THE PENNSYLVANIA STATE UNIVERSITY
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DEPARTMENTS OF MATHEMATICS AND ELECTRICAL ENGINEERING

TRANSFORMATION OPTICS METHODOLOGY REVIEW AND ITS
APPLICATION TO ANTENNA LENS DESIGNS

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SPRING 2014

A thesis
submitted in partial fulfillment
of the requirements
for a baccalaureate degrees
in Mathematics and Electrical Engineering
with honors in Mathematics and Electrical Engineering

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ABSTRACT

Transformation Optics (TO) is a mathematical method that is used to renormalize the space that enables arbitrary control over light propagation by relying on the coordinate invariant property of Maxwell’s equations. In other words, when applying coordinate transformations, Maxwell’s equations change in a way that they preserve the physics. This technique enables unprecedented device design flexibility. First, different methodologies of TO will be analyzed and discussed in terms of their advantages and disadvantages for electromagnetic applications. Second, an effective approach is proposed to design antenna lenses for controlling the far-field patterns, which only uses a single electromagnetic source or antenna feed located both outside and embedded inside the lens. By employing a complex coordinate transformation, a spatial distribution of gain and/or loss can be introduced into the lens. Consequently, the lens offers an extra degree of freedom that allows controlling not only the phase distribution of the electromagnetic field inside the lens but also the amplitude distribution. The complex coordinate transformation is then applied to a linear coordinate transformation enabled lens to achieve simpler material parameters and demonstrate the versatility of the proposed design approach. Several full-wave simulated lens examples and corresponding linear antenna array calculations are presented, demonstrating the capability of complex coordinate transformations for far-field pattern manipulation and near-field amplitude tapering across the lens aperture.
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ACKNOWLEDGEMENTS

This work could not have been done without the guidance of Professor Werner and Professor Mazzucato, and the help provided by Dr. Jiang. This project was partially supported by a Schreyer Honors College Research Grant.
Chapter 1

Introduction

A spatially changing refractive index leads to changes in light propagation characteristics. The most common example is a curved piece of glass that bends the light as the light experiences a change of refractive index through interface. A medium with continuous changing of refractive index can change the light direction continuously and create special effects, such as the mirage effect in deserts. It is realized that artificially made media that have spatially changing optical properties can bend light in any desired manner. Transformation Optics (TO) [1-6,17,51,52] is a mathematical method that allows us to design such artificial materials. It is achieved through the application of appropriate coordinate transformations from a physical space to a virtual space, which provide a pathway for realizing novel wave-medium interactions. The TO design methodology is based on the fact that Maxwell’s equations are coordinate invariant and can be interpreted as inhomogeneous and anisotropic compression and stretching of the constituent materials in the original space. This technique enables unprecedented design device flexibility. The most famous transformation-optics device is the invisibility cloak that can steer light around an object to make it seemly invisible. The reduction of scattering cross section of an object has been a subject of research interest for a long time. Before TO methods were introduced, early efforts to reduce scattering include anti-reflection coatings for glass [53], and absorbing-screen designs for conducting surfaces [54]. The interest of obtaining low observability was renewed in [55], where the total induced polarization vector for an electrically small dielectric object could theoretically be reduced to an arbitrarily low level if it was enclosed inside a plasmonic material shell that has a negative-permittivity value and spherical geometry. This approach to achieve low observability is referred to as the “scattering-cancelation” technique. In 2006, an invisibility-cloak design was presented to achieve near-zero observability using a very different technique [1-6,17,51,52], based on
applications of the TO methodology. This approach is then referred to as the “scattering-avoidance” technique. The “scattering-cancelation” technique can only be applied to electrically small objects while the “scattering-avoidance” technique can be applied to arbitrary size objects. Since the first experimental demonstration of an electromagnetic cloak in the microwave regime [56], research on cloaks has witnessed a tremendous amount of interest. The same coordinate-transformation technique that led to cloak designs was also applied to arrive at other novel devices, such as polarization rotators [57], field concentrators [58], beam splitters and shifters [59], beam collimators, flat lenses [59, 60] and phase transformers [61]. The coordinate-transformation approach was also extended to the cloaking of acoustic waves [62, 63] as well as matter waves [63], more specifically a cloaking system was constructed for cold atoms using optical lattices.

An “embedded” coordinate transformation is one of the TO methodologies that enables designers to control the interactions between waves and transformation devices in principle. By applying this methodology, engineers can then design devices that are more practical than invisibility cloaks. Additional device designs, other than the beam shifters and beam splitters [59], near-zone and far-zone focusing lenses [60] listed above, include beam benders [60]. The TO techniques has been applied to design lenses by creating the spatial distribution of the refractive index of a material rather than changing the interface geometry of lenses. The resulting lenses can be flat and avoid the typical aberrations of traditional lenses. Chapter 3 of this thesis introduces an “embedded” approach to design a flat lens. Conventional transformation optics (TO) [1–6,17,51,52] approaches provide the ability to control the wave propagation by specifying the spatial distribution of the real parts of the anisotropic permittivity and permeability tensors \( \varepsilon \) and \( \mu \), over a predefined region of physical (device) space, while at the same time minimizing their imaginary parts caused by material structures, such as capacitance and inductance. Such an approach has enabled several devices that control the far-field pattern of a radiating electromagnetic source such as multi-beam wave collimators [59, 60], far-zone and near-zone focusing flat optical lenses [60], and trapezoidal far-zone lenses with reduced reflection [64]. In contrast, the recently proposed complex transformation optics methodology [22] allows designers
to control not only the phase of the electromagnetic field but also its amplitude by purposely introducing non-vanishing imaginary parts in the coordinates of the transformed medium. Introducing imaginary parts is a mathematical artifact to achieve the damping. As a result, the imaginary components of the material parameters in the transformed medium can be designed to provide either loss or gain for controlling the wave amplitude that is not achievable by the conventional coordinate transformation approaches. Previously, the complex coordinate transformation technique [22] has been used to control the field amplitude inside an invisibility cloak. Here in this paper, we apply such an approach to the synthesis of flat lenses for controlling antenna far-field radiation patterns.

In this thesis, several well-known TO methodologies will be analyzed and discussed in terms of their advantages and disadvantages for electromagnetic applications in chapter 2. Then, an effective approach to design antenna lenses for controlling the far-field patterns, which only uses a single line source or antenna feed, will be introduced using complex coordinate transformation methodology. Because this method allows us to control not only the phase (direction where the wave travels) but also the amplitude of the electric field (near field), it provides us the ability to introduce a near field taper, such as a Gaussian Curve that is applied in this project. Since Far field radiation pattern is related to the near field by Fourier Transform, this methodology consequently allows us to control the far field radiation pattern. It will be shown in section 3.2 and 3.5 through numerical simulations that one can control the width of the main beam and the side lobe levels of the far-field pattern generated by the TO lens with a single source first located outside the lens then inside the lens. We apply the near field taper to a plane wave so that we would see the tapering effect easily. Therefore, to start with, we map a cylindrical radiated wave to a plane wave using conventional transformation optics. This coordinate transformation mapping math equations are different for source outside and source embedded cases because when the source is located outside, we only map a semi-circle to a line while when the source is embedded, we map the entire circle to two lines. It is then shown that how different tapering equations effect the near field, phase and radiation pattern. The complex coordinate transformation is then applied to a linear coordinate
transformation enabled lens to achieve simpler material parameters and demonstrate the versatility of the proposed design approach. Several full-wave simulated lens examples and corresponding linear antenna array calculations are presented, demonstrating the capability of complex coordinate transformations for far-field pattern manipulation and near-field amplitude tapering across the lens aperture.
Chapter 2

Review of Transformation Optics Design Methodologies and their Applications

Within the past few years, transformation optics has emerged as a new research area because it provides a general methodology and design tool for manipulating electromagnetic waves in a prescribed way. It is based on the form co-invariance of Maxwell’s equations under coordinate transformations. These methods provide an extremely useful and flexible set of design tools that employ spatial-coordinate transformations, where the compression and dilation of space in different coordinate directions are interpreted as a scaling of the material properties. In this chapter, several well-known TO methodologies will be analyzed and discussed in terms of their advantages and disadvantages for electromagnetic applications.

Section 2.1 Conformal Mapping

An invertible mapping is called a conformal mapping if it preserves angles. We say that \( f(z) = u(x,y) + jv(x,y) \) is invertible if the Jacobian matrix \( A = |\partial(u,v)/\partial(x,y)| \neq 0 \), in other words, \( u_xv_y - u_yv_x \neq 0 \). If \( f(z) \) is analytic, then \( f'(z) = u_x + jv_x \) exists. The Cauchy-Riemann condition tells us that \( u_x = v_y \) and \( u_y = -v_x \), therefore, by substituting these conditions into the Jacobian matrix, \( u_xv_y - u_yv_x = u_x^2 + v_x^2 = |f'(z)|^2 \). Hence, the mapping is invertible if \( f(z) = u(x,y) + jv(x,y) \) is analytic and \( f'(z) \neq 0 \).

Conformal mappings also preserve the solution of Laplace’s equation, which is a very useful result and one of the main reasons that many branches of physics use the theory of complex variables. In transformation optics, conformal mappings [17] are a useful
method that usually minimizes either anisotropic or inhomogeneous material property as presented in [8, 9].

As defined in [8], \( |A| = (u_x)^2 + (v_x)^2 = (u_x)^2 + (u_y)^2 \). This condition leads to simpler material parameters, permittivity \( \epsilon \) and permeability \( \mu \), which are given below:

\[
\epsilon = \mu = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1/|A|
\end{pmatrix}
\]  

(2-1)

This simplification is very important for realizing the transformation optics designs in practice.

**Section 2.1.1 Applications:**

Pendry [10] proposed a method to create a “cylindrical magnifying lens which produces the contents of the smaller cylinder in magnified yet undistorted form outside the larger cylinder by using a conformal transformation” [10]. Because conformal mappings preserve the solution of Laplace’s equation up to multiplicity, the values of \( \epsilon \) and \( \mu \) change in a way that corresponds to the equations and hence preserved the physics in their virtual domains. Conformal mappings also preserve the requirements of continuity of \( E_\parallel \) and \( D_\perp \) at the boundary with the same dielectric function. By utilizing these two properties, the cylindrical annulus lens, the crescent lens, and the kissing lens were designed by Pendry and Ramakrishna in [12]. Kwon and Werner proposed in [13] that using a conformal mapping can minimize reflection from the boundary in their two-dimensional flat focusing lens design. Because the electrostatic transformation does not alter the properties of the constituent material, this technique has been used in designing near-field prefect lenses and reflectors [12, 14]. In addition, it was proven in [15] that electrostatic eigenmodes and eigenvalues of a two-dimensional nanosystem with proper boundary conditions are invariant under conformal mappings, which suggests that transforming geometry is equivalent to transforming the expansion coefficients of the invariant eigen-modes. As suggested in [15], this method can be used to interpret divergent electric fields and broadband response in the region of the singularity.
Section 2.1.2 **Advantages:**

Since the variables in a conformal mapping, $u(x, y)$ and $v(x, y)$, are real-valued and positive, the implementation and fabrication of the designs are practical with currently available metamaterials. Conformal mappings not only can map a regular shaped virtual domain to some distorted physical domain, but also map any generalized quadrilateral to a physical domain by a rectangle of the same conformal module as demonstrated in [9]. With the reciprocal conformal transformation introduced in [8], the material parameters are now simplified compared to the original approach suggested in [10]. This suggests that metamaterial lenses may be implemented as a uniaxial homogeneous material with a reduced form of the material parameter tensor with the approximation introduced in [8] for rectangular lenses centered at the origin with a given width and height

$$
\epsilon = \mu = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix}
$$

These material parameter tensors are simple enough to be realized by using conventional metamaterial design techniques. These material parameters can also be utilized to produce a zero-index material lens with infinite effective phase velocity. That leads to equal phase fronts at the boundaries of the lens. Conformal mappings in the design of transformation media can help to reduce the complexity of the resulting material
parameters in the final optical and electromagnetic devices. This methodology allows for more practical implementations of transformation optics designs. Moreover, it has been shown that conformal mappings perform well beyond geometrical optics [35].

Section 2.1.3 Disadvantages:

As Pendry stated in [10], conformal mappings have their limit. Since Laplace’s equation is valid only in the electrostatic or magnetostatic, lenses defined by conformal mappings are valid if and only if all dimensions are significantly less than the wavelength of light. Hence, the drawback of Pendry’s invisibility cloak is that the invisibility expectation cannot be realized for broad band operation. This problem is later solved by the arbitrary coordinate transformation introduced in [11].

Section 2.2. Quasi-conformal Mapping

Quasi-conformal mapping can be considered as a generalization of conformal mapping can be appropriately defined in higher dimension and even in metric space. A quasi-conformal mapping is a homeomorphism:

$$\psi(u,v) = \xi(u,v) + j\eta(u,v)$$  \hspace{1cm} (2-3)

This transformation maps the \((u,v)\) space onto the \((\xi,\eta)\) space so that the real and imaginary parts of \(\psi\) satisfy Beltrami’s system of equations:

$$M\eta_v = p\xi_u + q\xi_v$$  \hspace{1cm} (2-4)

$$-M\eta_u = q\xi_u + r\xi_v$$  \hspace{1cm} (2-5)

where \(p, q,\) and \(r\) are functions of \(u\) and \(v\) with \(p > 0, r > 0\) which satisfy the equation \(pr - q^2 = 1\). The quasi-conformal quantity \(M\) is invariant and often referred to as the module or the aspect ratio of the virtual region. Let \(\xi\) and \(\eta\) be twice continuously differentiable, then the Jacobian \(J = u_\xi v_\eta - u_\eta v_\xi\) is non-zero in the virtual region. Therefore, the metrics \((u_\xi,u_\eta,v_\xi,v_\eta)\) and \((\xi_u,\xi_v,\eta_u,\eta_v)\) are uniquely related by

$$\xi_u = \frac{v_\eta}{J} \hspace{1cm} \xi_v = \frac{-u_\eta}{J} \hspace{1cm}$$

$$\eta_u = \frac{-v_\eta}{J} \hspace{1cm} \eta_v = \frac{u_\xi}{J}$$  \hspace{1cm} (2-6)
The parameter variables become the independent variables and the system can be expressed in the form of

\[
\xi_u = F(cn_v + b\eta_u)
\]
\[
-\xi_v = F(an_u + b\eta_v)
\]

where if we define \( J = \sqrt{g} \) and \( g = \bar{g}_{11} \bar{g}_{22} - \bar{g}_{12}^2 \), then

\[
a = \frac{\bar{g}_{22}}{\pm J} \quad b = \frac{\bar{g}_{12}}{\pm J} \quad c = \frac{\bar{g}_{11}}{\pm J}
\]

(2-7)

(2-8)

Note that \( ac - b^2 = 1 \), which is sufficient for ellipticity. The sign needs to be determined so that the Jacobian \( J > 0 \). These first order elliptic systems that represent conformal mapping of a parametric surface onto a square are in the form of Beltrami’s system for a quasi-conformal mapping of planar regions.

Quasi-conformal mapping can minimize the anisotropy factor [6] and its inverse, hence it leads to an isotropic dielectric material cloak that has less degradation in performance and brings the anisotropy factor close to unity. Note that a quasi-conformal mapping approaches a conformal mapping in the limit when \( M = m \), where \( M \) is the conformal module of the physical domain and \( m \) is the conformal module of the virtual domain, \( m = w/h \). In quasi-conformal transformations, Neumann boundary conditions lead to discontinuities at the boundaries and hence can cause undesirable reflections and refraction.

**Section 2.2.1 Applications:**

A carpet cloak that is designed to mimic a flat ground plane using quasi-conformal mapping was introduced by Li and Pendry in [6], and they also showed that by this technique, certain TO media can be realized using dielectric-only materials. A similar carpet cloak design using quasi-conformal mapping was realized in [16] by fabricating in a silicon nitride waveguide on a special nano-porous silicon oxide substrate with a low refractive index. A broadband lens that has a field-of-view (FOV) approaching 180° and zero f-number was designed by this mapping technique and demonstrated in [20]. A combination of conformal and quasi-conformal mapping is used to design electromagnetic devices that modify the omnidirectional radiation pattern of a point source [21].
applying quasi-conformal mapping, a 2D flattened Luneburg lens was suggested in [19] and realized in [20].

![Figure 2-2](image)

Figure 2-2. Reprint permitted by [6]. The transformed grid in physical system with inner cloak boundary for (a) transfinite grid and (b) the quasi-conformal grid.

**Section 2.2.2 Advantages:**

As mentioned in [6], the quasi-conformal mapping is the optimal technique to minimize the anisotropy of the cloak and also a useful technique for both $E$ and $H$ polarizations. In addition, dielectric-only media derived by quasi-conformal mapping can control wave propagating in both two-dimensions and three-dimensions. In three dimensional media, the quasi-conformal transformation is applied in a plane, while the transformation leaves the normal direction to this plane invariant. This technique can be applied to greatly improve conventional optical devices, such as Luneburg lenses. Landy, Kundtz and Smith have shown in [18] that the quasi-conformal mapping can be extended to the design of rotationally-symmetric TO devices that can be realized using simplified and uniaxial media. In addition, this technique allows the conformal module of the two domains to differ to a limited extent, whereas for a conformal mapping the conformal module ratio of the two domains is one. This leads to a material property that has very limited anisotropy that can be well approximated by a dielectric-only response [20]. Since
quasi-conformal mappings are more flexible than conventional conformal mappings, it is shown in [21] that complex radiation patterns can be achieved where the power radiated in each direction can be controlled to a certain extend.

**Section 2.2.3 Disadvantages:**

Although the quasi-conformal mapping technique is very powerful, it might not be practical for many photonic applications if the virtual domain is not chosen carefully. 3D quasi-conformal mapping media [18] are inherently anisotropic. Furthermore, we do not have full control of the density of rays crossing the transformed boundary, but this limitation can be overcome by increasing the degree of control over the angular power distribution as suggested in [21].

**Section 2.3. Complex Coordinate Transformation**

A complex number is an expression of \( z = x + jy \), where \( x \) and \( y \) are real numbers. We call \( x \) the real part of \( z \) and \( y \) the imaginary part. The set of complex numbers forms the complex plane, which we denote by \( \mathbb{C} \). The correspondence of \( z = x + jy \) and \( (x, y) \) is a one-to-one correspondence between complex numbers and point vectors in the Euclidean plane. We also use polar representations to denote complex numbers \( z = x + jy = r(cos \theta + jsin \theta) = re^{j\theta} \) where \( \theta = \text{arg} z \). The polar representation of \( z \) obeys the same algebraic rules that exponential numbers do.

Just as the real coordinate transformations allow the complicated lossless material parameters for a particular wave device to be determined, the complex coordinate transformation also allows devices to be realized that can manipulate wave amplitude [22] arbitrarily. Let \( \exp(j\omega t) \) represent the time variation, then the wave field component can be expressed as

\[
E(r) = E_0(r)e^{-jk(r)r}
\]

(2-9)
in a neighborhood of \( r = (x, y, z) \), where \( E_0 \) and \( k \) are the electric amplitude vector and the wave vector, respectively. The magnetic field component can be expressed as \( H(r) = \)
\( H_0(r) e^{-jk(r)\cdot r} \) in the same manner, where \( H_0 \) is the magnetic amplitude vector. In order to manipulate the amplitude of the wave as it propagates through the medium, we can define
\[
E_{\text{desired}}(r) = M(r)E(r)
\] (2-10)
where \( M(r) \) is a scalar function that corrects the amplitude to a desired value. Consider the complex coordinate transformation, where we map \( r=(x,y,z) \) to \( \rho(r) \), that maps \( E_{\text{desired}} \) to \( E(\rho) \). This complex coordinate mapping is in the form of \( \rho(r) = r + j\rho_{\text{im}}(r) \). Under this mapping, \( E_{\text{desired}} = [A^{-1}(\rho)]^T E(\rho) \), and it follows from transformation optics that the material tensor parameters are:
\[
\begin{align*}
\varepsilon' &= \frac{A \varepsilon A^T}{\det|A|} \\
\mu' &= \frac{A \mu A^T}{\det|A|}
\end{align*}
\] (2-11)

Then we can solve for \( \rho_{\text{im}}=0 \) at the boundary to ensure our manipulation of the field amplitude gives us the desired result by expanding (9), we obtain
\[
M(r)E_0(r) = [(I_3 + j\nabla \rho_{\text{im}})^{-1}]^T e^{k(r)\cdot \rho_{\text{im}}(r)}E_0(r)
\] (2-12)
where \( I_3 \) is the unity matrix. Similarly, we can follow the same process to obtain the manipulation of the magnetic field.

**Section 2.3.1 Applications:**

A range of transformation optics applications can be extended through the use of complex mapping. Complex coordinate transformations incorporated into a classical cylindrical invisibility cloak design has been shown to greatly reduce the electromagnetic field toward the inner boundary of the cloak to approximately zero in [22]. In addition, it is also shown in [22] that complex coordinates can be used to obtain arbitrarily shaped perfectly matched layers (PMLs) and reduce the scattering off the numerical domain edges in the interference pattern.
Figure 2-3. By mapping the virtual space (a) to a real space, one can create a classical cylindrical cloak (b). A modified cylindrical cloak based on complex coordinates that can greatly reduce horizontal wave propagation within the boundary is shown in (c), while the vertical wave propagation is not affected (d). Reprint with permission [22]

As illustrated in [22], both positive and negative values of material parameters are required, and we can hence control the loss and gain of the device. In particular, the material parameters are designed such that the wave is attenuated as it propagates along the radial direction and amplified as it propagates along the azimuthal dimension around the cloak. Moreover, recent literature has proposed that transformation optics may be able to deal with parity-time (PT) symmetric metamaterials [35] via complex analytic continuation of the spatial coordinates. It has been shown that complex-coordinate TO can be used for systematic generation, design and modeling of PT metamaterials for a broad range of applications.

Section 2.3.2 Advantages:

The classical invisibility cloaks have a non-negligible scattering caused by small deviations from the material parameters seen in Fig 2(b) and the PMC on the top and bottom boundaries makes the scattering more visible. Complex coordinates transformation techniques addresses this issue [22] based on the complex coordinate transformation $r \mapsto \xi(r) \mapsto \rho[\xi(r)]$ that greatly attenuates the fields toward the inner boundary (see Fig 2(c)).
Furthermore, by applying this technique, we can control the loss and gain of the device. This technique offers an extra level of freedom that enables us to control both wave propagation and amplitudes.

**Section 2.3.3 Disadvantage:**

This technique cannot reduce the incident fields in all directions because the complex transformation will have no effect on a wave whose vector is perpendicular to $\rho_{im}$. However, this approach is very broadband and works for all propagation directions of the incident wave except one.

**Section 2.4. Finite-Embedded Source Transformation**

The embedded coordinate transformation is so named because it is a technique that is similar to the “embedded Green’s function” method [68]. An embedded coordinate transformation is carried out locally for the transformation-optical medium and then embedded into free space. The beam shifter and the beam divider presented in [23] were designed by using this transformation, which provides a new way to develop optical devices that are not achievable from continuous transformations. Discontinuous transformations usually have some reflections at the boundary, and this issue is further discussed and addressed in [23, 24, 25]. It can be resolved by matching the boundary conditions for both the incident and transmitted field, or adopting 4-D (including a time variable) to discuss a generic boundary condition between the two transformation media. In general, it will lead to a time-dependent condition at the interface where the discontinuity occurs.
Figure 2-4. The left figure is a parallel beam shifter created by a linear spatial coordinate transformation. The coordinate transformation is carried out locally for the transformation-optical medium and then embedded into free space shown by the green arrow in the figure on the left. The figure on the right is a beam divider designed using by a second order nonlinear spatial coordinate transformation [23].

Section 2.4.1 Applications:

An embedded transformation is applied to a square cylindrical volume to rotate the direction of beam propagation by 90° as introduced in [26]. This transformation technique has also been adapted to other optical device designs such as beam compressors and expanders [27], cylindrical-to-planar wave converters [28] and wave collimators [29]. Unlike the beam splitter in [23], where an incident beam is split into two beams with the same polarization, Kwon and Werner presented a beam splitter based on the polarization state of the incident field in [30]. However, as suggested in [31], the coordinate transformations employed in [26-30] are not unified, therefore they cannot be easily adjusted for versatile functionalities. As an improvement, a universal transformation formula based on the embedded transformation technique was introduced which greatly increases the design flexibility. Later on, the concept of a beam controller in [31] was improved and the design of beam controller based on a nonlinear 2D embedded transformation was developed in [32].
Section 2.4.2 Advantages:

With finite-embedded transformations, it is possible to transfer field manipulation optical structures to a second medium. Therefore, this technique broadens the variety of materials that can be used to design devices and structures that are able to manipulate electromagnetic waves and operate without reflection [23]. The presented parallel beam shifter [23] could play an important role in connection with tunable metamaterials by allowing the scanning of a beam focus along a flat surface without introducing a beam aberration or changing the plane of focus. This property is very useful in the application where the distance between scanner and object is short. In conclusion, this transformation technique allows one to non-reversibly change the properties of electromagnetic waves in transformation media and maintain the changed electromagnetic properties in free space or, more generally, in a different medium. This is a new level of transformation optics that extends the idea of a continuous transformation.

Section 2.4.3 Disadvantage:

As discussed in [23], in order for this transformation technique to work for a design as a reflectionless device, the metric in and normal to the interface between the transformation optics medium and the non-transformation medium must be continuous to the surrounding space. This is a limitation, but if addressed and designed properly, embedded transformation devices could work well and exhibit reflectionless properties.
Section 2.5. Inverse Transformation

An electromagnetics problem in a virtual space is usually easier to solve, hence an inverse transformation technique first introduced in [33] is utilized to solve the problem in the virtual space first and then transform it back to the physical space. Consider a virtual space with an electromagnetic field \((E, H)\), and an original medium characterized by \((\epsilon, \mu)\). A relation between the virtual space and the physical space can be described by

\[
(x', y', z') = [X(x, y, z), Y(x, y, z), Z(x, y, z)]
\]  

(2-13)

By the coordinate invariant property of Maxwell’s equations, we can write the material properties in the physical space as

\[
\epsilon' = \frac{J\epsilon J^T}{\text{det}(J)}, \quad \mu' = \frac{J\mu J^T}{\text{det}(J)}
\]  

(2-14)

\[
E' = (J^T)^{-1} E, \quad H' = (J^T)^{-1} H
\]  

(2-15)

where \(J = \frac{\partial (x', y', z')}{{\partial (x, y, z)}}\) is the Jacobian matrix for the coordinate transformation.

Figure 2-6. Two types of transformations between a virtual space and the physical space near a boundary of a device designed using the TO method presented in [34].

Since the coordinate transformation in explicit form is not necessary, one can derive the tensors parameters using scalar field potentials by assuming that the field inside the cloak is an incident plane wave [34].
Section 2.5.1 Applications:
This transformation optics technique has been used to analyze the reflection at a boundary of a finite-embedded transformation optics medium in [33].

Section 2.5.2 Advantages:
The idea of this technique is to provide an alternative understanding of the reflectionless phenomenon at the boundary of designs obtained using finite-embedded transformation optics. For a given finite-embedded transformation optics medium, one can have many pairs of the original isotropic medium by using this technique.

Section 2.5.3 Disadvantages:
This inverse transformation can easily transform an isotropic medium in the virtual space back to an anisotropic physical space, but it is difficult to transform an anisotropic medium back to an isotropic medium because of the existence of some birefringence phenomenon in anisotropic media.

Section 2.6. EM Source Transformation
The transformation optics approach can not only map a normal free space to a new space with a different volume and space-distributed constitutive parameters, but it also can be applied to a space containing EM sources. It is shown in [36] that not only the permittivity and permeability tensors but also the EM sources can be transformed. The following coordinate transformation is obtained from [36].

Consider that a general coordinate transformation between the initial coordinate system and a transformed one, \((x', y', z') \mapsto (x, y, z)\) by \(\vec{r}' = \vec{F}(\vec{r})\), where \(\vec{r}' = x' \hat{x} + y' \hat{y} + z' \hat{z}\) and \(\vec{r} = x \hat{x} + y \hat{y} + z \hat{z}\), and \(\vec{F}\) is a deformation mapping from the initial space to the new closed transformed space. Equivalently,

\[
\begin{align*}
    x' &= f(x, y, z) \\
    y' &= g(x, y, z) \\
    z' &= h(x, y, z)
\end{align*}
\]  

(2-16)
and let $\mathbf{J} = \frac{\partial (f, g, h)}{\partial (r, \theta, \phi)}$ be the Jacobian matrix. Under these conditions the desired EM constitutive parameters obtained from Maxwell’s equations are

$$\bar{\varepsilon} = \det(\mathbf{J}) \bar{\varepsilon}' \cdot (\mathbf{J}^T)^{-1} \quad (2-17)$$

$$\bar{\mu} = \det(\mathbf{J}) \bar{\mu}' \cdot (\mathbf{J}^T)^{-1} \quad (2-18)$$

$$\rho(\mathbf{r}) = \det(\mathbf{J}) \rho'(\mathbf{F}(\mathbf{r})) \quad (2-19)$$

$$\gamma(\mathbf{r}) = \det(\mathbf{J}) \gamma' \cdot (\mathbf{F}(\mathbf{r})) \quad (2-20)$$

where $\rho(\mathbf{r})$ and $\gamma(\mathbf{r})$ are functions of the sources. The surface current, current and charge in the transformed coordinates can be represented as

$$\bar{J}_s(\mathbf{r}) = \bar{J} \cdot \bar{J}'(\mathbf{F}(\mathbf{r})) \quad (2-21)$$

$$\bar{I}(\mathbf{r}) = \bar{I}(\mathbf{F}(\mathbf{r})), q(\mathbf{r}) = q'(\mathbf{F}(\mathbf{r})). \quad (2-22)$$

Let $\rho(\mathbf{r}_0)$ and $\gamma(\mathbf{r}_0)$ be the sources in domain $V$. It is shown in [36] that by applying the coordinate transformation $\mathbf{F}$, the point $\mathbf{r}' = \mathbf{r}(\mathbf{r}_0)$ in the initial coordinate will be transformed to $\mathbf{r}' = \mathbf{r}_0$ in the new coordinates according to the Green’s function. Hence, by applying the proper transformation $\mathbf{F}$, a given source can be transformed into any shape and position. This is a useful principle that can be applied to active EM designs, including antenna design in particular.

**Section 2.6.1 Applications:**

This EM source transformation can be used to obtain a conformal antenna with the same radiation pattern as a dipole but with a spherical shape [36]. Kundtz et al. implemented numerical simulations to confirm the restoration of dipole radiation patterns from both a bent dipole, which is a partially closed cylindrical scatterer, and a distorted “pinwheel” antenna [37]. It also confirms through numerical simulations that a complex current distribution can be made to radiate as simple ones when surrounded by a properly designed TO medium. Popa et al. then demonstrated that the source transformation approach can be applied to manipulate arrays of sources so that it can be tested experimentally [38]. Furthermore, they have shown with numerical simulation that conformal antenna arrays can be designed using a TO technique to create a single array that behaves the same as a uniform linear array.
Section 2.6.2 Advantages and disadvantages:

This method demonstrates that the TO technique is not restricted to source free cases. Under the transformation, not only the space but also the sources can be transformed. This allows the TO methodology to be applied to the design of conformal antenna arrays. Although this method might cause the resulting parameters of the required TO materials to be complex, it is still within the range of possibility to fabricate such designs using existing RF metamaterial technology. In addition, this approach in principle can be applied to arbitrary complex conformal arrays.

Section 2.7. Non-Euclidean Transformation Mapping

All the previous proposals for invisibility require special materials that have strict properties. Leonhardt and Tyc further developed the idea of conformal mapping and showed in [39] that the problem of infinite speed of light required to cloak an object in a broadband frequencies is not achievable and hence it prevents a device from working in a broadband spectrum. This problem can be avoided using a different approach, which is to apply the concept of non-Euclidean geometry. In addition, non-Euclidean transformation
optics relaxes the material requirements and also can lead to a broadband spectrum of invisibility. The mathematical equations below are obtained from [40].

Starting by assuming the medium is impedance matched to vacuum, and an anisotropic linear medium with an electric permittivity tensor $\varepsilon^{ij}$ that is equivalent to its magnetic permeability $\mu^{ij}$. Then the inverse matrix of the metric tensor, $g^{ij}$, is given by

$$g^{ij} = \frac{\varepsilon^{ij}}{\det \varepsilon}$$  \hspace{1cm} (2-23)

where the $\det \varepsilon$ denotes the determinant of $\varepsilon^{ij}$. By the inversion relation of the above equation, we find that

$$\varepsilon^{ij} = \sqrt{g} g^{ij}$$  \hspace{1cm} (2-24)

where $g$ is the determinant of $g^{ij}$. So far, all the variables are given in Cartesian coordinates. In the non-Euclidean cloaking device, variations of bipolar coordinates are adapted to the geometrical shape and profile of the material. In flat Cartesian space, the natural solutions of Maxwell’s equations are plane waves. To describe the transformed plane waves, a coordinate independent form of Cartesian plane waves can be represented as

$$E_i = \varepsilon_i e^{i \phi}, \phi = \int k_i \, dx^i - \omega t$$  \hspace{1cm} (2-25)

with the dispersion relation

$$g^{ij} k_i k_j = \frac{\omega^2}{c^2}$$  \hspace{1cm} (2-26)

the transversality

$$g^{ij} k_i \varepsilon_j = 0$$  \hspace{1cm} (2-27)

and the continuity of the amplitude

$$\nabla^i \varepsilon_i = 0.$$  \hspace{1cm} (2-28)

The four equations above give a good approximation for the regime of geometric optics. Two convenient mathematical ingredients for the non-Euclidean cloak are stereographic projection and bipolar cylindrical coordinates $(\tau, \sigma, z)$ [40]. We can relate the bipolar cylindrical coordinates to Cartesian coordinates by

$$x = \frac{a \sinh \tau}{\cosh \tau - \cos \sigma}, \quad y = \frac{a \sin \sigma}{\cosh \tau - \cos \sigma}, \quad z = z$$  \hspace{1cm} (2-29)

where $a$ is the size of the cloak.
Our goal is to map the inner branch of the device onto the surface \( \{X',Y',Z'\} \) of a sphere with radius \( r_0 \) as shown in Fig. 6 A. Let \( \sigma' \) be the angle of longitude such that we map the \( \{X',Y',Z'\} \) onto the \( \{X,Y,Z\} \) sphere. The mapping is best described in terms of a stereographic projection. The spherical coordinates of the \( \{X',Y',Z'\} \) sphere can be represented in terms of a complex number \( z' \) as

\[
X' + jY' = \frac{2r_0 z'}{1 + |z'|^2}, \quad Z' = r_0 \frac{|z'|^2 - 1}{|z'|^2 + 1} \quad \text{(2-30)}
\]

where \( \sigma' = e^{i\sigma'} \cot(\frac{\theta}{2}) \), in spherical coordinates. Similarly, we can express

\[
X + jY = \frac{2r_0 z}{1 + |z|^2}, \quad Z = r_0 \frac{|z|^2 - 1}{|z|^2 + 1} \quad \text{(2-31)}
\]

The Mobius transformation \( z' = \frac{z}{1+z}, \quad z = \frac{z'}{1-z'} \) represents the desired transformation because \( z'(z) \) is analytic and hence conformal and \( z'(z) \) maps the equatorial region \( z=-1 \) onto the north pole \( z' = \infty \), and the south pole \( z=0 \) onto \( z'=0 \). Then the last step is to match \( \theta \) and \( \tau \) at the branch cut \( \sigma' = \pm \pi \), where both \( \theta \) and \( \tau \) are parameters in bipolar coordinates and \( \tau \) runs from \( (-\infty, \infty) \). Details of matching these two parameters as well as the derivation for the 3D case can be found in [40], where the most important steps are listed in [41]. It is explained in [41] that the non-Euclidean region of the device is an image of a sphere in the virtual space with uniform refractive index. If the great circle is chosen to be the equator, then the solution to the wave equation will be the spherical harmonics \( Y_{lm}(\theta,\varphi) \), which is a solution to the Helmholtz equation on the sphere, where \( l \) is a non-negative integer and \( m \) is also an integer satisfying \(-l < m < l\). Then the corresponding wavevectors can be represented as an \( l \)-dependent form

\[
k_l = \frac{\sqrt{l(l+1)}}{r} \quad \text{(2-32)}
\]

where \( r \) is the radius of the sphere. Then the corresponding frequencies and wavelengths can be represented as

\[
\omega_l = \frac{c}{r} \sqrt{l(l+1)}, \quad \lambda_l = \frac{2\pi r}{\sqrt{l(l+1)}} \quad \text{(2-33)}
\]

Waves that satisfy this frequency and wavelength condition correspond to those that work perfect for the cloaking devise. It has been proven by numerical simulation that the larger the \( l \), the better the cloaking device will work.
Figure 2-8. Images from [39] reprinted with permission. Non-Euclidean cloak device in 2D. (A) is the light propagation through a virtual space that consists of a plane attached with a surface of a sphere along the line. The light never crosses the red line on the sphere. (B) The magenta line is the boundary of the device, and the black lines consist of an expended interior area for the grid of the sphere. (C) Placing a mirror around the equator of the virtual sphere can create the same illusion as in (A) with an invisible northern hemisphere. (D) Similarly, expanding the red line where light never crosses through to create a hidden space.

Section 2.7.1 Applications:

Non-Euclidean TO techniques can be used to design a cloak as demonstrated in [39 - 41]. They can also be used to confine light waves on a subwavelength scale [42] and to design a thin metamaterial lens that can achieve wide-beam radiation by embedding a simple source [43].

Section 2.7.2 Advantages and Disadvantages:

Adding the mirror and creating an invisible region causes the cloak to work almost perfectly for the resonant wavelengths. When the wavelength decreases, the performance of the cloaking device improves gradually and becomes perfect in the limit of geometrical
Section 2.8. Nonlocal Transformation Optics

Nonlocal character arises from a spatial dispersion of the EM constitutive relationships [44] and it is usually a negligible effect for most natural media. But it is now of growing interest because it is critical for many homogenized metamaterial models [45] of practical interest. For most metamaterials, their spatial dispersion can provide additional degrees of freedom for controlling the propagation of electromagnetic waves [46]. Nonlocal transformation optics proposed in [47] is based on coordinate transformations in the wave number domain because the nonlocal constitutive relationships are most easily dealt with. The following mathematical procedures regarding nonlocal transformations optics were obtained from [47,48].

For simplicity, assume that the distribution of electric and magnetic sources \((J', M')\) are radiating an EM field \((E', H')\) in a vacuum. Then in the wave number domain \((k')\), the relevant Maxwell’s equation can be expressed as

\[
j k' \times \vec{E}'(k') = j \omega \mu_0 \vec{H}'(k') - \vec{M}'(k') \tag{2-34}
\]

\[
j k' \times \vec{H}'(k') = -j \omega \varepsilon_0 \vec{E}'(k') + \vec{J}'(k') \tag{2-35}
\]

The boldface symbols identify vector quantities and the tilde identifies the wave number domain quantities. Under a real-valued coordinate transformation to a new wave number domain \(k\),

\[
k' = \Lambda^T(k) \cdot k = \vec{F}(k) \tag{2-36}
\]

Then the corresponding field \((E, H)\), sources \((J, M)\) and constitutive relationships to those in a vacuum are

\[
\{E, \vec{H}\}(k) = \Lambda^{-T}(k) \cdot \{E', \vec{H}'\} \vec{F}(k) \tag{2-37}
\]

\[
\{J, \vec{M}\}(k) = det^{-1}[\Lambda(k)] \Lambda(k) \cdot \{J', \vec{M}'\} \vec{F}(k) \tag{2-38}
\]

\[
\{\vec{e}, \vec{m}\}(k) = det^{-1}[\Lambda(k)] \Lambda(k) \cdot \vec{A}(k) \tag{2-39}
\]

Note that the source transformations above may be used to design a desired response systematically in a fictitious curved-coordinate spectral (wave number) domain. This
response might be equivalent to the one gained from an actual physical space that is filled with the nonlocal material properties described by the equation above.

**Section 2.8.1 Applications:**

This nonlocal TO technique can be applied to a wave-splitting refraction scenario [47].

**Section 2.8.2 Advantages:**

This TO technique provides engineers and physicists with a useful tool to interpret equi-frequency contour (EFC) deformations. This interpretation is equally insightful as the geometrical characteristics of the EFCs that determine the kinematical properties of the wave propagation and refraction [49]. This approach [47] may open intriguing venues in dispersion engineering and/or metamaterials. It can also be used to address nonreciprocal effects, which are of great interest, by using non-center-symmetric coordinate transformations.

**Section 2.8.3 Disadvantages:**

As mentioned in [47], the practical applicability of this technique crucially relies on the synthesized anisotropic nonlocal homogenized metamaterial that possess the material parameters which meet the required relation given above. This can be technologically challenging compared to metamaterial designs required by the standard spatial-domain TO.
Chapter 3

Transformation-Optics Antenna Lens Designs Using Complex Coordinate Transformations

The transformation-optics design approach relies on the invariance of Maxwell’s equations under coordinate transformations [17,51,52] and an interpretation of the regular electric permittivity and the magnetic permeability in the transformed coordinate system as a set of material parameters in the original coordinate system [17, 51, 52]. The coordinate system used in this design is Cartesian, while the technique can be generalized to other orthogonal coordinate systems accordingly.

Consider Maxwell’s curl equations with the time-harmonic electric field, $E$, and the magnetic field, $H$, that exist in a Cartesian coordinate system. Away from the source point, the time-harmonic field satisfies the following Maxwell’s equations:

\[ \nabla \times E = -j \omega \mu H \quad (3-1) \]
\[ \nabla \times H = j \omega \epsilon E \quad (3-2) \]

The two divergence equations can be derived from the two equations above and the electric and magnetic flux densities, $B$ and $D$, are related to the field quantities via the electric permittivity tensor, $\epsilon$, and the magnetic-permeability tensor, $\mu$.

\[ B = \mu H \quad (3-3) \]
\[ D = \epsilon E \quad (3-4) \]

Consider a transformation from the $(x, y, z)$ system to the $(x', y', z')$ system described by

\[ x' = x'(x, y, z) \]
\[ y' = y'(x, y, z) \]
\[ z' = z'(x, y, z) \quad (3-5) \]
They take the same form in the transformed coordinate system as in the original coordinate system, because Maxwell’s equations are form-invariant under spatial transformation.

\[ \nabla' \times \mathbf{E}' = -j\omega\mu'\mathbf{H}' \quad (3-6) \]
\[ \nabla' \times \mathbf{H}' = j\omega\epsilon'\mathbf{E}' \quad (3-7) \]

Moreover, the material property tensors, \( \mu' \) and \( \epsilon' \) are related to the \( \mu \) and \( \epsilon \) in the original space via

\[ \mu' = \frac{A\mu A^T}{\det(A)} \quad (3-8) \]
\[ \epsilon' = \frac{A\epsilon A^T}{\det(A)}, \quad (3-9) \]

where \( A \) is the 3x3 Jacobian matrix of the transformation from \( r \) to \( r' \).

\[ r = \hat{x}x + \hat{y}y + \hat{z}z \quad (3-10) \]
\[ r' = \hat{x}'x' + \hat{y}'y' + \hat{z}'z' \quad (3-11) \]

### Section 3.1. Applying Complex Coordinate Transformation to a Spatial Coordinate Transformation

![Diagram](image)

Figure 3-1. Spatial coordinate transformation used for a 2D far-zone focusing lens design. (a) The original coordinate system and (b) the transformed system. Reprinted with permission from [26].
Consider a line source, located outside the optical device as shown in Fig. 9(a), radiating at the coordinate origin in the original coordinate system. If the arc portion of the circle, where the amplitude and phase are all equal for the cylindrical field, is mapped to a plane surface as in Fig. 9(b), then one such mapping [59] can be written as

\[
\begin{align*}
    x' &= x, \\
    y' &= \frac{l}{\sqrt{a^2 - x'^2 - g}} (y - g) + g, \\
    z' &= z.
\end{align*}
\]  

(3-12)

The material parameters of the device are then obtained by (3-9)

\[
\begin{align*}
    \epsilon'_{xx} &= \frac{\sqrt{a^2 - x'^2 - g}}{l}, \\
    \epsilon'_{yy} &= \epsilon'_{zz} = \frac{x'(y' - g)}{l\sqrt{a^2 - x'^2}}, \\
    \epsilon'_{xy} &= \frac{1}{\sqrt{a^2 - x'^2 - g}} \left[ \frac{x^2(y' - g)^2}{l(a^2 - x'^2)} + l \right], \\
    \epsilon'_{xz} &= \frac{\sqrt{a^2 - x'^2 - g}}{l}.
\end{align*}
\]  

(3-13)

As shown in Fig. 11(a), an antenna collimator lens was designed using the real coordinate transformation method [59], which converts a diverging cylindrical wave into a directive planar beam for obtaining directive radiation. In order to manipulate the amplitude of the wave as it propagates through the medium, we can change the material parameters by multiplying the source electric field by a smooth scalar function that represents the desired amplitude correction factor [22]

\[
M(x, y, z) = e^{-\alpha f(x,y,z)},
\]  

(3-14)

where \( \alpha \) is the amplitude control constant, such that

\[
E_{desired}(x,y,z) = M(x,y,z)E(x,y,z)
\]  

(3-15)
Hence, the field amplitude can be controlled through a coordinate transformation \( r(x, y, z) \rightarrow \rho(x, y, z) \) that maps \( E_{\text{desired}}(r) \) to \( E(\rho) \). Based on the mapping of the flat focusing lens design [11], the amplitude correction factor was set to \( M(x') = e^{-ax'} \), resulting in \( \rho_{\text{in}}(x) = -xak_0^{-1}x' \), where \( x' \) is the same as defined in the mapping equation above. The resulting mapping equations are

\[
\begin{align*}
x'' &= x - jak_0^{-1}x \\
y'' &= \frac{l}{\sqrt{a^2 - x^2 - g}}(y - g) + g \\
z'' &= z
\end{align*}
\]

and the resulting material parameters are

\[
\begin{align*}
\varepsilon''_{xx} &= \frac{(\sqrt{a^2 - x^2 - g})(1 - jak_0^{-1})}{l} \\
\varepsilon''_{xy} &= \frac{x(y - g)}{(\sqrt{a^2 - x^2 - g})\sqrt{a^2 - x^2}} \\
\varepsilon''_{yy} &= \frac{l}{\sqrt{a^2 - x^2 - g}}\left[\frac{x^2(y - g)^2}{(\sqrt{a^2 - x^2 - g})^2(a^2 - x^2)} + 1\right](1 - jak_0^{-1})^{-1} \\
\varepsilon''_{zz} &= \frac{\sqrt{a^2 - x^2 - g}}{l}(1 - jak_0^{-1})^{-1}
\end{align*}
\]

(3-16)

(3-17)

It should be noted that all the other material parameters not included in Eq. (3-17) have zero values. It should also be noted that when \( \alpha=0 \), we obtain the same equations for both the mapping and material parameters as the original transformation without the amplitude correcting factor. Moreover, when \( \alpha < 0 \), loss is introduced, whereas when \( \alpha > 0 \), gain is added. In such a way, a tapered aperture field distribution can be formed on the top surface of the lens. By changing the value of \( \alpha \), different degrees of tapering can be achieved, which in turn controls the far-field properties of the system.

Similarly, the same design principle can be applied to a source embedded collimating lens. The mapping is shown in Fig.10.
The mathematical transformation equations are given in [26] as

\[ \begin{align*}
    x' &= \frac{wx}{a}, \\
    y' &= \frac{ly}{\sqrt{a^2 - x^2}}, \\
    z' &= z,
\end{align*} \tag{3-18} \]

for the region in Fig. 10(a) inside the boundary of a circle with radius \(a\). The resulting material parameters are

\[ \begin{align*}
    \epsilon'_{xx} &= \frac{\sqrt{w^2 - x'^2}}{l}, \\
    \epsilon'_{yy} &= \frac{\sqrt{w^2 - x'^2}}{l}, \\
    \epsilon'_{xy} &= \epsilon'_{yx} = \frac{x'y'}{l\sqrt{w^2 - x'^2}}, \\
    \epsilon'_{yy} &= \frac{x'^2y^2}{l(lw^2 - x'^2)^{3/2}} + \frac{l}{\sqrt{w^2 - x'^2}}, \\
    \epsilon'_{zz} &= \frac{a^2\sqrt{w^2 - x'^2}}{w^2 l},
\end{align*} \tag{3-19} \]

Now we apply the same amplitude correction factor \(M(x') = \exp(-\alpha x')\) to achieve a smooth taper in the x-direction. The new transformation is obtained as

\[ \begin{align*}
    x'' &= \frac{wx}{a} - j\alpha k_0^{-1} \frac{wx}{a}, \\
    y'' &= \frac{ly}{\sqrt{a^2 - x^2}}.
\end{align*} \]

Figure 3-2. Spatial coordinate transformation used for the 2D wave collimator design. (a) Original coordinate system and (b) transformed system. Reprinted with permission from [26].
and the resulting material parameters are

\[
\begin{align*}
\epsilon''_{xx} &= \frac{\sqrt{w^2 - x'^2}}{l}(1 - j\alpha k_0)^{-1} \\
\epsilon''_{xy} &= \frac{x'y'}{l\sqrt{w^2 - x'^2}} \\
\epsilon''_{yy} &= \left(\frac{x'^2y'^2}{l(\sqrt{w^2 - x'^2})^3} + \frac{l}{\sqrt{w^2 - x'^2}}\right)(1 - j\alpha k_0)^{-1} \\
\epsilon''_{zz} &= \frac{a^2 \sqrt{w^2 - x'^2}}{w^2l}(1 - j\alpha k_0)^{-1}^{-1}
\end{align*}
\] (3-21)

As before, all the other material parameters not listed above have zero values. It should be noted that when \(\alpha=0\), we obtain the same equations for both the mapping and material parameters as the original transformation without the amplitude correcting factor. When \(\alpha > 0\), loss is introduced, whereas when \(\alpha < 0\), gain is added. Therefore, a tapered aperture field distribution can be formed on both the top and bottom surfaces of the lens. By changing the value of \(\alpha\), different degrees of tapering can be achieved, which can be used to control the far-field properties of the system.

In addition to this design, we split the lens into four sub-lenses and assign different \(\alpha\) values to each of them as shown in Fig. 14. By doing so, it allows us to obtain arbitrary control of the radiation pattern.
Figure 3-3. Configuration of a flat lens designed for far-zone focusing at 3 GHz where a single source is placed outside the lens. All the dimensions are in meters. The units of the color bar are arbitrary. (a) Snapshot of the electric field distribution with $\alpha=0$ and (b) $\alpha=-5$. (c) Near-field magnitude and phase distribution of the aperture field on the top surface of the lens and the sources in a 41-element linear array. (d) Far-field gain patterns for the TO lenses with $\alpha=0$ and $\alpha=-5$ and the conventional antenna array.
Figure 3-4. (a) Near-field magnitude, (b) phase, and (c) far-field gain patterns for TO lenses with $\alpha=-1$, $\alpha=-3$ and $\alpha=-5$ at 3 GHz.
Figure 3-5. Configuration of a flat lens designed for far-zone focusing at 3 GHz where a single source is embedded inside of the lens. All the dimensions are in meters. The units of the color bar are arbitrary. (a) Snapshot of the electric field distribution with $\alpha=0$ and (b) $\alpha=-2$. Far-field gain patterns for TO lenses with (c) $\alpha=0$ and (d) $\alpha=-2$ at 3 GHz compared to a conventional antenna array.

Figure 3-6. Configuration of a flat lens designed for far-zone focusing at 3 GHz where a single source is embedded inside of the lens. All the dimensions are in meters. The units of the color bar are arbitrary. Snapshot of the electric field distribution of four sub-lenses with assigned alpha values (i) $\alpha=0$, (ii) $\alpha=-2$, (iii) $\alpha=+2$ and (iv) $\alpha=-2$. 
Section 3.2. Numerical Results for Spatial Complex TO

Fig. 12(b) shows a snapshot of the electric field distribution for the TO lens, where the source is located outside of the lens, with $\alpha = -5$ obtained using COMSOL. The system efficiency for the lenses with $\alpha = 0$ and $\alpha = -5$ are 100% and 64.45%, respectively. The side lobe levels for the two cases are -10.97dB and -15.67dB, respectively. As a comparison, we calculated the gain of a conventional linear antenna array with quarter-wavelength equally spaced elements that has the same aperture size as the lens. Each element is assigned the same magnitude and phase corresponding to those of the same position along the lens. The 41-element linear array has a side lobe level of -24.25dB, which is lower than that of the $\alpha = -5$ lens. The gain values for the $\alpha = 0$, $\alpha = -5$ lenses and the linear array are 12.18dB, 8.67dB and 9.31dB, respectively. In addition, the 3dB beam width of the $\alpha = 0$, $\alpha = -5$ lenses and the antenna array are 3.86°, 5.99°, and 5.6°, respectively. It should be noted that the linear array, which requires a set of RF front-end sub-systems behind each antenna element, has a bidirectional radiation pattern, while the TO lens provides only a single directive beam pointing in the desired broadside direction.

In order to investigate the impact $\alpha$ has on the far-field pattern, we further compared the near-field aperture magnitude and phase distribution as well as the far-field gain patterns for lenses with $\alpha = -1$, $\alpha = -3$ and $\alpha = -5$. As shown in Fig. 12(a) and Fig. 12(b), as $\alpha$ decreases, the aperture field magnitude has an increased tapering while the phase becomes smoother. From the far-field gain patterns displayed in Fig. 12 (c), the side lobe level is reduced from -11.98dB to -15.67dB when $\alpha$ is decreased from -1 to -5. At the same time, the gain is also reduced from 11.54dB to 8.67dB. The system efficiency for the three lenses with $\alpha = -1$, $\alpha = -3$ and $\alpha = -5$ are 86.55%, 72.22%, and 65.46%, respectively. The 3dB beamwidths are 5.1°, 6.1° and 6°, respectively. In general, larger negative values of $\alpha$ reduce the side lobe level, increase the 3dB beamwidth, and reduce the gain. The best compromise design corresponds to the $\alpha = -3$ case.

Fig. 13(a) shows a snapshot of the electric field distribution for the TO lens, where the source is embedded in the middle of the lens, with $\alpha = 0$ generated using
COMSOL. This is equivalent to the real coordinate transformation and without a taper effect, while Fig. 13(b) shows a snapshot of the electric field distribution for the lens with \( \alpha=-2 \). As a comparison, we calculated the gain of a conventional linear antenna array with both quarter-wavelength and half-wavelength equally spaced elements that has the same aperture size as the lens. Each element is assigned the same magnitude and phase corresponding to those of the same position along the lens. Fig. 13(c) and (d) show the results of a far field radiation pattern comparison for the \( \alpha=0 \) and \( \alpha=-2 \) lens. The 3dB beamwidth of the \( \alpha=0 \) lens, corresponding half-wavelength array and quarter-wavelength array is 6°, 5.8° and 5.6°, respectively. The side lobe level is -10.97dB, -10.91dB and -10.95dB, respectively. The gain is 13.24dB, 12.93dB and 12.92dB, respectively. The 3dB beamwidth of the \( \alpha=-2 \) lens, the corresponding half-wavelength and quarter-wavelength array is 6.6°, 6.4° and 6.4°, respectively. The side lobe levels are -13.034dB, -13.015dB and -12.763dB, respectively. The gain is 17.93dB, 17.85dB and 17.82dB, respectively. It should be noted that the end fire of the lens has a lower radiated power than that of the antenna arrays.

Section 3.3. Applying Complex Coordinate Transformation For Amplitude and Phase Smoothing

When mapping the cylindrical radiated wave to a plane wave, the mapping equations we used in the above section have a finite width \( w \), and because we truncate the lens to \( w \), there will be some irregular variation in amplitude as well as in phase. It is noticed that the even power of \( x' \) in the \( M \) function have the property of smoothing out these irregular variations. Consider the source embedded design, now instead of applying \( M(x')=exp(-\alpha x') \) or \( M(x')=exp(-\alpha x'^2) \), we apply \( M(x')=exp(-\alpha x'^2) \). Note that when this choice of \( M \) is applied, the domain \( x=[-0.4, +0.4] \) is squared in \( M \) and hence the desired effect is shifted to only the positive side of the lens. Therefore, in order to obtain the symmetric effect for both negative and positive \( x \), we split the lens into two parts to
calculate and assign the material parameters. The following is the modified mapping and
the resulting material parameters.

\[
\begin{align*}
x' &= \frac{wx}{a} - j\alpha k_0^{-1} \left(\frac{wx}{a}\right)^2, x' \leq 0 \\
y' &= \frac{y}{\sqrt{a^2 - x'^2}} \\
z' &= z
\end{align*}
\]

\[
\begin{align*}
x'' &= \frac{wx}{a} + j\alpha k_0^{-1} \left(\frac{wx}{a}\right)^2, x'' \geq 0 \\
y'' &= \frac{y}{\sqrt{a^2 - x'^2}} \\
z'' &= z
\end{align*}
\]

The corresponding material parameters are then determined by the following
expressions, where \(l\) stands for left and \(r\) stands for right:

\[
\begin{align*}
le_{xx}'' &= \left[\frac{w}{a} - j2\alpha k_0^{-1} \left(\frac{wx}{a}\right)^2\right] \cdot \frac{\sqrt{w^2 - x'^2}}{l} \\
le_{xy}'' &= \frac{x'y'}{l\sqrt{w^2 - x'^2}} \\
le_{yy}'' &= \left[\frac{x'^2y'^2}{\sqrt{w^2 - x'^2}} \cdot l + \frac{l}{\sqrt{w^2 - x'^2}}\right] \cdot \left[\frac{w}{a} - j2\alpha k_0^{-1} \left(\frac{w}{a}\right)^2 x'\right]^{-1} \\
le_{zz}'' &= \left[\frac{w}{a} - j2\alpha k_0^{-1} \left(\frac{w}{a}\right)^2 x'\right]^{-1} \cdot \frac{\sqrt{w^2 - x'^2}}{l} \\
re_{xx}'' &= \left[\frac{w}{a} + j2\alpha k_0^{-1} \left(\frac{wx}{a}\right)^2\right] \cdot \frac{\sqrt{w^2 - x'^2}}{l} \\
re_{xy}'' &= \frac{x'y'}{l\sqrt{w^2 - x'^2}} \\
re_{yy}'' &= \left[\frac{x'^2y'^2}{\sqrt{w^2 - x'^2}} \cdot l + \frac{l}{\sqrt{w^2 - x'^2}}\right] \cdot \left[\frac{w}{a} + j2\alpha k_0^{-1} \left(\frac{w}{a}\right)^2 x'\right]^{-1} \\
re_{zz}'' &= \left[\frac{w}{a} + j2\alpha k_0^{-1} \left(\frac{w}{a}\right)^2 x'\right]^{-1} \cdot \frac{\sqrt{w^2 - x'^2}}{l}
\end{align*}
\]

(3-22)

As shown in Fig. 15, the more positive the \(\alpha\), the smoother the amplitude and phase.
It is indeed shown that even power of \(x'\) in the M function can help solve the finite
truncation problem of the lens. By applying the even power complex transformation, we
can get back close to the uniform amplitude and phase and hence achieve higher directivity.
Section 3.4. Applying Complex Coordinate Transformation to Anisotropic Homogeneous Lenses

The key reason that Transformation Optics is a valid approach that allows designers to arbitrarily control wave propagation is that Maxwell’s equations are coordinate invariant. Hence, there are infinitely many mappings that can convert a cylindrical radiated wave into a plane wave. The equations in the above sections are just two of the possible cases. Those equations are referred to as spatial mappings. However, spatial TO usually leads to extremely complicated material parameters that make it almost impossible for practical implementation. As discussed in Chapter 2, quasi-conformal mapping [66] has the advantage of producing designs that are relatively easy to fabricate. That methodology creates homogeneous and anisotropic material tensor parameters and the simplification process can be explained by the Linear Transformation Optics or K-Space TO [65,67] that
uses the fact that upon coordinate transformation, the wave vector $\mathbf{k}'$ can be related to the wavenumber vector $\mathbf{k}$ in the original space by

$$\mathbf{k}' = (A^{-1})^T \mathbf{k}. \quad (3-25)$$

Based on the dispersion theory explained in [100], we introduce a highly directional and smooth wave in the y-direction that has material tensor parameters

$$\epsilon' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.01 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (3-26)$$

In general, for any mapping from the original free space $(x,y,z)$ to $(f(x,y,z), g(x,y,z), z)$, we have a Jacobian $A_1$ such that the material parameters can be calculated using the classical equation $\bar{\epsilon} = \frac{A_1 \bar{\epsilon}_o A_1^T}{\det(A_1)}$ as follows:

$$\bar{\epsilon}' = \frac{A_1 \bar{\epsilon}_o A_1^T}{\det(A_1)} = \begin{pmatrix} \epsilon'_{xx} & \epsilon'_{xy} & \epsilon'_{xz} \\ \epsilon'_{yx} & \epsilon'_{yy} & \epsilon'_{yz} \\ \epsilon'_{zx} & \epsilon'_{zy} & \epsilon'_{zz} \end{pmatrix}, \quad (3-27)$$

where $\bar{\epsilon}_o$ is the free space material tensor $\bar{\epsilon}_o = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Now we can apply Complex Coordinate TO to any mapping as

$$\begin{align*}
\begin{cases} 
  x'' = x' - j \alpha k_0^{-1} x'^3 \\
  y'' = y' \\
  z'' = z'
\end{cases}
\end{align*} \quad (3-28)$$

where the Jacobian for this specific $x'', y'', z''$ mapping is $A_2 = \begin{pmatrix} 1 - j3 \alpha k_0^{-1} (x')^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Then if the material relation is applied to $A_2$, we obtain

$$\bar{\epsilon}'' = \frac{A_2 \bar{\epsilon} A_2^T}{\det(A_2)} \quad (3-29)$$

$$\bar{\epsilon}'' = \frac{1}{1 - j3 \alpha k_0^{-1} (x')^2} \begin{pmatrix} 1 - j3 \alpha k_0^{-1} (x')^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{pmatrix} \begin{pmatrix} 1 - j3 \alpha k_0^{-1} (x')^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$ 

(3-30) It can be verified that the material parameters (3-21) obtained in the above sections
are the same as if (3-30) were applied to (3-19) directly. Fig. 16 shows the amplitude, phase and far field radiation pattern when applying (3-30) to (3-26). The resulting material parameter tensor is

\[
\epsilon'' = \frac{A_2 \epsilon A_2^T}{\det(A_2)} = \begin{pmatrix}
1 - j3\alpha k_0^{-1}x'^2 & 0 & 0 \\
0 & 0.01 & 0 \\
0 & 1 - j3\alpha k_0^{-1}(x'^2) & 1
\end{pmatrix} \quad (3-31)
\]
Figure 3-8. Configuration of a flat lens designed for far-zone focusing at 3 GHz where a single source is embedded inside of the lens. All the dimensions are in meters. (a) The amplitude change from $\alpha=0$ to $\alpha=180$ and (b) the phase change from $\alpha=0$ to $\alpha=180$. (c) Far-field gain patterns for TO lenses from $\alpha=0$ to $\alpha=180$. 
As Maxwell’s equations show, the dispersion relations and the wave trajectory in the physical space remain the same provided that \( \varepsilon''_{xx} \varepsilon''_{zz} \) and \( \varepsilon''_{yy} \varepsilon''_{zz} \) are held constant [67]. Thus the material parameter tensor (3-31) can be further simplified as

\[
\varepsilon'' = \frac{A_2 \varepsilon t A_2^T \det(A_2)}{\det(A_2)} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0.01 & 0 & 0 \\
0 & 0 & (1 - j3\alpha k_0^{-1}(x'^2))^2 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\] (3-32)

The equation (3-32) only contains a \( y \) component, and therefore when fabricating, we only need to implement the antenna lens using multiple layers of metamaterial split ring resonators in the \( y \) direction to achieve highly directional wave propagation.
(b) Linear Complex TO Equivalent eps (epsyy' = 0.01)

![Graph showing amplitude vs. arc length with different lens angles.]

(c) Linear Complex TO Equivalent Eps (epsyy' = 0.01)

![Graph showing phase vs. arc length with different lens angles.]

Section 3.5. Numerical Results for Linear Complex TO

Fig. 16(a) shows the snapshot of the electric field distribution for the TO lens, where the source is located in the middle of the lens, with $\alpha$ from 0 to 180 obtained using COMSOL. The side lobe levels shown in Fig. 16(c) for $\alpha=0$, $\alpha=50$, $\alpha=100$ and $\alpha=150$ are -9.3dB, -11.73dB, -15.68dB and -18.97dB, respectively. In addition, the 3dB beam width of the $\alpha=0$, $\alpha=50$, $\alpha=100$ and $\alpha=150$ lenses are 5.8°, 6.3°, 6.8° and 7.2°, respectively.

Fig. 17 is a set of plots generated for the equivalent material parameter case in equation (3-32). Fig 17 (a) shows a snapshot of the real (left panels) and imaginary (right panels) material parameters inside of the lens with $\alpha=150$. It should be noted that inside of this lens, we have both imaginary and real material spatial distribution which provide the desired gain/loss effect. Fig.17 (b) shows the electric field distribution for the TO lens, where the source is embedded with $\alpha$ from 0 to 250 obtained using COMSOL. The side lobe levels shown in Fig. 17(d) for $\alpha=0$, $\alpha=50$, $\alpha=100$ and $\alpha=150$ are -9.3dB, -10.13dB, -12.66dB and -15.11dB, respectively. In addition, the 3dB beam width of the $\alpha=0$, $\alpha=50$, $\alpha=100$ and $\alpha=150$ lenses are 5.8°, 6.3°, 6.8° and 7.2°, respectively.
$\alpha=100$ and $\alpha=150$ lenses are 5.8°, 5.9°, 6.3° and 6.4°, respectively. The equivalent material parameters in this case are easier to fabricate and has the same effect in controlling the far field radiation pattern and side lobe level. However, the equivalent material parameters are obtained based on the theory that is utilized when the material parameters are purely real. Therefore, a different degree of radiation pattern control from the equivalent material parameters and the original linear material parameters is expected.

Section 3.6. Conclusion

In summary, we have presented a method to control both the amplitude and phase of the aperture field distribution for a far-field focusing lens using a complex coordinate transformation. By adjusting the amplitude control parameter, both the near-field and far-field properties of the radiating system fed by a single electromagnetic source can be manipulated. We have also shown that this complex coordinate transformation can be applied to any material parameters and obtain gain/loss distribution along the lens. $M(x') = \exp(-\alpha x')$ works well with the source outside loss case, while $M(x') = \exp(-\alpha x'^3)$ works well with the source embedded gain case. In addition, $M(x') = \exp(-\alpha x'^2)$ has the property to flatten the variation in amplitude and phase due to the finite truncation effect. When applying the complex coordinate transformation to a homogeneous, anisotropic lens derived from a linear transformation, we created more realistic and easier to fabricate material parameters that exhibit the same control on the far field radiation pattern. This work would benefit future TO-based lens designs with extra degrees of freedom for controlling the field distribution and associated wave propagation.
Appendix A

Flow Chart for Applying Complex Transformation Optics to Any Mapping

Theoretical Computation

Any mapping can add Complex TO to it.

Say the material parameters calculated from the 2D TO mapping is

\[ \bar{\varepsilon}' = \begin{pmatrix} \varepsilon'_{xx} & \varepsilon'_{xy} & \varepsilon'_{xz} \\ \varepsilon'_{yx} & \varepsilon'_{yy} & \varepsilon'_{yz} \\ \varepsilon'_{zx} & \varepsilon'_{zy} & \varepsilon'_{zz} \end{pmatrix} \]

\[ \begin{align*}
& x = x \\
& y = y \\
& z = z
\end{align*} \]

\[ \begin{align*}
& x' = f(x, y, z) \\
& y' = g(x, y, z) \\
& z' = z
\end{align*} \]

\[ M = e^{-\alpha x'^2} \]

Apply the Complex TO

\[ \begin{align*}
& x'' = x' - j a k_0^{-1} x'^2 \\
& y'' = y' \\
& z'' = z'
\end{align*} \]

Jacobian A1

\[ \bar{\varepsilon}' = A_1 \bar{\varepsilon}' A_1^T \frac{1}{\det(A_1)} = \begin{pmatrix} \varepsilon'_xx & \varepsilon'_xy & \varepsilon'_xz \\ \varepsilon'_yx & \varepsilon'_yy & \varepsilon'_yz \\ \varepsilon'_zx & \varepsilon'_zy & \varepsilon'_zz \end{pmatrix} \]

Jacobian A2

\[ \bar{\varepsilon}'' = \frac{1}{\det(A_2)} A_2 \bar{\varepsilon}' A_2^T \]

\[ \bar{\varepsilon}'' = \begin{pmatrix} \varepsilon''_{xx} & \varepsilon''_{xy} & \varepsilon''_{xz} \\ \varepsilon''_{yx} & \varepsilon''_{yy} & \varepsilon''_{yz} \\ \varepsilon''_{zx} & \varepsilon''_{zy} & \varepsilon''_{zz} \end{pmatrix} \]

\[ \bar{\varepsilon}'' = \begin{pmatrix} 1 - j 3 a k_0^{-1} (x')^2 & 0 & 0 \\ 0 & 1 - j 3 a k_0^{-1} (x')^2 & 0 \\ 0 & 0 & 1 - j 3 a k_0^{-1} (x')^2 \end{pmatrix} \]
BIBLIOGRAPHY


EDUCATION:

The Pennsylvania State University, University Park, PA
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WORK EXPERIENCE:

Teaching Intern 2013 Fall Semester
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PUBLICATIONS:

- Transformation-Optics Antenna Lens Design Using Complex Coordinate Transformation 2014
- Monte Carlo Simulation and Option Price: with Prof. Anna Mazzucato 07/2011
- New Young Talent: Excellent as I Know It China Chinese Drama Publishing House 2007

CURRENT PROJECT:

Under Professor Werner in the Computational Electromagnetics and Antenna Research Lab
- Designing lens material using Complex Coordinate Transformation Optics in order to control not only the phase of wave propagation but also the amplitude.
- Writing a review paper on Transformation Optics Methods and its advantages, disadvantages, and applications with respect to math.
Volunteer Experience:
Penn State University Health Service 2011-2013
Physical Therapy Volunteer
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Student member of IEEE and SPIE society
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Women in Math Research Fellowship 2011-2012
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