Research Statement

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1. Introduction

In the classical theory of smooth dynamical systems, one studies actions of 1-parameter groups by diffeomorphisms of a manifold: in discrete time, one studies the $\mathbb{Z}$-action generated by a diffeomorphism $f\colon M \to M$ and, in continuous time, the $\mathbb{R}$-action or *flow* generated by a system of ordinary differential equations or a vector field on a manifold. A more general theory of smooth dynamical systems can be defined as the study of actions of general (non-compact) groups by diffeomorphisms on a manifold.

Given a smooth group action, a classical problem is to understand the existence and properties of invariant geometric structures (e.g. probability measures, closed subsets, and foliations). A major focus of my research is the study of invariant probability measures for differentiable group actions. A classical problem in dynamics is the following: *Given a group action on a manifold, classify the invariant (or stationary) probability measures.* For actions of 1-parameter groups (exhibiting some hyperbolicity) one expects many invariant measures, whereas for actions of "larger groups" (e.g. higher-rank abelian groups) there are situations in which one expects relatively few invariant measures. (This phenomenon is outlined in Section 2.) My research in this direction is outlined in Section 3 where the following rigidity phenomenon is observed:

> Given a smooth group action on a surface and a positive-entropy stationary measure μ, either the action preserves a μ-measurable line bundle, or μ satisfies some absolute continuity properties.

The precise absolute continuity property depends on additional hypotheses: in the general case the measure will be *SRB* (Theorem 4), whereas under additional hypotheses the measure itself will be absolutely continuous (Theorem 6.) These results can be seen as an extension of recent measure rigidity results for affine actions (c.f. A) to a general setting of actions by diffeomorphisms on surfaces.

In Section 2, a less classical problem is considered: *Given a probability measure on a manifold, classify the group of measure preserving diffeomorphisms.* A complete description for (positive-entropy) measures on the torus $\mathbb{T}^2$, invariant under an *Anosov* diffeomorphism, is given in Theorem 1. Here such measures are shown to be very rigid:

> If μ is not absolutely continuous and not the *measure of maximal entropy*, the group of μ-preserving diffeomorphisms is virtually cyclic.

Partial results extending Theorem 1 to measures supported on other surfaces are also outlined in Section 2.

Another motivating problem in classical dynamics is the classification of dynamical systems up to some notion of equivalence (e.g. a continuous change of coordinates). A classical result classifying—up to a continuous coordinate change—all *Anosov* diffeomorphisms on tori (and infra-nilmanifolds) is given by Theorem B in Section 4. In Section 4, I outline new results extending Theorem B to actions of certain discrete groups:

> Given an Anosov action of a higher-rank lattice on a torus, under certain hypotheses on the *linear data,* we construct a conjugacy between the action and its linear model; furthermore, this conjugacy is often smooth.

Also in Section 4, I describe new criterion characterizing when an action of a lattice on a closed manifold has an invariant measure.

In Section 5, some current projects as well as plans for future research are outlined.

2. Groups of measure preserving diffeomorphisms

Consider a self-map of a metric space $f\colon X \to X$. A Borel probability measure μ on X is *f-invariant* if, for any measurable $A \subset X$,

$$\mu(A) = \mu\left(f^{-1}(A)\right). \tag{1}$$

If X is compact and f is continuous, the Krylov–Bogolyubov theorem guarantees the existence of at least one f-invariant Borel probability measure. An invariant probability μ is *ergodic*[1] if it is indecomposable in the sense that

$$\mu \neq t\nu_1 + (1-t)\nu_2$$

for any $t \in (0,1)$ and invariant probabilities ν_1, ν_2.[2]

Let $f\colon M \to M$ be a diffeomorphism of a manifold M and let μ be an f-invariant measure. Associated to the pair (f,μ) is a number, called the *metric entropy of f with respect to μ*, denoted by $h_\mu(f)$. The number $h_\mu(f)$ measures, in some sense, the complexity or degree of "chaos" of the transformation f (with respect to the measure μ). Positivity of entropy $h_\mu(f)$ indicates exponential complexity.

When X is a manifold and f is a diffeomorphism exhibiting *hyperbolicity* there are typically many mutually singular, invariant probability measures with positive metric entropy $h_\mu(f)$. In particular, this holds for Anosov diffeomorphisms (see Definition 1) and *expanding maps* (e.g. the map $x \mapsto 2x \bmod 1$ of the circle $\mathbb{R}/\mathbb{Z}$). One may ask to what degree such measures uniquely determine the ambient dynamics.

Problem 1. *Given a Borel probability measure μ supported on a compact manifold M, describe the group* $\mathrm{Diff}^r_\mu(M)$ *of μ-preserving C^r diffeomorphisms.*

Problem 1 is partially motivated by *measure rigidity* phenomenon for actions of higher-rank abelian groups. In contrast to the case of a single expanding map $x \mapsto 2x \bmod 1$, Rudolph showed that any measure on $\mathbb{R}/\mathbb{Z}$, ergodic under the abelian semi-group action generated by

$$f\colon x \mapsto 2x \bmod 1 \quad \text{and} \quad g\colon x \mapsto 3x \bmod 1,$$

is either Lebesgue or has zero entropy (for each generator) [Rud]. This dichotomy—that given an action of a higher-rank abelian group, either every ergodic invariant measure is absolutely continuous or has zero entropy—has been extended to more general settings including algebraic actions on tori [KS, EL] and non-algebraic, nonuniformly hyperbolic actions on tori and general manifolds [KK, KKRH].

2.1. **Outline of my results.** In [Bro1], I consider Problem 1 for measures on the 2-torus invariant under an Anosov diffeomorphism. We recall the following definition.

Definition 1 (Anosov diffeomorphisms)**.** A diffeomorphism $f\colon M \to M$ of a compact manifold M is called *Anosov* if, for any Riemannian metric, there are constants $C > 0$, $0 < \kappa < 1$ and a continuous Df-invariant splitting of the tangent bundle $T_xM = E^s_f(x) \oplus E^u_f(x)$ such that for every $x \in M$ and $n \in \mathbb{N}$

$$\begin{aligned} \|D_x f^n v\| &\le C\kappa^n \|v\|, \quad \text{for } v \in E^s_f(x) \\ \|D_x f^{-n} u\| &\le C\kappa^n \|u\|, \quad \text{for } u \in E^u_f(x). \end{aligned} \tag{2}$$

($D_x f^n\colon T_xM \to T_{f^n(x)}M$ denotes the differential of the n-fold composition $f^n = f \circ f \circ \cdots \circ f$.)

In [Bro1], I prove that for many positive entropy, invariant measures for an Anosov $f\colon \mathbb{T}^2 \to \mathbb{T}^2$, the group of measure-preserving diffeomorphisms is essentially the 1-parameter subgroup generated by f. Combining the result in [Bro1] with an improvement[3] in [BRH], we have the following precise statement.

Theorem 1 ([Bro1, BRH])**.** *Let μ be a singular*[4] *Borel probability measure on $\mathbb{T}^2$ with full support. Assume for $r \ge 2$, that* $\mathrm{Diff}^r_\mu(\mathbb{T}^2)$ *contains an ergodic, Anosov element $f\colon \mathbb{T}^2 \to \mathbb{T}^2$ with positive metric entropy $h_\mu(f)$. In addition, assume μ is not the* measure of maximal entropy[5] *for f. Then, the subgroup of* $\mathrm{Diff}^r_\mu(\mathbb{T}^2)$ *generated by f has finite index in* $\mathrm{Diff}^r_\mu(\mathbb{T}^2)$. *In particular,* $\mathrm{Diff}^r_\mu(\mathbb{T}^2)$ *is virtually-$\mathbb{Z}$.*

[1] Alternatively, μ is ergodic if all measurable invariant subsets null or co-null.

[2] Given an invariant probability μ, we may decompose μ into its *ergodic components*. Thus, it is not very restrictive to assume an invariant measure is ergodic when stating theorems.

[3] The original result in [Bro1] required an additional hypothesis that the Lyapunov exponents of μ have different absolute value.

[4] with respect to to the Riemannian volume

[5] The *unique* f-invariant probability maximizing the expression $h_\nu(f)$ over all f-invariant probabilities ν.

2.2. **Extensions to other surfaces.** The only surface admitting an Anosov diffeomorphism is the torus $\mathbb{T}^2$. To study Problem 1 for other surfaces, one must work in the setting of nonuniformly hyperbolic dynamics. Let $f\colon M \to M$ be a C^2 diffeomorphism of a closed surface M and let μ be an ergodic, f-invariant measure. Assume $h_\mu(f) > 0$. It follows from [LY] that f is *nonuniformly hyperbolic*: there are numbers $-\infty < \lambda^s < 0 < \lambda^u < \infty$ and a μ-measurable, Df-invariant splitting $T_xM = E^u_f(x) \oplus E^s_f(x)$ with

$$\begin{aligned} \lim_{n\to\pm\infty} \frac{1}{n} \log \|D_x f^n v\| &= \lambda^s, \quad \text{for } v \in E^s_f(x) \\ \lim_{n\to\pm\infty} \frac{1}{n} \log \|D_x f^n u\| &= \lambda^u, \quad \text{for } u \in E^u_f(x). \end{aligned} \tag{3}$$

(Compare with the uniform estimates in (2).) The μ-a.e. defined, μ-measurable line fields $E^s_f(x)$ and $E^u_f(x)$ are called the *Lyapunov subspaces* and the numbers λ^s and λ^u are the *Lyapunov exponents.*

In the setting that $f\colon \mathbb{T}^2 \to \mathbb{T}^2$ is Anosov, the line fields $E^u_f(x)$ and $E^s_f(x)$ are defined for every x, continuous in x, and are uniquely integrable. The main observation used to prove Theorem 1 is the following:

Proposition 2. *If μ is not absolutely continuous, then there is a finite index subgroup $\Gamma \subset \mathrm{Diff}^2_\mu(\mathbb{T}^2)$ such that one of the continuous line fields E^u_f or E^s_f is invariant under the action of Γ.*

By the unique integrability of the line fields E^u_f and E^s_f, it follows that each $g \in \Gamma$ preserves a corresponding integral *foliation*. The proof of Theorem 1 in [Bro1] depends heavily on the existence of this invariant foliation as well as the conjugacy between f and its linear model (see Theorem B, below).

Outside of the Anosov setting, the dynamically defined line fields $E^u_f(x)$ and $E^s_f(x)$ are only measurable and not, a priori, uniquely integrable. In this more general setting, new tools are needed to establish a result analogous to Theorem 1. However, we have the following generalization of Proposition 2. Given a probability measure μ on M and a subgroup $\Gamma \subset \mathrm{Diff}^r_\mu(M)$, we say a μ-measurable line field $V \subset TM$ is Γ-*invariant* if for all $g \in \Gamma$

$$D_x g V(x) = V(g(x)) \quad \mu\text{-a.s.}$$

Theorem 3 ([BRH])**.** *Let M be a closed surface and μ a Borel probability measure on M. Assume that μ is not absolutely continuous and that there exists $f \in \mathrm{Diff}^2_\mu(M)$ with $h_\mu(f) > 0$. Then there is a μ-measurable line field $V \subset TM$ and a finite index $\Gamma \subset \mathrm{Diff}^2_\mu(M)$ such that V is Γ-invariant.*

In fact, the line field V in the conclusion of Theorem 3 will be one of the Lyapunov subspace for f.

3. Stationary measures for random dynamics

3.1. **Background.** Given a compact manifold M and a diffeomorphism $f\colon M \to M$ there exists at least one f-invariant probability measure. However, for actions of larger (non-amenable) groups, there need not be any invariant measure. For that reason, we define a weaker notion of invariance: Given a probability ν on $\mathrm{Diff}^2(M)$, we say a probability measure μ on M is ν-*stationary* if, for any measurable $A \subset M$, we have

$$\mu(A) = \int f^{-1}(A)\, d\nu(f). \tag{4}$$

(Compare to (1).)

My work is motivated by number of recent results classifying all stationary measures for certain homogeneous actions. For instance, consider a subgroup $\Gamma \subset \mathrm{SL}(n,\mathbb{Z})$. We have a natural action of Γ on the torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ given by

$$\Gamma \ni A\colon x + \mathbb{Z}^n \mapsto Ax + \mathbb{Z}^n.$$

We say the action of Γ on $\mathbb{R}^n$ is *strongly irreducible* if no proper, nontrivial subspace of $\mathbb{R}^n$ has finite Γ-orbit.

Theorem A ([BQ])**.** *Let ν be a finitely supported measure on $\mathrm{SL}(n,\mathbb{Z})$ and let $\Gamma_\nu \subset \mathrm{SL}(n,\mathbb{Z})$ be the subgroup generated by the support of ν. Assume the action of Γ_ν on $\mathbb{R}^n$ is strongly irreducible. Then*

(1) the only ergodic ν-stationary measures on $\mathbb{T}^n$ are Haar or finitely supported;
(2) in particular, all ν-stationary measures are Γ-invariant.

In [BQ], the authors obtain similar results for actions by translations on quotients of semisimple Lie groups.

In a recent paper [EM], the authors consider a similar problem. Here, the natural action of the upper-triangular subgroup of $\mathrm{SL}(2,\mathbb{R})$ on moduli space is considered, and the authors prove that every upper-triangular-invariant measure is, in fact, $\mathrm{SL}(2,\mathbb{R})$-invariant and homogeneous in an appropriate sense. The techniques in [EM] motivated the techniques used to prove the results outlined below.

3.2. **Setup for my results.** In joint work with Federico Rodriguez Hertz, I have recently obtained analogues of the above rigidity results in non-affine settings. Let M be a closed *surface* and let ν be a compactly supported probability measure on $\mathrm{Diff}^2(M)$. We consider the composition of diffeomorphism chosen randomly according to ν. Let μ be an ergodic, ν-stationary measure.[6] To state the results, some notation is needed.

- Given a sequence $\omega = (f_0, f_1, f_2, \dots) \in \left(\mathrm{Diff}^r(M)\right)^{\mathbb{N}}$ and $n \geq 0$ we define a *cocycle*
$$f_\omega^0 = \mathrm{Id}, \quad f_\omega^1 = f_0, \quad f_\omega^n := f_{n-1} \circ f_{n-2} \circ \cdots \circ f_1 \circ f_0.$$
- Equip the space of all sequences $\left\{\omega = (f_0, f_1, f_2, \dots) \in \left(\mathrm{Diff}^r(M)\right)^{\mathbb{N}}\right\}$ with the product measure $\nu^{\mathbb{N}}$; we then study the asymptotic properties of f_ω^n for typical ω.
- Write $\mathscr{X}^+(M,\nu)$ to denote random process on M defined by the setup above.
- As in the case of a single diffeomorphism, given a ν-stationary measure μ we define a *metric entropy*[7] of the random process $\mathscr{X}^+(M,\nu)$, denoted $h_\mu(\mathscr{X}^+(M,\nu))$.
- Assume that $h_\mu(\mathscr{X}^+(M,\nu)) > 0$. Then there are numbers $-\infty < \lambda^s < 0 < \lambda^u < \infty$ and, for $\nu^{\mathbb{N}}$-a.e. sequence $\omega = (f_0, f_1, f_2, \dots)$ and μ-a.e. $x \in M$, a one-dimensional subspace $E_\omega^s(x) \subset T_xM$ such that, for all $v \in E_\omega^s(x)$ and $w \in T_xM \smallsetminus E_\omega^s(x)$,
$$\lim_{n\to\infty} \frac{1}{n} \log \left\| D_x f_\omega^n(v) \right\| = \lambda^s \qquad \lim_{n\to\infty} \frac{1}{n} \log \left\| D_x f_\omega^n(w) \right\| = \lambda^u.$$
(Compare to (3) in the case of a single diffeomorphism.)
- We call $E_\omega^s(x)$ the *stable Lyapunov subspace at x for the word ω*. We say the family $\{E_\omega^s(x)\}$ is *non-random* if it is independent of the word ω; that is, $\{E_\omega^s(x)\}$ is *non-random* if there is a μ-measurable subbundle $V \subset TM$ so that for μ-a.e. $x \in M$ and $\nu^{\mathbb{N}}$-a.e. $\omega = (f_0, f_1, f_2, \dots) \in \left(\mathrm{Diff}^r(M)\right)^{\mathbb{N}}$
$$E_\omega^s(x) = V(x).$$
- We say μ is *SRB*[8] if $h_\mu(\mathscr{X}^+(M,\nu)) = \lambda^u$.

3.3. **My results.** In [BRH] the following dichotomy is proved:

Theorem 4. *Let M be a closed surface. Let ν be a compactly supported Borel probability measure on $\mathrm{Diff}^2(M)$ and let μ be an ergodic, ν-stationary probability measure with $h_\mu(\mathscr{X}^+(M,\nu)) > 0$. Then either*

(1) the stable distribution $E_\omega^s(x)$ is non-random, or
(2) μ is SRB.

We obtain the following corollary that is in some sense an analogue of Theorem A in our setting. Given a probability ν on $\mathrm{Diff}^2(M)$ and a ν-stationary measure μ, a μ-measurable line field $V \subset TM$ is *ν-a.s. invariant* if
$$D_x g V(x) = V(g(x)) \tag{5}$$
for ν-a.e. $g \in \mathrm{Diff}^2(M)$ and μ-a.e. $x \in M$.

Corollary 5. *Let ν and μ be as in Theorem 4. Assume there are no ν-a.s. invariant, μ-measurable line fields. Then μ is SRB.*

[6] The compactness of M guarantees the existence of at least one stationary measure. As in the case of a single diffeomorphism, we may decompose μ into ergodic components.

[7] This is defined, for example, in [Kif].

[8] This condition is equivalent to the *sample measures* of μ having absolutely continuous conditional measures along unstable manifolds; see [LQ].

Because we do not work in the setting of affine actions, we can not promote the SRB property in Corollary 5 to absolute continuity or almost-sure invariance of the measure as is done in [BQ] and [EM]. However, under the additional hypothesis that ν-a.e. g preserves a fixed volume, we can upgrade certain stationary measures to ν-a.s. invariant, absolutely continuous measures.

Theorem 6. *Let M be a closed surface, m a smooth volume, and $\Gamma \subset \mathrm{Diff}^2_m(M)$ a subgroup of m-preserving diffeomorphisms. Let ν be a compactly supported probability measure on Γ and let μ be an ergodic, ν-stationary measure. Assume*

(1) $h_\mu(\mathscr{X}^+(M,\nu)) > 0$, and
(2) there are no ν-a.s. invariant, μ-measurable line fields.

Then μ is equal to (an ergodic component of) m and, in particular, μ is invariant for ν-a.e. $g \in \Gamma$.

4. Rigidity of smooth lattice actions

4.1. **Motivation.** Given a homeomorphism $f\colon \mathbb{T}^n \to \mathbb{T}^n$ there is a unique matrix $A \in \mathrm{GL}(n,\mathbb{Z})$ such that every lift $\tilde{f}\colon \mathbb{R}^n \to \mathbb{R}^n$ of f is of the form

$$\tilde{f}(x) = Ax + \psi(x) \tag{6}$$

for some $\mathbb{Z}^n$-periodic function ψ. Since A has integer entries and $\det A = \pm 1$, A induces an automorphism of the quotient $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. We denote by $L_A\colon \mathbb{T}^n \to \mathbb{T}^n$ the induced action:

$$L_A(x + \mathbb{Z}^n) := Ax + \mathbb{Z}^n.$$

We call the matrix A the *linear data of f*. We have the following classical theorem.

Theorem B ([Fra, Man])**.** *Assume A has no eigenvalues of modulus one. Then there exists a continuous $h\colon \mathbb{T}^n \to \mathbb{T}^n$ such that*

$$L_A \circ h = h \circ f. \tag{7}$$

Furthermore, if f is an Anosov diffeomorphism, then h is a homeomorphism and

$$L_A = h \circ f \circ h^{-1}. \tag{8}$$

A map h satisfying (7) is called a *semiconjugacy*; h satisfying (8) is called a *conjugacy*. Theorem B provides a complete classification of Anosov diffeomorphisms on tori[9] up to *continuous change of coordinates*.

One may ask to what extent Theorem B generalizes to an action of an arbitrary discrete group Γ. We restrict our consideration to the following setting: Let Γ be an irreducible lattice[10] in a higher-rank, semisimple Lie group G. For a concrete example, take $\Gamma = \mathrm{SL}(n,\mathbb{Z})$ and $G = \mathrm{SL}(n,\mathbb{R})$ for $n \geq 3$. Fix such a lattice Γ and let $\alpha\colon \Gamma \to \mathrm{Diff}^\infty(\mathbb{T}^n)$ be an action. Given $\gamma \in \Gamma$, let $\rho(\gamma) \in \mathrm{GL}(n,\mathbb{Z})$ be the unique matrix A satisfying (6) with $f = \alpha(\gamma)$. Then $\rho\colon \Gamma \to \mathrm{GL}(n,\mathbb{Z})$ is a representation called the *linear data of α*. Since each matrix $\rho(\gamma)$ is integer valued with $\det \rho(\gamma) = \pm 1$, we have an induced action $\tilde{\rho}\colon \Gamma \to \mathrm{Diff}^\infty(\mathbb{T}^n)$ given by

$$\tilde{\rho}(\gamma)(x + \mathbb{Z}^n) = \rho(\gamma)x + \mathbb{Z}^n$$

Question 1 (Existence of a semiconjugacy)**.** *Given an action $\alpha\colon \Gamma \to \mathrm{Diff}^\infty(\mathbb{T}^n)$, when is there a finite index subgroup[11] $\Gamma' \subset \Gamma$ and continuous $h\colon \mathbb{T}^n \to \mathbb{T}^n$ with*

$$\tilde{\rho}(\gamma) \circ h = h \circ \alpha(\gamma) \qquad \textit{for all } \gamma \in \Gamma'? \tag{9}$$

By Theorem B, if a semi-conjugacy (9) exists and, if $\alpha(\gamma)$ is Anosov for some $\gamma \in \Gamma$ (recall Definition 1), then $h\colon \mathbb{T}^n \to \mathbb{T}^n$ is automatically a homeomorphism. We ask when the regularity of h can be improved.

Question 2 (Smoothness of conjugacies)**.** *If there is $\gamma \in \Gamma$ with $\alpha(\gamma)$ Anosov, when is a homeomorphism $h\colon \mathbb{T}^n \to \mathbb{T}^n$ satisfying (9) C^∞?*

[9] More generally, Theorem B extends to Anosov diffeomorphisms of infra-nilmanifolds.

[10] Recall a lattice is a discrete subgroup with finite covolume.

[11] The restriction to a finite index subgroup is natural when looking for a conjugacy between general discrete group actions.

Questions 1 and 2 have been studied previously under the additional hypothesis that the non-linear action α preserves a measure [KLZ, MQ]. Our results addressing Questions 1 and 2 do not assume the existence of an invariant measure. However, we ask when such an invariant measure exists.

Question 3. *Given a smooth action of Γ on a manifold M, when is there a Γ-invariant probability measure?*

We note that such Γ are non-amenable so a smooth action of Γ on a compact manifold need not have an invariant measure. For instance, for the natural action of $\mathrm{SL}(n,\mathbb{Z})$ on the sphere S^{n-1} there is no $\mathrm{SL}(n,\mathbb{Z})$-invariant measure.[12]

4.2. **Main corollary of new results.** In collaboration with Federico Rodriguez Hertz and Zhiren Wang, we have established new results answering each of the above questions. As a primary example, we have the following corollary of our work.

Theorem 7. *Let $\Gamma = \mathrm{SL}(n,\mathbb{Z})$ for $n \geq 5$. Let $\alpha\colon \Gamma \to \mathrm{Diff}^\infty(\mathbb{T}^n)$ be an action and assume the linear data $\rho\colon \Gamma \to \mathrm{GL}(n,\mathbb{Z})$ of α is the* standard representation.

(a) There is a continuous $h\colon \mathbb{T}^n \to \mathbb{T}^n$ and a subgroup $\Gamma' \subset \Gamma$ of finite index with

$$\tilde{\rho}(\gamma) \circ h = h \circ \alpha(\gamma) \quad \forall \gamma \in \Gamma'. \tag{10}$$

Moreover, if there exists $\gamma \in \Gamma$ with $\alpha(\gamma)$ Anosov, then $h\colon \mathbb{T}^n \to \mathbb{T}^n$ is a C^∞ diffeomorphism.

(b) There is an α-invariant Borel probability measure μ.

We remark that we show the existence of the α-invariant Borel probability measure after establishing the existence of the semi-conjugacy (10). Additionally, in Theorem 7, we can show that the measure μ is absolutely continuous.

4.3. **Outline of results: existence and smoothness of (semi-)conjugacies.** The existence of a semiconjugacy in Theorem 7(a) follows from the following more technical theorem. Let $\alpha\colon \Gamma \to \mathrm{Diff}^\infty(\mathbb{T}^n)$ be an action with linear data $\rho\colon \Gamma \to \mathrm{GL}(n,\mathbb{Z})$. We assume[13] that the linear data $\rho\colon \Gamma \to \mathrm{GL}(n,\mathbb{Z})$ extends to representation $\rho\colon G \to \mathrm{GL}(n,\mathbb{R})$. Let $\{\chi_j\}$ denote the *roots* of G and $\{\lambda_i\}$ denote the *weights* of the representation $\rho\colon G \to \mathrm{GL}(n,\mathbb{R})$. We say ρ is *non-resonant* if

$$\chi_j \neq c\lambda_i$$

for all i, j and $c \in \mathbb{C}$.

Theorem 8. *Let G be a connected, semisimple Lie group of rank ≥ 2 and let $\Gamma \subset G$ be an irreducible lattice. Let $\alpha\colon \Gamma \to \mathrm{Diff}^\infty(\mathbb{T}^n)$ be an action. Assume:*

(1) the linear data, $\rho(\Gamma)$, contains a matrix with no eigenvalue of modulus 1;
(2) the linear data, $\rho\colon \Gamma \to \mathrm{GL}(n,\mathbb{Z})$, extends to a representation $\rho\colon G \to \mathrm{GL}(n,\mathbb{R})$;
(3) the representation $\rho\colon G \to \mathrm{GL}(n,\mathbb{R})$ is non-resonant;
(4) the action $\alpha\colon \Gamma \to \mathrm{Diff}^\infty(\mathbb{T}^n)$ lifts[14] to an action $\tilde{\alpha}\colon \Gamma \to \mathrm{Diff}^\infty(\mathbb{R}^n)$.

Then, there is a continuous $h\colon \mathbb{T}^n \to \mathbb{T}^n$ with

$$\tilde{\rho}(\gamma) \circ h = h \circ \alpha(\gamma) \quad \forall \gamma \in \Gamma.$$

For irreducible lattices in many semisimple Lie groups, and for Anosov representations $\rho\colon \Gamma \to \mathrm{GL}(n,\mathbb{Z})$ of certain dimensions, the technical hypotheses in Theorem 8 hold after passing to finite index subgroups. In particular, conditions (1)–(4) hold in the setting of Theorem 7.

Remark 9 (Smoothness of conjugacies). If we assume that there exists some $\gamma \in \Gamma$ so that $\alpha(\gamma)$ is Anosov, then the semiconjugacy $h\colon \mathbb{T}^n \to \mathbb{T}^n$ guaranteed by Theorem 8 is a homeomorphism. We ask when h can

[12] The action is the natural action of $\mathrm{SL}(n,\mathbb{Z})$ on S^{n-1} interpreted as the space of rays in $\mathbb{R}^n$.

[13] By Margulis Superridigity, this holds for large classes of lattices.

[14] A sufficient condition that guarantees this lifting property is the vanishing of some second group cohomology $H^2_\rho(\Gamma,\mathbb{Z}^n)$; this condition, in particular, only depends on the linear data ρ of the action α.

be upgraded to a C^∞-diffeomorphism. If the element $\gamma \in \Gamma$ is in "general position" in Γ, the smoothness of h follows from [RHW]. In the context of Theorem 7, it follows from [KLZ] that the presence of a single Anosov element implies an Anosov element in general position. In many settings considered in Theorem 8, we can show that the presence of a single Anosov element implies the existence of an Anosov element in general position.

4.4. **Outline of results: existence of invariant measures.** Consider for simplicity the case $\Gamma = \mathrm{SL}(n,\mathbb{Z})$ as a lattice in $G = \mathrm{SL}(n,\mathbb{R})$ for $n \geq 3$. Given a smooth action $\alpha\colon \Gamma \to \mathrm{Diff}^\infty(M)$ on a closed manifold M, the existence of an α-invariant measure on M is equivalent to the existence of a G-invariant measure on an auxiliary manifold X^α on which G acts naturally.[15] Note that G is *non-amenable* so need not a priori preserve any probability measure. The space X^α is a fiber-bundle over the homogeneous space $\mathrm{SL}(n,\mathbb{R})/\mathrm{SL}(n,\mathbb{Z})$ with fibers M. This fibration is preserved by the G-action.

Denote by $P \subset \mathrm{SL}(n,\mathbb{R})$ the subgroup of upper triangular matrices and by $H \subset P$ the diagonal subgroup $H = \{\mathrm{diag}(\exp(t_1), \exp(t_2), \ldots, \exp(t_n)) \mid \sum t_i = 0\}$ and consider the restriction of the action of G on X^α to the subgroups H and P. Since P is solvable (in particular, amenable), it follows (from a generalization of the Krylov–Bogolyubov Theorem) that there exists an ergodic, P-invariant probability ν on X^α projecting to Haar measure on G/Γ. We have $H \simeq \mathbb{R}^{n-1}$ and, since $H \subset P$, ν is also H-invariant. Such a ν will be a candidate for a G-invariant measure; our characterization of G-invariance is stated in terms of the *Lyapunov exponent functionals* for the action of H on (X^α, ν).

Proposition (Higher-rank Oseledec's multiplicative ergodic theorem)**.** *Given $H \simeq \mathbb{R}^{n-1}$ acting on (X^α, ν), there are linear functionals*

$$\eta_j\colon H \to \mathbb{R}$$

and a DH-invariant, ν-measurable splitting

$$T_x X^\alpha = \bigoplus E_x^j$$

such that for ν-a.e. $x \in X^\alpha$, every $v \in E_x^j \smallsetminus \{0\}$, and every $s \in H$

$$\lim_{n\to\infty} \frac{1}{n} \log \left\| Ds^n(v) \right\| = \eta_j(s).$$

(Compare to (3) in the rank-one case.)

We decompose the Lyapunov exponent functionals η_j into 2 families: write $\{\lambda_i\}$ for those tangent to the fibers of X^α and $\{\chi_j\}$ for those tangent to G-orbits of X^α. We say the action of H on (X^α, ν) is *non-resonant* if for all j, i and $c \in \mathbb{R}$,

$$\chi_j \neq c\lambda_i.$$

Theorem 10. *Let ν be as above and assume the action of H on (X^α, ν) is non-resonant. Then ν is G-invariant.*

In the case of $\mathrm{SL}(n,\mathbb{Z})$ acting on $\mathbb{T}^n$ we can verify—using the semiconjugacy in Theorem 7(a) and arguments from [KK]—that the non-resonance condition holds for any candidate measure ν. Since the existence of a G-invariant measure ν on X^α is equivalent to the existence of an α-invariant measure μ on M, we obtain Theorem 7(b).

Theorem 10 extends naturally to actions of lattices in more general higher-rank Lie groups with H and P replaced by appropriate Cartan and Borel subgroups.

5. Current and future projects

Low dimensional extensions of Theorem 1. It is expected that results analogous to Theorem 1 hold for measures invariant under more general rank-one Anosov actions and for measures supported on more general hyperbolic sets. I have obtained some extensions in low-dimensional settings. In the first setting, I consider the case that μ is an invariant probability for an Anosov flow in a 3-manifold. In the second setting,

[15] X^α is the suspension space: $X^\alpha = (G \times M)/(g,x) \sim (g\gamma, \alpha(\gamma^{-1})(x))$ equipped with the left G-action $a\cdot(g,x) = (ag,x)$ for $a \in G$.

I consider the case that μ is supported on a hyperbolic surface attractor. In both of these settings, I have obtained results analogous Theorem 1: for most (fully supported) measures μ, the group of μ-preserving diffeomorphisms is, essentially, a 1-parameter group. These results will be forthcoming [Bro2].

Hypotheses for smooth rigidity. In Theorem 8, if one has a continuous semiconjugacy $h\colon \mathbb{T}^n \to \mathbb{T}^n$ between Γ actions α and $\tilde{\rho}$, and if there is a $\gamma \in \Gamma$ with $\alpha(\gamma)$ Anosov, the map h is automatically a (Hölder) homeomorphism and hence a conjugacy. If the element $\gamma \in \Gamma$ such that $\alpha(\gamma)$ is Anosov can be chosen in "general position," it follows from [RHW] that h is C^∞. The difficultly here is that the element $\gamma \in \Gamma$ with $\alpha(\gamma)$ Anosov may, a priori, have small centralizer in Γ. To apply [RHW] one needs a higher-rank abelian subgroup of Γ containing an Anosov element. We have made partial progress in removing the "general position" hypothesis and expect to be able to remove it entirely in the near future.

Removing entropy hypotheses. We note that—unlike all measure rigidity results for abelian actions—in Theorem A there is no positive entropy hypothesis. It is hopeful that, for similar reasons, the entropy hypotheses in Theorems 1 and 4 can be removed. Indeed, adding the hypotheses that there are no ν-a.s. invariant, μ-measurable line fields in these settings is a natural analogue of the strong irreducibility hypotheses in Theorem A. I have some partial results removing the entropy hypotheses. For instance, if one assumes in Theorem 4 that ν is supported on two Anosov elements, whose stable and unstable distributions satisfy certain geometric conditions, then we can remove positive entropy hypotheses using the techniques in [BQ]. I am hopeful that techniques from [EM] can be extended to the general setting considered [BRH] to remove the positive entropy hypotheses in Theorems 1 and 4.

Higher dimensional extensions. Obtaining extensions of Theorem 1 to Anosov diffeomorphism in higher dimensional manifolds seems promising in the near future. Although many arguments in [Bro1] heavily use the low ambient dimension, the arguments used to prove Theorem 4 should extend to higher dimensional settings. After conversations with Alex Eskin, I believe the inductive procedure in [EM] can be adapted, at least under certain hypotheses, to higher dimensional versions of the context considered in [BRH]. This would yield higher dimensional versions of Theorems 4 and 3. A major tool that would be needed in this setting is the theory of *nonuniform, non-stationary, normal forms.*

Promoting invariant measurable line fields to additional rigidity. In the setting of Theorem 1, one proves existence of an invariant *continuous* line field $V \subset TM$ which, by dynamical reasons, is uniquely integrable. In the setting of Theorem 3, assuming the measure μ is not SRB, one obtains only a *measurable* invariant line field $V \subset TM$ which is not, a priori, uniquely integrable. The existence of a measurable invariant line field in Theorem 3 and the non-randomness of the distributions $E^s_\omega(x)$ in Theorem 4 likely imply additionally constraints on the group $\mathrm{Diff}_\mu(M)$. For instance, in [BK] it is shown under strong hypotheses that the non-randomness of the distribution $E^s_\omega(x)$ implies unique integrability of associated Pesin manifolds. It is possible that this approach can be extended to more general settings.

Stiffness of stationary measures for lattice actions. A question that motivated the work outlined in Section 4 is the question of *stiffness* of stationary measures for lattice actions. Let Γ be a higher-rank lattice.

Question 4. *Given an action $\alpha\colon \Gamma \to \mathrm{Diff}^\infty(M)$, a finitely supported measure ν on Γ, and an ergodic ν-stationary measure μ, under what conditions is μ necessarily α-invariant?*

For many natural lattice actions, there are stationary measures that are not invariant (e.g. the natural $\mathrm{SL}(n,\mathbb{Z})$ action on S^{n-1}). However, under appropriate hypotheses (including some *non-resonance* conditions) we expect that stationary measures are Γ-invariant. This is planned future collaboration with Zhiren Wang and Federico Rodriguez Hertz.

References

[BQ] Y. Benoist and J.-F. Quint. *Mesures stationnaires et fermés invariants des espaces homogènes*, Ann. of Math. (2) **174**(2011), 1111–1162.

[BK] M. Brin and Y. Kifer. *Dynamics of Markov chains and stable manifolds for random diffeomorphisms*, Ergodic Theory Dynam. Systems **7**(1987), 351–374.

[Bro1] A. W. Brown. *Smooth stabilizers for measures on the torus*, Discrete Contin. Dyn. Syst. Accepted for publication, 2013.

[EL] M. Einsiedler and E. Lindenstrauss. *Rigidity properties of $\mathbb{Z}^d$-actions on tori and solenoids*, Electron. Res. Announc. Amer. Math. Soc. **9**(2003), 99–110 (electronic).

[EM] A. Eskin and M. Mirzakhani. *Invariant and stationary measures for the SL(2,R) action on Moduli space*. Preprint, 2013. arXiv:1302.3320.

[Fra] J. Franks. *Anosov diffeomorphisms on tori*, Trans. Amer. Math. Soc. **145**(1969), 117–124.

[KK] B. Kalinin and A. Katok. *Measure rigidity beyond uniform hyperbolicity: invariant measures for Cartan actions on tori*, J. Mod. Dyn. **1**(2007), 123–146.

[KKRH] B. Kalinin, A. Katok, and F. Rodriguez Hertz. *Nonuniform measure rigidity*, Ann. of Math. (2) **174**(2011), 361–400.

[KLZ] A. Katok, J. Lewis, and R. Zimmer. *Cocycle superrigidity and rigidity for lattice actions on tori*, Topology **35**(1996), 27–38.

[KS] A. Katok and R. J. Spatzier. *Invariant measures for higher-rank hyperbolic abelian actions*, Ergodic Theory Dynam. Systems **16**(1996), 751–778.

[Kif] Y. Kifer. *Ergodic theory of random transformations*, volume 10 of *Progress in Probability and Statistics*. Birkhäuser Boston Inc., Boston, MA, 1986.

[LY] F. Ledrappier and L.-S. Young. *The metric entropy of diffeomorphisms. II. Relations between entropy, exponents and dimension*, Ann. of Math. (2) **122**(1985), 540–574.

[LQ] P.-D. Liu and M. Qian. *Smooth ergodic theory of random dynamical systems*, volume 1606 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1995.

[Man] A. Manning. *There are no new Anosov diffeomorphisms on tori*, Amer. J. Math. **96**(1974), 422–429.

[MQ] G. A. Margulis and N. Qian. *Rigidity of weakly hyperbolic actions of higher real rank semisimple Lie groups and their lattices*, Ergodic Theory Dynam. Systems **21**(2001), 121–164.

[RHW] F. Rodriguez Hertz and Z. Wang. *Global rigidity of higher rank Anosov algebraic actions*. Preprint, 2013. arXiv:1304.1234.

[Rud] D. J. Rudolph. $\times 2$ *and* $\times 3$ *invariant measures and entropy*, Ergodic Theory Dynam. Systems **10**(1990), 395–406.

Papers in preparation and in progress

[Bro2] A. W. Brown. *Smooth stabilizers for uniformly hyperbolic measures and basic sets*. In progress, 2013.

[BRH] A. W. Brown and F. Rodriguez Hertz. *Measure rigidity for random surface diffeomorphisms*. In preparation, 2013.

[BRHW] A. W. Brown, F. Rodriguez Hertz, and Z. Wang. *Topological, smooth, and measure rigidity for lattice actions*. In progress, 2013.