimate
\[
\| f(a) - \tilde{f}(x) \|_1 \leq \| f(a) - u_j \|_1 + \| u_j - \tilde{f}(x) \|_1 \\
\leq \varepsilon + \sum_{n=1}^{\infty} \int_{T} \| u_j - u_n \|_E \cdot \chi_n(x) \, d\mu \\
= \varepsilon + \sum_{n=1}^{\infty} \int_{T} g_{2j3n} \cdot \chi_n(x) \, d\mu \\
= \varepsilon + \sum_{n=1}^{\infty} g_n(x) \int_{T} g_{2j3n} \, d\mu = \varepsilon + \sum_{n=1}^{\infty} q_n(x) \| u_j - u_n \|_1 \leq 3\varepsilon.
\]
Since \( \varepsilon \) was arbitrary, this completes the proof in the case where \( L^1(T; E) \) is separable.

When \( X \) is separable, only minor modifications of the above arguments are needed. Consider again the open covering \( \{ B(x_i, r_i); x \in X \setminus A \} \) and a locally finite refinement \( \{ V_i; i \in I \} \). Notice that in this case the set \( I \) is necessarily countable, since \( X \setminus A \) is separable. For each \( i \), choose \( x_i \in V_i \), \( y_i \in A \) such that \( d(x_i, y_i) < 2d(x_i, A) \). It now suffices to define the countable set \( D = \{ f(y_i); i \in I \} \subset L^1(T; E) \) and arrange its elements into a sequence, say \( D = \{ u_n; n \geq 1 \} \). From this point on, the proof runs exactly as in the previous case.

6. Proof of Theorem 2. The following proof is an adaptation of the arguments given in [5].

Assume first that \( L^1(T; E) \) is separable. Fix \( \varepsilon > 0 \). For every \( x \in X \), choose a number \( \delta(x) \in ]0, \varepsilon/6[ \) such that \( F(x') \subseteq B(F(x), \varepsilon/6) \) whenever \( x' \in B(x, \delta(x)) \). Let \( \{ V_i; i \in I \} \) be an open nbhd-finite refinement of the covering \( \{ B(x, \delta(x)/2); x \in X \} \) of \( X \). For each \( i \), choose \( x_i \in X \) such that \( V_i \subseteq B(x_i, \delta(x_i)/2) \) and select \( u_i \in F(x_i) \). For \( i, j \in I \), choose also \( v_{i,j} \in F(x_j) \) such that
\[
(6.1) \quad \| u_i - v_{i,j} \|_1 \leq \varepsilon/6 + \inf \{ \| u_i - v \|_1; v \in F(x_j) \} \leq \varepsilon/6 + d_{L^1} (u_i, F(x_j)).
\]
Let \( D = \{ y_n; n \geq 1 \} \) be a countable dense subset of \( F(X) \). For every \( i \in I \) select a \( y_{v(i)} \in D \) for which \( \| u_i - y_{v(i)} \|_1 < \varepsilon/6 \). The set \( D' \) of all functions \( g \in L^1(T; R) \) of the form \( g(t) = \| y_m(t) - y_n(t) \| \), \( m, n \geq 1 \), is countable. Arrange its elements into a sequence, say \( D' = \{ g_k; k \geq 1 \} \). Let \( \{ p_i(\cdot); i \in I \} \) be a continuous partition of unity subordinate to the covering \( \{ V_i \} \). For every \( n \geq 1 \), define the open set \( W_n = \bigcup \{ V_i; v(i) = n \} \) and let \( q_n(x) = \sum_{v(i)=n} p_i(x) \). Clearly, \( \{ q_n(\cdot); n \geq 1 \} \) is a continuous partition of unity subordinate to the nbhd-finite open covering \( \{ W_n \} \). Define
\[
(6.2) \quad \lambda_n(x) = \sum_{m \leq n} q_m(x) \quad (n \geq 0, \ x \in X).
\]
For every \( x \in X \), take an open neighborhood \( U_x \) of \( x \) which intersects finitely many sets \( V_i \). Setting \( I(U_x) = \{ i \in I; U_x \cap V_i \neq \emptyset \} \), this of course means that \( N(x) = \# I(U_x) \) is a finite integer. For every couple of indices \( i, j \in I(U_x) \), choose a \( y_{v(i,j,x)} \in D \) such that

\[
\| y_{v(i,j,x)} - \nu_{i,j} \|_1 < \varepsilon/(6N(x)).
\]

Let \( M(x) \) be an integer so large that the set \( \{ g_k; 1 \leq k \leq M(x) \} \) contains the finite set of functions \( \{ \| y_{v(i)} - y_{v(i,j,x)} \|_E; i, j \in I(U_x) \} \subseteq D' \). Applying Lemma 3 to the collection of neighborhoods \( \{ U_x; x \in X \} \) and integers \( M(x) \), we get the existence of a continuous function \( \tau: X \to \mathbb{R}^+ \) such that

\[
\tau(x) \geq \min \{ M(x); x \in U_x \}.
\]

Recalling (6.2), the map \( f_\varepsilon: X \to L^1(T; E) \) can now be defined by setting

\[
f_\varepsilon(x) = \sum_{n \geq 1} y_n \cdot \chi_n(x).
\]

Here \( \{ \Phi(\tau, \lambda) \} \) is the family of sets constructed in Lemma 1 relative to the sequence \( (g_k)_{k \geq 1} \) of elements of \( D' \), and

\[
\chi_n(x) = \chi_{\Phi(\tau(x), \lambda_n(x)) \Phi(\tau(x), \lambda_n - 1(x))}.
\]

It is easily checked that \( f_\varepsilon \) is continuous and takes values inside \( \text{dec}[F(X)] \).

To show that \( f_\varepsilon \) is an \( \varepsilon \)-approximate selection, fix \( x \in X \) and define \( I(x) = \{ i \in I; p_i(x) \neq 0 \} \), \( J(x) = \{ n \geq 1; q_n(x) \neq 0 \} \). Notice that \( \# J(x) \leq \# I(x) < +\infty \). Since \( I(x) \) is finite, there exists an \( \hat{i} \in I(x) \) such that \( \hat{\delta} = \delta(x_{\hat{i}}) = \max \{ \delta(x_i); i \in I(x) \} \). For every \( i \in I(x) \) we have \( x_i \in B(x_{\hat{i}}, \delta) \), hence

\[
F(x_{\hat{i}}) \subseteq B(F(x_{\hat{i}}), \varepsilon/6).
\]

Take a point \( z \in X \) such that \( x \in U_z \) and \( M(z) = \min \{ M(x); x \in U_x \} \). For every \( n \in J(x) \), select an index \( i_n \in I(x) \subseteq I(U_z) \) such that \( v(i_n) = n \). Define

\[
w = \sum_{n \geq 1} y_{v(i_n,i,z)} \cdot \chi_n(x), \quad w' = \sum_{n \geq 1} \nu_{i_n,i} \cdot \chi_n(x).
\]

Notice that \( w' \in F(x_{\hat{i}}) \). For every \( n \in J(x) \), using (6.1), (6.3) and (6.6) we obtain

\[
\| y_n - y_{v(i_n,i,z)} \|_1 \leq \| y_n - u_{i_n} \|_1 + \| u_{i_n} - \nu_{i_n,i} \|_1 + \| \nu_{i_n,i} - y_{v(i_n,i,z)} \|_1
\]

\[
\leq \varepsilon/6 + [\varepsilon/6 + d_{L^1}(u_{i_n}, F(x_{\hat{i}}))] + \varepsilon/(6N(z)) \leq \frac{3}{4} \varepsilon.
\]

Relying on the properties of the sets \( \Phi(\tau, \lambda) \) and recalling that by (6.4), \( \tau(x) \geq M(z) \), from (6.7) we deduce the estimates

\[
\| f_\varepsilon(x) - w \|_1 = \sum_{n \geq 1} \int \| y_n - y_{v(i_n,i,z)} \|_E \cdot \chi_n(x) \, d\mu
\]

\[
= \sum_{n \geq 1} q_n(x) \| y_n - y_{v(i_n,i,z)} \|_1 \leq \frac{3}{4} \varepsilon,
\]
Putting together (6.8) and (6.9), one has

\[
d_{X \times L^1}(\{x, f_\varepsilon(x)\}, (x_i, w)) \\
\leq d_X(x, x_i) + ||f_\varepsilon(x) - w||_1 + ||w - w'||_1 < \varepsilon/6 + 2\varepsilon/3 + \varepsilon/6 = \varepsilon.
\]

Hence \( (x, f_\varepsilon(x)) \in B(\text{graph}(F), \varepsilon) \). This completes the proof in the case where \( L^1(T; E) \) is separable.

When \( X \) is separable, a slight modification of the above arguments is needed. The nbd-finite open covering \( \{V_i; i \in I\} \) of \( X \) is now countable, because of the separability assumption. It is therefore possible to define the countable set \( D = \{u_i; i \in I\} \cup \{v_{ij}; i, j \in I\} \) and arrange it into a sequence, say \( D = \{y_n; n \geq 1\} \). After this choice of the set \( D \), the rest of the proof goes exactly as in the previous case.

7. Proof of Theorem 3. In what follows, the main arguments are taken from [7]. We list first some preliminary results.

**Proposition 1.** For every family \( \mathcal{K} \) of nonnegative measurable functions \( u: T \to \mathbb{R}^+ \), there exists a measurable function \( v: T \to \mathbb{R}^+ \) such that

(i) \( v \leq u \) \( \mu \)-a.e. for all \( u \in \mathcal{K} \),

(ii) if \( w \) is a measurable function such that \( w \leq u \) \( \mu \)-a.e. for all \( u \in \mathcal{K} \), then \( w \leq v \) \( \mu \)-a.e.

Furthermore, there exists a sequence \( (u_n) \) in \( \mathcal{K} \) such that

\[
v(t) = \inf \{u_n(t); n \geq 1\} \quad \text{for a.e. } t \in T.
\]

If the family \( \mathcal{K} \) is directed downwards (i.e., if for any \( u, u' \in \mathcal{K} \) there exists \( w \in \mathcal{K} \) such that \( w \leq u \) and \( w \leq u' \) \( \mu \)-a.e.), then the sequence \( (u_n) \) can be chosen to be decreasing.

For the proof, see Neveu [14, p. 121].

By (ii), the function \( v \) is unique up to \( \mu \)-equivalence. It represents the greatest lower bound of \( \mathcal{K} \) in the sense of \( \mu \)-a.e. inequality, and is denoted by \( \text{ess inf} \{u; u \in \mathcal{K}\} \).

**Proposition 2.** Let \( K \) be a nonempty closed decomposable subset of \( L^1(T; E) \) and let \( \psi(t) = \text{ess inf} \{||u(t)||_E; u \in K\} \). Then, for every \( v_0 \in L^1(T; R) \) such that \( v_0(t) > \psi(t) \) \( \mu \)-a.e., there exists an element \( u_0 \in K \) such that

\[
||u_0(t)||_E < v_0(t) \quad \mu \text{-a.e.}
\]
Proof. Notice that the set \( \mathcal{K} = \{ \| u(t) \|_E; u \in K \} \) is a decomposable subset of \( L^1(T; \mathbb{R}) \). Therefore, it is directed downwards. Using Proposition 1, take a sequence \( (u_n)_{n \geq 1} \) in \( K \) such that

\[
\| u_m(t) \|_E \geq \| u_n(t) \|_E \quad \forall m < n, \ t \in T,
\]

\[
\psi(t) = \lim_{n \to \infty} \| u_n(t) \|_E \quad \mu\text{-a.e.}
\]

Let now \( v_0 \) be given, with \( v_0(t) > \psi(t) \) \( \mu\text{-a.e.} \), and define the increasing sequence of sets: \( T_0 = \emptyset \), \( T_n = \{ t \in T; \| u_n(t) \|_E < v_0(t) \} \), \( n \geq 1 \). Observe that \( \mu(T \setminus \bigcup_{n \geq 0} T_n) = 0 \). Define the sequence \( (w_n) \) by setting

\[
w_n(t) = \begin{cases} u_k(t) & \text{if } t \in T_{k-1} \setminus T_{k-2}, \ k = 1, \ldots, n-1, \\ u_n(t) & \text{if } t \in T \setminus \bigcup_{k < n} T_k. \end{cases}
\]

Since \( K \) is decomposable, each \( w_n \) belongs to \( K \). Moreover, the sequence \( w_n(t) \) is eventually constant for a.e. \( t \in T \), and \( \| w_n(t) \|_E \leq \| u_1(t) \|_E \) \( \mu\text{-a.e.} \); hence, by the Dominated Convergence Theorem, \( w_n \) converges in \( L^1(T; E) \) to some function \( u_0 \). Clearly, \( u_0 \in K \) because \( K \) is closed. Finally, if \( t \in T_n \setminus T_{n-1} \) for some \( n \), then \( \| u_0(t) \|_E = \| u_n(t) \|_E < v_0(t) \). Therefore, \( u_0 \) satisfies (7.1). \( \blacksquare \)

**Proposition 3.** Let \( X \) be a metric space and let \( F: X \to D(L^1(T; E)) \) be a l.s.c. map with closed decomposable values. For all \( x \in X \), set \( \psi_x(t) = \text{ess inf} \{ \| u(t) \|_E; u \in F(x) \} \). Then the multivalued map \( P: X \to L^1(T; \mathbb{R}) \) defined as

(7.2) \[
P(x) = \{ v \in L^1(T; \mathbb{R}); \ v(t) > \psi_x(t) \ \mu\text{-a.e.} \}
\]

is lower semicontinuous.

**Proof.** Let \( C \) be an arbitrary closed subset of \( L^1(T; \mathbb{R}) \). It suffices to show that, if \( P(x_n) \subseteq C \) for some sequence \( (x_n)_{n \geq 1} \) converging to \( x_0 \), then also \( P(x_0) \subseteq C \). To this purpose, fix any \( v_0 \in P(x_0) \) and take, by Proposition 2, a function \( u_0 \in F(x_0) \) such that \( \| u_0(t) \|_E < v_0(t) \) \( \mu\text{-a.e.} \). Because of the lower semicontinuity of \( F \), there exists a sequence \( u_n \in F(x_n) \) such that \( u_n \to u_0 \) in \( L^1(T; E) \). Then, for every \( n \geq 1 \), the function \( v_n = \| u_n \|_E + v_0 - \| u_0 \|_E \) belongs to \( P(x_n) \) which is contained in \( C \). Since the sequence \( (v_n) \) converges to \( v_0 \) in the norm of \( L^1(T; \mathbb{R}) \) and \( C \) is closed, this implies \( v_0 \in C \). \( \blacksquare \)

**Proposition 4.** Let \( X \) be a metric space and let \( G: X \to D(L^1(T; E)) \) be a l.s.c. map with closed decomposable values. Assume that \( g: X \to L^1(T; E) \) and \( \varphi: X \to L^1(T; R) \) are continuous functions such that, for every \( x \in X \), the set

\[
H(x) = \{ u \in G(x); \ \| u(t) - g(x)(t) \|_E < \varphi(x)(t) \ \mu\text{-a.e.} \}
\]

is nonempty. Then the map \( H: X \to D(L^1(T; E)) \) is l.s.c. with decomposable values.
Proof. For every \( x \in X \), \( H(x) \) is the intersection of two decomposable sets, hence it is decomposable. To check the lower semicontinuity of \( H \), let \( C \) be any closed subset of \( L^1(T; E) \). It suffices to show that, for any sequence \( (x_n) \) in \( X \) converging to a point \( x_0 \), if \( H(x_n) \subseteq C \) for all \( n \geq 1 \), then \( H(x_0) \subseteq C \).

To this purpose, fix any \( u_0 \in H(x_0) \). Because of the lower semicontinuity of \( G \), there exists a sequence \( u_n \in G(x_n) \) such that \( u_n \to u_0 \) in \( L^1(T; E) \). By possibly taking a subsequence, we can assume that \( u_n(t), g(x_n)(t), \varphi(x_n)(t) \) converge to \( u_0(t), g(x_0)(t), \varphi(x_0)(t) \) respectively, \( \mu \text{-a.e. in } T \). Applying Egorov's theorem to these sequences w.r.t. the measure \( \varphi(x_0) \cdot \mu \), for each \( i \geq 1 \) we obtain a measurable set \( T_i \subseteq T \) such that \( u_n, g(x_n) \) and \( \varphi(x_n) \) converge uniformly on \( T_i \) and \( \int_{T \setminus T_i} \varphi(x_0) d\mu < 1/i \). For each \( k \geq 1 \), consider the sets

\[
T^k_i = \{ t \in T_i ; \| u_0(t) - g(x_0)(t) \|_E < \varphi(x_0)(t) - 1/k \}.
\]

Notice that \( \bigcup_{k \geq 1} T^k_i = T_i \) and \( T^k_i \subseteq T^{k+1}_i \). Hence, for every \( i \geq 1 \), there exists a \( k(i) \) such that

\[
\int_{T_i \setminus T^{k(i)}_i} \varphi(x_0) d\mu < 1/i.
\]

Define \( T'_i = T^{k(i)}_i \). The sets \( T'_i \) have the following properties:

(7.3)
\[
\int_{T \setminus T'_i} \varphi(x_0) d\mu < 2/i,
\]

(7.4)
\[
\| u_0(t) - g(x_0)(t) \|_E < \varphi(x_0) - 1/k(i) \quad \forall t \in T'_i.
\]

By (7.4) and by the uniform convergence on \( T'_i \), for all \( i \geq 1 \) there exists some \( n_i \) such that

(7.5)
\[
\| u_n(t) - g(x_n)(t) \|_E < \varphi(x_n)(t) \quad \forall t \in T'_i, \ n \geq n_i.
\]

We can also assume that the sequence \( (n_i)_{i \geq 1} \) is strictly increasing. For each \( n \), choose an arbitrary \( w_n \in H(x_n) \) and set, for \( n_i \leq n < n_{i+1} \),

\[
v_n = u_n \cdot \chi_{T'_i} + w_n \cdot \chi_{T \setminus T'_i}.
\]

Since \( H(x_n) \) is decomposable, \( v_n \in H(x_n) \). We claim that \( v_n \to u_0 \) in \( L^1(T; E) \), which implies \( u_0 \in C \).

Indeed, for \( n_i \leq n < n_{i+1} \), (7.3) and (7.5) yield

\[
\| v_n - u_0 \|_1 \leq \int_{T \setminus T'_i} \| w_n - g(x_n) \|_E d\mu + \int_{T \setminus T'_i} \| g(x_n) - g(x_0) \|_E d\mu + \int_{T \setminus T'_i} \| u_n - u_0 \|_E d\mu
\]

\[
\leq \varphi(x_n) d\mu + \| g(x_n) - g(x_0) \|_1 + \int_{T \setminus T'_i} \varphi(x_0) d\mu + \| u_n - u_0 \|_1
\]

\[
\leq [2/i + \| \varphi(x_n) - \varphi(x_0) \|_1] + \| g(x_n) - g(x_0) \|_1 + 2/i + \| u_n - u_0 \|_1.
\]

As \( n \to +\infty \), we also have \( i \to +\infty \), hence our claim is proved. \( \blacksquare \)
The next result, concerning the existence of approximate selections, is the core of the whole proof of Theorem 3.

**Proposition 5.** Let \( X \) be a separable metric space and let \( G: X \to D(L^1(T; E)) \) be a l.s.c. map with closed decomposable values. Then, for every \( \varepsilon > 0 \), there exist continuous maps \( f_\varepsilon: X \to L^1(T; E) \) and \( \varphi_\varepsilon: X \to L^1(T; R) \) such that \( f_\varepsilon \) is an \( \varepsilon \)-approximate selection of \( G \), in the sense that, for each \( x \in X \), the set

\[
G_\varepsilon(x) = \{ u \in G(x); \| u(t) - f_\varepsilon(x)(t) \|_E < \varphi_\varepsilon(x)(t) \mu\text{-a.e.} \}
\]

is nonempty, and \( \| \varphi_\varepsilon(x) \|_1 < \varepsilon \). Moreover, the map \( x \to G_\varepsilon(x) \) is l.s.c. with decomposable values.

**Proof.** Fix \( \varepsilon > 0 \). For every \( \bar{x} \in X \) and \( \bar{u} \in G(\bar{x}) \), the multivalued map \( Q \) defined as

\[
Q(x) = \{ v \in L^1(T; R); v(t) \geq \text{ess inf} \{ \| u(t) - \bar{u}(t) \|_E; u \in G(x) \} \text{ for a.e. } t \in T \}
\]

is l.s.c. with closed convex values.

To see this, define \( F(x) = \{ u - \bar{u}; u \in G(x) \} \). Then the map \( F \) is also l.s.c. with closed decomposable values. By Proposition 3, the multivalued map \( P \) defined in (7.2) is l.s.c. Hence \( Q \) is also l.s.c., because \( Q(x) \) is the closure of \( P(x) \), for all \( x \in X \).

It is therefore possible to apply Michael's theorem to \( Q \) and obtain a continuous selection \( \varphi_{\bar{x}, \bar{u}} \) such that \( \varphi_{\bar{x}, \bar{u}}(x) \in Q(x) \) for all \( x \in X \) and \( \varphi_{\bar{x}, \bar{u}}(\bar{x}) \equiv 0 \). The family of sets

\[
\left\{ \{ x \in X; \| \varphi_{\bar{x}, \bar{u}}(x) \|_1 < \varepsilon/4 \}; \bar{x} \in X, \bar{u} \in G(\bar{x}) \right\}
\]

is an open covering of the separable metric space \( X \), therefore it has a countable nbd-finite open refinement \( \{ V_n; n \geq 1 \} \). Let \( \{ p_n(\cdot) \} \) be a continuous partition of unity subordinate to the covering \( \{ V_n \} \) and let \( \{ h_n(\cdot) \} \) be a family of continuous functions from \( X \) into \( [0, 1] \) such that \( h_n \equiv 1 \) on \( \text{supp}(p_n) \) and \( \text{supp}(h_n) \subseteq V_n \). For every \( n \geq 1 \), choose \( x_n, u_n \) such that \( V_n \subseteq \{ x; \| \varphi_{x_n,u_n}(x) \|_1 < \varepsilon/4 \} \) and set \( \varphi_n = \varphi_{x_n,u_n} \). The functions \( \varphi_n \) have the following properties:

\[
\varphi_n(x)(t) \geq \text{ess inf} \{ \| u(t) - u_n(t) \|_E; u \in G(x) \},
\]

\[
p_n(x) \| \varphi_n(x) \|_1 \leq p_n(x) \cdot \varepsilon/4 \quad (x \in X, n \geq 1).
\]

Lemma 2, applied to the sequences \( \{ \varphi_n \} \) and \( \{ h_n \} \), and to the function \( l: l(x) = \sum_{n \geq 1} h_n(x) \), yields a continuous function \( \tau: X \to R^+ \) and a family \( \{ \Phi(\tau, \lambda) \} \) of measurable subsets of \( T \) satisfying (a), (b) and (c').
It is now possible to construct the functions \( f_{\varepsilon} \) and \( \varphi_{\varepsilon} \). Set \( \lambda_0 \equiv 0, \lambda_n(x) = \sum_{m \leq n} p_m(x) \), and define

\[
f_{\varepsilon}(x) = \sum_{n \geq 1} u_n \cdot \chi_n(x), \quad \varphi_{\varepsilon}(x) = \varepsilon/4 + \sum_{n \geq 1} \varphi_n(x) \cdot \chi_n(x),
\]

where

\[
\chi_n(x) = \chi_{\Phi(x), \lambda_n(x)} \cap 1_{\Phi(x), \lambda_n-1(x)}.
\]

Clearly, \( f_{\varepsilon} \) and \( \varphi_{\varepsilon} \) are continuous, because the above summations are locally finite.

Let \( G_{\varepsilon} \) be defined by (7.6). To check that the values of \( G_{\varepsilon} \) are nonempty, fix any \( x \in X \). For every \( n \geq 1 \), use Proposition 2 and select \( u^n_x \in G(x) \) such that

\[
\|u^n_x(t) - u_n(t)\|_E < \varepsilon/4 + \operatorname{ess inf} \{\|u(t) - u_n(t)\|_E; \ u \in G(x)\}
\]
\( \mu \text{-a.e. in } T \). Then

\[
u_x = \sum_{n \geq 1} u^n_x \cdot \chi_n(x)
\]
lies in \( G(x) \), because \( G(x) \) is decomposable. We claim that \( u_x \in G_{\varepsilon}(x) \). Indeed, (7.8) and (7.10) yield

\[
\|u_x(t) - f_{\varepsilon}(x)(t)\|_E \leq \sum_{n \geq 1} \|u^n_x(t) - u_n(t)\|_E \cdot \chi_n(x)(t)
\]

\[
< \varphi_{\varepsilon}(x)(t) \quad \mu \text{-a.e. in } T.
\]

Hence \( G_{\varepsilon}(x) \neq \emptyset \). Being the intersection of two decomposable sets, \( G_{\varepsilon}(x) \) is also decomposable. The lower semicontinuity of \( G_{\varepsilon} \) follows from Proposition 4.

To conclude the proof of Proposition 5, it now suffices to show that \( \|\varphi_{\varepsilon}(x)\|_1 < \varepsilon \) for every \( x \). Set \( I(x) = \{n \geq 1; p_n(x) > 0\} \) and notice that \( 1 \leq \# I(x) \leq l(x) \). From (c') in Lemma 2 and (7.9) we deduce

\[
\|\varphi_{\varepsilon}(x)\|_1 = \varepsilon/4 + \sum_{n \geq 1} \int_T \varphi_n(x) \cdot \chi_n(x) \, d\mu
\]

\[
< \varepsilon/4 + \sum_{n \in I(x)} \left[ p_n(x) \|\varphi_n(x)\|_1 + \varepsilon/(2l(x)) \right] \leq \varepsilon/4 + \left[ \varepsilon/4 + \frac{\# I(x) \cdot \varepsilon}{2l(x)} \right] \leq \varepsilon. \quad \blacksquare
\]

At this stage, everything is ready for the completion of the proof of Theorem 3.

Let the function \( F \) be given. Construct two sequences of continuous maps \( f_n; X \to L^1(T; E) \) and \( \varphi_n; X \to L^1(T; R) \), and a sequence of l.s.c. multifunctions \( G_n \) with decomposable values, such that, for all \( x \in X \) and \( n \geq 1 \),
(i) $G_n(x) = \{u \in F(x); \|u(t) - f_n(x)(t)\|_E < \varphi_n(x)(t) \text{ } \mu\text{-a.e.}\} \neq \emptyset$,
(ii) $\|f_n(x)(t) - f_{n-1}(x)(t)\|_E \leq \varphi_n(x)(t) + \varphi_{n-1}(x)(t) \text{ } \mu\text{-a.e. in } T \text{ } (n \geq 2)$,
(iii) $\|\varphi_n(x)\|_1 < 2^{-n}$.

To do this, define $f_1$ and $\varphi_1$ by applying Proposition 5 with $G = F, \varepsilon = 1/2$. Let now $f_m, \varphi_m$ and $G_m$ be defined so that (i)–(iii) hold for all $m = 1, \ldots, n - 1$. To construct $f_n$ and $\varphi_n$, apply again Proposition 5 with $\varepsilon = 2^{-n}$, defining $G(x)$ to be the closure of $G_{n-1}(x)$, for all $x$. By induction, the maps $f_n, \varphi_n$ and $G_n$ can be defined for all $n \geq 1$. By (ii), the sequence $(f_n)_{n \geq 1}$ is Cauchy in the $L^1$-norm, hence it converges uniformly to some continuous function $f: X \rightarrow L^1(T; E)$. By (i) and (iii), $d_{L^1}(f_n(x), F(x)) < 2^{-n}$. Since $F(x)$ is closed, this implies that $f(x) \in F(x)$ for all $x \in X$, hence $f$ is a selection of $F$.

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References


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