HEAT KERNELS, SOLVABLE LIE ALGEBRAS, AND THE MEAN REVERTING SABR STOCHASTIC VOLATILITY MODEL

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Abstract. We use commutator techniques and calculations in solvable Lie groups to investigate certain evolution Partial Differential Equations (PDEs for short) that arise in the study of stochastic volatility models, pricing contingent claims on risky assets. In particular we derive the exact kernel of the fundamental solution of a degenerate PDE, which corresponds to a singular small-diffusion limit of the full SABR model with mean reversion. Although our results are motivated by the SABR model with mean reversion, they often apply in a more general setting.

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1. INTRODUCTION

We study certain parabolic partial differential equations (PDEs for short) that arise in some concrete applications of probability theory to option pricing [13, 46, 48]. Our main result is an explicit approximation formula with a rigorous error bound of solutions to the Cauchy problem for the so-called $\lambda$SABR PDE, a linear
parabolic equation in two space dimensions. Most of our results however apply to a more general setting.

After suitable substitutions, the \( \lambda \)SABR PDE takes the form

\[
\partial_t u - \kappa (\theta - \sigma) \partial_{\sigma} u - \frac{\sigma^2}{2} \left[ (\partial_{\sigma}^2 u - \partial_x u) + 2 \nu \rho \partial_x \partial_{\sigma} u + \nu^2 \partial_{\sigma}^2 u \right] = 0
\]

for the function \( u(t, \sigma, x) \), where \( t \geq 0, \sigma > 0 \) and \( x \in \mathbb{R} \). Thus, throughout the paper we will denote by \( L \) the following strongly, but not uniformly elliptic operator:

\[
L := \frac{\sigma^2}{2} \left[ (\partial_{\sigma}^2 u - \partial_x u) + 2 \nu \rho \partial_x \partial_{\sigma} u + \nu^2 \partial_{\sigma}^2 u \right] + \kappa (\theta - \sigma) \partial_{\sigma} u.
\]

The distributional kernel of the solution operator for \( \partial_t - L \) is not known in explicit form. Our method of approximation is akin to a splitting method, but it does not employ any time discretization. Rather, it utilizes certain commutator properties between the operators into which \( L \) is decomposed. We first write the operator \( L \) as

\[
L = A + \frac{\sigma^2}{2} B + \nu L_1 + \nu^2 L_2,
\]

where

\[
A := \kappa (\theta - \sigma) \partial_{\sigma}, \quad B := \partial_{\sigma}^2 - \partial_x,
\]

\[
L_1 := \rho \sigma^2 \partial_x \partial_{\sigma}, \quad \text{and} \quad L_2 := \frac{1}{2} \sigma^2 \partial_{\sigma}^2.
\]

We then study separately these operators and their combinations, based on the commutator identities that they satisfy. One of the main results of this paper is the fact that

\[
L_0 := A + \frac{\sigma^2}{2} B
\]

generate strongly continuous (or \( c_0 \)) semi-groups on suitably weighted Sobolev spaces (see the discussion below) provided that we restrict the variable \( \sigma \) to a bounded interval, \( \sigma \in I := (\alpha, \beta) \), where \( 0 < \alpha < \theta < \beta < \infty \). These conditions on \( I \), which are not restrictive for our purposes, can be lifted, provided additional conditions are satisfied by the solution, such as boundary conditions. Under these conditions, \( L \) becomes a uniformly strongly elliptic operator. The operators \( L, L_0, B \) are parabolic, while \( A \) is of hyperbolic type. Classical arguments give that \( L, A, B \) generate \( c_0 \) semigroups and, in fact, the semigroups generated by \( L \) and \( B \) are analytic.

Throughout the paper, if \( T \) is a linear operator whose closure generates a \( c_0 \) semigroup, we shall denote the semigroup it generates by \( e^{tT}, t \geq 0 \), as usual. We recall that then the solution to the abstract Initial Value Problem (IVP) \( \partial_t u - Tu = 0, u(0) = h \), is given by \( u = e^{tT} h \).

We stress that \( L_0 \) is a degenerate operator, in the sense that the diffusion matrix associated to \( L_0 \) is not of full rank, and the the operator \( \partial_t - L_0 \) is not hypoelliptic, in particular the distributional kernel of the fundamental solution of \( \partial_t - L_0 \) does not agree with a smooth function for \( t > 0 \). Therefore, the existence of the semigroup does not follow from standard arguments. We thus employ a different strategy, which allows us to establish that \( L_0 \) generates a \( c_0 \) semigroup \( e^{tL_0} \) with an explicit kernel. The explicit formula for the distribution kernel of \( e^{tL_0} \) is obtained from
those of the semigroups $e^{tB}$ and $e^{tA}$. The key observation is that the operators $A$ and $\frac{\sigma^2}{2}B$ generate a solvable, finite-dimensional Lie algebra. See also [3, 4, 77] and the references therein for further results on degenerate parabolic equations.

The main equation that we study, Equation (1), is obtained by time reversion from the backward Kolmogorov equation for the probability density function associated to a two-dimensional stochastic process (the “$\lambda$SABR-process”) in the variable $\sigma$ and $x$ [13, 48]. The parameter $\nu > 0$ is the covariance of the one-dimensional $\sigma$-process, which is $\rho$-correlated to the one-dimensional $x$-process. Unlike the standard SABR process [46, 47], the $\lambda$SABR process includes a mean reverting to $\theta$ term in $\sigma$, which accounts for the term $\kappa(\theta-\sigma)\partial_\sigma u$ in Equation (1). This mean reverting term seems to affect significantly the analysis of the $\lambda$SABR PDE, Equation (1), and this is one of the reasons why we were interested in this problem in the first place. See also [30, 36, 49, 60] for recent results on models with mean-reversion.

Having an explicit formula for $e^{tL_\sigma}$ is important in obtaining an accurate, yet easily computable, approximation of the solution operator $e^{tL}$, one of the main goals of this work, even when the difference $e^{tL}h - e^{tL_\sigma}h$ may not be very small. This fact is especially relevant for model parameter calibration using inference methods, which requires solving the forward problems many times in order to sample the parameter space sufficiently. We estimate our error in exponentially weighted Sobolev spaces. Exponential weights arise naturally in applications, as the initial data $h$ for (1) is typically of the form $h(\sigma, x) := |e^{x-K}|_+$, where $|y|_+ = (y + |y|)/2$ denotes the positive part of the number $y \in \mathbb{R}$. This particular type of initial data arises in the pricing of the so-called European call options after an exponential change of variable of the price of the underlying risky asset (we refer to [37, 41, 79, 56] for a more detailed discussion of options). From a mathematical point of view, this exponential change of variable makes the equation more tractable, since the coefficients of $L$ become constant in $x$ and no boundary condition in $x$ is needed. We note that the initial value $h := |e^{x-K}|_+$ considered above is constant, hence analytic in $\sigma$, a fact exploited for the error analysis in this work (see Equation (6) below and the statement of one of our main results, Theorem 4.11).

The properties of the semigroups generated by $L$ and $L_\sigma$ allow us to estimate the difference $e^{tL}h - e^{tL_\sigma}h$. More specifically, we derive an error estimate of the form:

\[
\|e^{tL}h - e^{tL_\sigma}h\|_{L^2_\nu} \leq C\nu \left(\|\partial_\nu h\|_{L^2_\nu} + \|h\|_{L^2_\nu}\right),
\]

for $\nu \in (0, 1]$ and with a constant $C$, possibly dependent on $L$ and $\kappa$, but not on $h$ and $\nu$ (see Theorem 4.11 for a complete statement). Above, $L^2_\nu$ denotes an exponentially weighted Lebesgue space (see Section A.4). The method of proof of this estimate is a perturbative argument based on heat kernels estimates, following the method developed by two of the authors in [17, 18]. This method extends the work on Henry-Labordère on heat kernel asymptotics [49, 50]. A similar method was developed by Pascucci et al. [73, 75]. Heat kernel asymptotics were employed in this context also by Gatheral et al. [42, 43]. See also [20, 33, 52, 58, 65, 68, 72]. In the process, we also establish several mapping properties for the semi-groups generated by $L_0$ and $L$ on weighted spaces.

More generally, it is possible to show that $e^{tL}h$ can be written as a regular perturbation expansion in powers of $\nu$ and to derive a combinatorial formula for the higher-order terms (this result is the focus of current work by the authors,
see also [83]), provided the initial data $h$ is analytic in $\sigma$ (as it is the case for the examples of interest in this work). The limit $\nu \to 0$ can be viewed as a small-diffusion limit or vanishing-viscosity limit. Such a limit arises in applications and it can also be used to construct a certain type of weak solutions to typically non-linear PDEs, henceforth called viscosity solutions [26, 25] (see also [80] for a treatment of unbounded terms). Small diffusion limits been studied using a variety of methods, from purely PDE methods, to microlocal (semi-classical) analysis techniques, to probabilistic methods. In particular, the theory of random perturbations of dynamical systems yields precisely an expansion of the probability density function in powers of $\nu$ (see [38, 39, 40], the Levinson case, see also [82]), even in the presence of boundaries under suitable conditions, but generally requires stronger hypotheses on the coefficients of the operators, such as a uniform Lipschitz condition (but see [16, 19], and references therein, for more singular coefficients, among others). We should remark that the presence of the mean-reverting term $(\theta - \sigma)\partial_\sigma$ does not allow to transform the operators $L$ and $L_0$ into operators with strictly positive, bounded diffusion coefficients, by a change of variable even when the correlation parameter $\rho = 0$ (as in the normal SABR model, for instance).

As it is well known, there is a large body of work concerning the generation of strongly continuous and analytic semigroups by linear operators (we refer the reader in particular to the excellent monographs [5, 9, 69, 76], see also [15] for kolmogorov equations with singular coefficients). The presence of exponential weights makes the analysis somewhat less standard even for the strongly parabolic operator $L$. The study of linear parabolic equations on weighted spaces can often be recast as an analysis of heat kernels on manifolds, typically (as it is the case for polynomial and exponential weights) complete manifolds of bounded geometry. In this context, Amann’s work [3, 7] provides deep results on semigroup generation on rather general noncompact manifolds and manifolds with boundary. Our approach is more direct and more elementary, owing to the explicit form of the operator $L$ and of the weights. We also mention that fundamental solutions for degenerate equations related to $\partial_t u - L_0 u = F$, but in the context of ultraparabolic equations satisfying Hörmander’s conditions for hypoellipticity, have been studied by many authors, starting with the seminal work of Kolmogorov [61] (see [31, 32, 74, 63] for some recent, relevant works connected to financial applications). Our operator $\partial_t - L_0$ does not satisfy Hörmander’s condition, however.

The paper is organized as follows. Section 2 contains standard results (except for the use of exponentially weighted spaces and the fact that we work on certain non-compact spaces) on the generation of analytic semigroups by the operators $L$ and $B$, which are both strongly elliptic, using the Lumer-Phillips theorem. Section 3 deals with the semi-groups generated by $L_0$, which is degenerate parabolic. An explicit formula for $e^{tL_0}$ is obtained by combining the results for the operators $A$ and $B$, more specifically by exploiting the commutator identities that $A$ and $f(\sigma)B$ satisfy and Lie group ideas. The properties of the semi-group generated by the operator $A$, which is of transport type, are also briefly discussed. The last section, Section 4, contains the derivation of mapping properties and norm estimates for the semi-groups, which are utilized in the proof of the error estimate (6). Lastly, in the appendix (Section A), we review a few needed facts on evolution equations and semi-groups of operators and we introduce the exponentially weighted spaces used in this work.
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2. The semi-group generated by \( L \) and \( B \)

In this section, we give a short proof that the operators \( L \) and \( B \), defined in (3) and (4) respectively, generate analytic semigroup on exponentially weighted Sobolev spaces (introduced in Appendix A.4), using the Lumer-Phillips theorem and other results recalled in the Appendix. The methods we use are standard in classical Sobolev spaces, although the theory is somewhat less developed in exponentially weighted spaces (see, however, [6, 8, 45]). The operators \( L \) and \( B \) are very particular cases of the ones considered in [7], and allow for a more direct and elementary approach. This approach has the advantage of yielding additional mapping properties for the operators \( L \) and \( B \). For the reader’s sake, we give full details for the operator \( L \).

Our approach is to study the semi-groups generated by \( L \) and \( B \) as particular cases of the solution semi-group for the abstract problem

\[
\partial_t u - Pu = F, \quad u(0) = h \in X,
\]

where \( P \) is a (usually unbounded) operator on a Banach space \( X \) with domain \( D(P) \). For the ASABR PDE, Equation (1), one takes \( P = L \) acting on \( L^2_\lambda(\Omega) := e^{\lambda|x|}L^2(\Omega), \Omega = I \times \mathbb{R}, F = 0, \) and \( h(\sigma,x) := |e^x - K|^+ \), for \( I = [\alpha, \beta] \subset (0, \infty) \). General results on this abstract problem are reminded in Appendix A, for the benefit of the reader. For instance, we will prove that the operator \( L \) is quasi-dissipative on weighted Sobolev spaces \( H^m_\lambda(\Omega) := e^{\lambda|x|}H^m(\Omega) \) (discussed also in the Appendix). Here \( (x) := \sqrt{1 + x^2} \) (the “Japanese bracket”). The space

\[
K_0 := H^2_\lambda(I \times \mathbb{R}) \cap \{u = 0 \text{ on } \partial I \times \mathbb{R}\}
\]

will be the common domain of several operators and will play a central role in our analysis. See again Appendix, especially Subsection A.4, for a more detailed discussion of the weighted Sobolev spaces that we are using.

2.1. Operators with totally bounded coefficients. Let \( \Omega = \mathbb{R} \) or \( \Omega = I \times \mathbb{R} \), with \( I \subset \mathbb{R} \) an interval.

Definition 2.1. A function \( f : \Omega \to \mathbb{C} \) is totally bounded if it is smooth and bounded and all its derivatives are also bounded.

We have the following simple lemmas by a direct calculation.

Lemma 2.2. Let \( P \) be an order \( m \) differential operator on \( \Omega \) with totally bounded coefficients. Then \( P \) defines continuous a map \( H^s_\lambda(\Omega) \to H^{s-m}_\lambda(\Omega) \), for every \( s \geq m \).

Lemma 2.3. Let \( P := \sum_{|\alpha| \leq m} a_\alpha \partial^\alpha \) be an order \( m \) differential operator on \( \Omega \) with totally bounded coefficients. If \( w(\sigma,x) = e^{\lambda|x|} \), as before, then \( w^{-1}Pw \) also has totally bounded coefficients and the same terms of order \( m \) as \( P \).

We formulate the following result in slightly greater generality than needed for the proof of the existence of the semi-group generated by \( L \), for further possible applications. We recall the definition of a second order uniformly strongly elliptic
differential operator on $\Omega = \mathbb{R}$ or $\Omega = I \times \mathbb{R}$, with real coefficients, in the form that we use in this paper.

**Definition 2.4.** Let $P = a_{xx}(\sigma, x)\partial^2_{xx} + 2a_{x\sigma}(\sigma, x)\partial_{x\sigma} + a_{\sigma\sigma}(\sigma, x)\partial^2_{\sigma\sigma} + b(\sigma, x)\partial_{\sigma} + c(\sigma, x)\partial_{x} + d(\sigma, x)$ be a differential operator with real coefficients on $I \times \mathbb{R}$. We say that $P$ is uniformly strongly elliptic if it has bounded coefficients and if there exists $\epsilon > 0$ such that $a_{xx} \geq \epsilon$ and $a_{xx}a_{\sigma\sigma} - a^2_{\sigma\sigma} \geq \epsilon$.

If $\Omega = \mathbb{R}$, we simply set $a_{\sigma\sigma} = a_{xx} = c = 0$. We have the following standard regularity results, where we continue to assume that $\Omega = I \times \mathbb{R}$ or $\Omega = \mathbb{R}$.

**Theorem 2.5.** Let $P$ be second order, uniformly strongly elliptic differential operator with totally bounded coefficients on $\Omega$. Assume $u \in H^1_0(\Omega)$ is such that $Pu \in H^{m-1}_0(\Omega)$. If $\Omega = I \times \mathbb{R}$, we also assume that $u$ vanishes at the endpoints of $I$. Then $u \in H^{m+1}_0(\Omega)$. Moreover, there exists $C > 0$, independent of $u$, such that $\|u\|_{H^{m+1}_0(\Omega)} \leq C(\|Pu\|_{H^{m-1}_0(\Omega)} + \|u\|_{H^1_0(\Omega)})$.

This result was proved in the greater generality of Lie manifolds in [8], Theorem 0.1. A direct proof of can also be obtained by first reducing to the case $\lambda = 0$, that is, $w = 1$, using Lemma 2.3 and then either by using a dyadic partition of unity (see [10, 71] and the references therein) or by using divided differences (this approach is sometimes called Nirenberg’s trick after [2]), which is facilitated in this case since the boundary is straight (see, for instance, [67]). Theorem 2.5 holds in the greater generality of manifolds with bounded geometry. We obtain the following consequence.

**Corollary 2.6.** Let $P$ be second order, uniformly strongly elliptic differential operator with totally bounded coefficients on $\mathbb{R}$. Then $\|u\|_{L^2_\lambda} + \|P^ku\|_{L^2_\lambda}$ defines an equivalent norm on $H^{2k}_{\lambda}(\mathbb{R})$.

See also [6, 21, 28, 45, 62, 70] for more related results on PDEs on manifolds with bounded geometry. Similar ideas are used in analysis on polyhedral domains (see [8, 11, 23, 24] for some relevant results in this direction).

**Definition 2.7.** Let $\mathcal{P}$ denote the set of second-order differential operators

$$T := a_{xx}(\sigma, x)\partial^2_{xx} + 2a_{x\sigma}(\sigma, x)\partial_{x\sigma} + a_{\sigma\sigma}(\sigma, x)\partial^2_{\sigma\sigma} + b(\sigma, x)\partial_{\sigma} + c(\sigma, x)\partial_{x} + d(\sigma, x)$$

with real, totally bounded coefficients on $I \times \mathbb{R}$, satisfying

$$a_{xx}, a_{\sigma\sigma}, a_{xx}a_{\sigma\sigma} - a^2_{\sigma\sigma} \geq 0.$$

To an operator $T \in \mathcal{P}$, we associate the matrix of highest-order coefficients

$$M_T := \begin{bmatrix} a_{xx} & a_{\sigma\sigma} \\ a_{x\sigma} & a_{\sigma\sigma} \end{bmatrix}.$$  

Then, $\xi^4M_T\xi$, $\xi \in \mathbb{R}^2$, is the principal symbol of $T$.

**Proposition 2.8.** If $w(\sigma, x) = e^{\lambda(x)}$ and $T \in \mathcal{P}$, then $w^{-1}Tw \in \mathcal{P}$. Let $M_T$ be as in Equation (9), then there exists $C > 0$ such that

$$(Tu, u)_{L^2_\lambda(I \times \mathbb{R})} \leq -\int_{I \times \mathbb{R}} (M_T \nabla u, \nabla u)e^{-2\lambda(x)} \, d\sigma dx + C\|u\|_{L^2_\lambda(I \times \mathbb{R})}^2, \quad u \in \mathcal{K}_0,$$

and hence, $T$ with domain $\mathcal{K}_0 := H^2_{\lambda}(I \times \mathbb{R}) \cap \{u = 0 \text{ on } \partial I \times \mathbb{R}\}$ is quasi dissipative on $L^2_{\lambda}(I \times \mathbb{R})$. 
Proof. The fact that \( w^{-1}Tw \) is of the same form as \( T \) follows from Lemma 2.3. In view of Equation (40), we can assume that \( \lambda = 0 \), that is, \( w := e^{\lambda(x)} = 1 \). The rest of the proof is then a well-known direct calculation, which we include for the benefit of the reader, since we assume a slightly weaker condition than ellipticity on \( T \). First, \( f^*(\xi) = (\xi, f) \) in the definition of quasi-dissipativity, Definition A.4, as \( L^2_\lambda \) is a Hilbert space. Also, by changing \( b, c, \) and \( d \), we can assume that \( T \) is in divergence form:

\[
Tu = \partial_x(a_{xx}\partial_x u) + \partial_\sigma(a_{\sigma x}\partial_\sigma u) + \partial_x(a_{\sigma x}\partial_\sigma u) + \partial_\sigma(a_{\sigma x}\partial_\sigma u) + b\partial_xu + c\partial_\sigma u + du.
\]

Then a standard energy estimate gives:

\[
2\Re(Tu, u) \leq -\int_{I \times \mathbb{R}} (M\nabla u, \nabla u) \, d\sigma dx \, d\sigma dx - \int_{I \times \mathbb{R}} (\partial_x b + \partial_\sigma c + d)|u|^2 \, d\sigma dx
\]

\[
= -\int_{I \times \mathbb{R}} (M\nabla u, \nabla u) \, d\sigma dx + C\|u\|^2 \leq -\delta \int_{I \times \mathbb{R}} \|\nabla u(x)\|^2 \, d\sigma dx + C\|u\|^2,
\]

where \( C = \|\partial_x b + \partial_\sigma c + d\|_{\infty} \), and \( \delta \geq 0 \) is the smallest eigenvalue of \( M_{T} \). Above, we utilized that \( u \in K_0 \), which justifies the integration by parts, and the fact that \( M_T \) is non-negative, given that \( T \in \mathcal{P} \) by hypothesis. The quasi-dissipativity of \( T \) follows immediately from Definition A.4.

Under the hypotheses of Proposition 2.8, it follows immediately that there exists a constant \( C > 0 \) such that

\[
|(Tu, u)| \leq C\|u\|_{H^1(I \times \mathbb{R})}.
\]

This result follows by using again Lemma 2.3 and the total boundedness of the coefficients.

Under the condition of strong, uniform ellipticity on \( T \), the previous energy estimates gives Garding’s inequality (stated for negative-definite operators).

**Corollary 2.9.** Let \( T \) be as in the statement of Proposition 2.8. Assume also that there exists \( \epsilon > 0 \) such that \( a_{xx}a_{\sigma \sigma} - a_{x\sigma}^2 \geq \epsilon \). Then there exist \( C_1 > 0 \) and \( C_2 \) such that

\[
\Re(Tu, u) \leq -C_1\|u\|^2_{H^2_\lambda(I \times \mathbb{R})} + C_2\|u\|^2_{L^2_\lambda(I \times \mathbb{R})}.
\]

Also, if \( u \in H^1(I \times \mathbb{R}) \cap \{u|_{\partial I \times \mathbb{R}} = 0\} \) satisfies \( Tu \in L^2_\lambda(I \times \mathbb{R}) \), then \( u \in H^2_\lambda(I \times \mathbb{R}) \). Consequently, \( T - \mu_0 : K_0 \to L^2_\lambda(I \times \mathbb{R}) \) is invertible for \( \mu_0 > C_2 \).

**Proof.** Garding’s inequality is an immediate consequence of Equation (10). The rest is a direct consequence of this inequality, which we briefly recall.

Again, by Lemma 2.3, we can assume that \( \lambda = 0 \). Let \( \Omega := I \times \mathbb{R} \). By the Lax-Milgram Lemma applied to \( T - \mu_0, \mu_0 > C_2 \), and by replacing \( T \) with \( T - \mu_0 \), if necessary, we can assume that \( T : H^1(I \times \mathbb{R}) \cap \{u|_{\partial \Omega} = 0\} \to H^{-1}(I \times \mathbb{R}) \) is invertible. Strong uniform ellipticity of \( T \) implies elliptic regularity (Theorem 2.5), which in turn gives:

\[
T : K_0 \to L^2(I \times \mathbb{R})
\]

is both injective and surjective, and hence it is invertible. \( \square \)

We obtain as a consequence the following theorem.
Theorem 2.10. Let $T$ be as in the statement of Corollary 2.9. Then $T$ generates an analytic semi-group $e^{tT}$ on $L^2(I \times \mathbb{R})$. In particular, if $I = (\alpha, \beta)$ is a bounded interval with $0 < \alpha \leq \beta < \infty$, then $L$ as given in (1) satisfies the hypothesis of Corollary 2.9, and hence it generates an analytic semi-group on $L^2(I \times \mathbb{R})$.

Proof. Corollaries 2.9 and Equation (11) show that the $T$ satisfies the assumptions of Lemma A.7 (that is, $T$ is continuous and quasi dissipative). Since $T - \mu_0$ is invertible for $\mu_0$ large, again by Corollary 2.9, we are in position to use Corollary A.8 to conclude that $T$ generates an analytic semi-group. If $I$ is bounded, then $L$ has totally bounded coefficients. Since $\alpha > 0$, $L$ is also uniformly strongly elliptic, and the first part of the result applies. \hfill \Box

Corollary 2.11. Let $T$ be as in Theorem 2.10 and $h \in L^2(I \times \mathbb{R})$, for some $\lambda \in \mathbb{R}$. Then $u(t) := e^{tT}h$ is a strong solution of $\partial_t u - Tu = 0$, $u(0) = h$. It is also a classical solution on $(0, \tau]$, for all $\tau > 0$. Moreover, $u(t)$ does not depend on $\lambda$.

Proof. We have that $T$ generates an analytic semi-group $S(t) = e^{tT}$. Moreover, elliptic regularity gives $D(T^k) \subset H^{2k}(I \times \mathbb{R})$, for all $k \in \mathbb{Z}_+$. The Sobolev embedding theorem then gives us that the assumptions of Lemma A.14 and Proposition A.17 are satisfied. These facts prove the first part of the result.

The independence of $u$ on $\lambda$ follows from the fact that the map $L^2(I \times \mathbb{R}) \to L^2(I \times \mathbb{R})$ is injective and continuous for all $\lambda' \leq \lambda''$ and from the uniqueness of strong solutions. \hfill \Box

Remark 2.12. The assumption that $I$ be a bounded interval in the second half of Theorem 2.10, is essential for our method to apply. Our method does not apply, for instance, if $I = (0, \infty)$. The problem lies in the fact that, at $\sigma = 0$, we lose uniform ellipticity and, at $\sigma = \infty$, the coefficient $\theta - \sigma$ becomes unbounded. However, if $\kappa = 0$, we do obtain that $L$ generates an analytic semi-group using the results in [70]. The degeneracy at $\sigma = 0$ and $\sigma = \infty$ could be addressed by introducing appropriate weights in $\sigma$. For the applications of interest in this work, it is enough to consider $\sigma$ in a bounded interval, bounded away from zero.

We now consider the operator $B := \partial_x^2 - \partial_x$ (recall Equation 4). The analysis of $B$ is a special case of the analysis of the operator $L$. We collect and state the main result for clarity.

Theorem 2.13. $T = a\partial_x^2 + b\partial_x + c$ be a uniformly strongly elliptic operator with totally bounded coefficients. Then, $T$ generates an analytic semi-group on $L^2(I \times \mathbb{R})$.

In particular, $B$ generates an analytic semi-group on $L^2(\mathbb{R})$.

Since $D(B^k) = H^{2k}(\mathbb{R})$, we also have the following corollary.

Corollary 2.14. The operator $B$ generates an analytic semi-group on $H^1(\mathbb{R})$, for all $j$.

Remark 2.15. As for $L$, $h \in L^2(I \times \mathbb{R})$, $u(t) := e^{tT}h$ is a strong solution of $\partial_t u - Tu = 0$, $u(0) = h$, a classical solution on any interval $(0, \tau]$, $\tau > 0$, and $u(t)$ does not depend on $\lambda$. In view of the independence of $\lambda$, the semi-group $e^{tB}$ on $L^2$ is given by the usual explicit formula

\begin{equation}
    e^{tB}h(x) = \frac{1}{\sqrt{4\pi t}} \int e^{-\frac{|x-y|^2}{4t}} h(y) \, dy.
\end{equation}
To treat the IVP for Equation (1), we will need to consider families of operators. In particular, we will show that the operator \( P = \frac{\alpha^2}{2} B \), acting on functions of \( \sigma \) and \( x \), that appears in the SABR PDE, also generates an analytic semigroup. If \( p : I \rightarrow [0, \infty) \) is bounded and continuous, we shall write \( pB \) for the operator \( (pBv)(\sigma) = p(\sigma)Bv(\sigma) \in L^2_\lambda(\mathbb{R}) \) and \( e^{pB} \) for the operator

\[
(e^{pB}v)(\sigma) := e^{p(\sigma)B}v(\sigma) \in L^2_\lambda(\mathbb{R}).
\]

where \( v : I \rightarrow H_\lambda^1(\mathbb{R}) \) is a suitable measurable function. We thus regard both \( pB \) and \( e^{pB} \) as a family of operators parameterized by \( \sigma \in I \) and acting on \( L^2_\lambda(\mathbb{R}) \)-valued measurable functions defined on \( I \).

For the proof we utilize the following well-known results.

**Lemma 2.16.** Let \( \xi \in \mathcal{C}([0,1] ; X) \) and \([0,1] \ni t \rightarrow V(t) \in \mathcal{L}(X) \) be strongly continuous. Then the map \([0,1] \ni t \rightarrow V(t)\xi(t) \in X \) is continuous.

**Proposition 2.17.** Let \( T \) be a differential operator as in Theorem 2.13. Let \( I \subset \mathbb{R} \) be an interval and \( p : I \rightarrow [0, \infty) \) be a bounded continuous function. Then \( e^{pT} \), defined by \( (e^{pT}h)(\sigma) := e^{p(\sigma)T}h(\sigma) \in L^2_\lambda(\mathbb{R}) \), \( \sigma \in I \), defines a \( c_0 \) semi-group on \( L^2_\lambda(I \times \mathbb{R}) \) with generator \( pT \).

**Proof.** Since \( T \) generates a \( c_0 \) semi-group, \( e^{p(\sigma)T}h(s) \) depends continuously on \( \sigma \in I \) whenever \( h \in L^2_\lambda(I \times \mathbb{R}) \) is continuous in \( \sigma \). Since \( \| e^{T} \| \) is uniformly bounded for \( t \) in a bounded interval, we obtain that the family of operators \( e^{T}p(\sigma)T \) thus defines a bounded operator on \( L^2_\lambda(I \times \mathbb{R}) \). \( \square \)

To deal with higher regularity, we need the following extension of Lemma 2.16.

**Lemma 2.18.** Let \( J := (0,1) \) and assume that \( \xi \in \mathcal{C}^1(J ; X) \), that \( T \) is the generator of \( c_0 \) semi-group \( V(t) \) on \( X \), and that one of the following two conditions is satisfied:

(i) \( \xi(t) \in D(T) \) and the map \([0,1] \ni t \rightarrow T\xi(t) \in X \) is continuous;

(ii) the semi-group \( V(t) \) generated by \( T \) is an analytic semi-group.

Then \( V(t)\xi(t) \in \mathcal{C}^1(J ; X) \) with differential \( TV(t)\xi(t) + V(t)\xi'(t) \).

**Corollary 2.19.** Let \( f : [\alpha, \beta] = T \rightarrow [\epsilon, \infty) \), \( \epsilon > 0 \). Assume that \( f \), \( f' \), and \( f'' \) are (defined and) continuous. Let \( K_1 := H_\lambda^2(T \times \mathbb{R}) \). Then \( e^{fB} \) maps \( K_1 \) to itself. Moreover, \( e^{fB} \) defines a \( c_0 \) semigroup on \( K_1 \), generated by \( fB \) as an operator with domain

\[
\{ \xi \in K_1, B\xi \in K_1 \} \subset H_\lambda^{2,4}(I \times \mathbb{R}) := H^2(I; H_\lambda^4(\mathbb{R})).
\]

**Proof.** The first part is an immediate consequence of Lemma 2.18(ii) and of Remark A.9. The second part follows using also Corollary 2.14. \( \square \)

### 3. The semi-group generated by \( L_0 \)

In this section, we discuss the derivation of an explicit formula for the distributional kernel of the operator \( e^{L_0} \) using Lie algebra techniques. Besides being of independent interest, in this work we utilize the explicit formula for \( e^{L_0} \) to approximate \( e^{tL} \), for which no closed form are available. This is achieved by means of a perturbative expansion in the parameter \( \nu \), the so-called “volvol” or “volatility of the volatility.” We recall that \( L_0 = A + \frac{\alpha^2}{2} B \) and \( L = L_0 + \nu L_1 + \nu^2 L_2 \), with \( L_i \) independent of \( \nu \) (see Equations (3) and (4)).
There is an added difficulty in our problem, namely, the fact that \( L_0 \) is not strongly elliptic, and \( \partial_t - L_0 \) is not hypoelliptic in the sense of Hörmander [55] (although \( L_0 \) is). As a matter of fact, this expansion is only valid under additional regularity assumptions on the initial data \( h \), which will be discussed in Section 4.

The explicit formula for \( e^{tL_0} \) is derived from the corresponding formulas for \( e^{tA} \) and \( e^{it\alpha B} \), where the later is defined using Proposition 2.17. The existence of the group \( e^{tA} \), \( t \in \mathbb{R} \), follows directly from the transport character of the operator \( A = (\theta - \sigma)\partial_\sigma \), as recalled below.

We thus assume that \( I = (\alpha, \beta) \) satisfies \( 0 < \alpha < \theta < \beta < \infty \), as in Proposition 2.17. We will make the further assumption that \( \kappa > 0 \).

This last assumption implies that the characteristics of the operator \( A \) are incoming at \( \sigma = \alpha \) and \( \sigma = \beta \), as long as \( \alpha < \theta < \beta \) and \( \kappa > 0 \). Therefore, no boundary conditions need to be imposed at \( \sigma = \alpha \) and \( \sigma = \beta \) (cf. the seminal work of Feller [34, 35]). The case \( \kappa < 0 \) is similar provided one imposes suitable boundary conditions. However, this case will not be needed for our purposes.

We next briefly discuss \( e^{tA} \) and its properties. These will be used in deriving an explicit formula for \( e^{tL_0} \).

### 3.1. The generation property for \( L_0 \)

Let \( I = (\alpha, \beta) \subset \mathbb{R} \) and \( A := \kappa(\theta - \sigma)\partial_\sigma \), as before. We consider first the transport equation

\[
\partial_t v - Av = 0,
\]

where \( v \) depends on \( \sigma \) and, possibly, on some parameters. This equation is solved explicitly by the method of characteristics. For \( s \in \mathbb{R} \), let

\[
\delta_t(s) := \theta(1 - e^{-\kappa t}) + se^{-\kappa t},
\]

be the characteristic line starting at \( s \), that is, \( \delta(t, s) = \delta_t(s) \), \( t \in \mathbb{R} \), \( s \in \mathbb{R} \), is the flow map generated by \( A \). Then, \( \delta_t \circ \delta_s = \delta_{t+s} \). In addition, by the assumptions on \( I \), \( \delta_t(I) \subset I \) for \( t \geq 0 \).

By property of the flow, for any \( h \in L^1_{\text{loc}}(I) \), there is a unique weak solution of (14), which is a classical solution if \( h \in C^1(I) \), and given by the formula:

\[
v(t, \sigma) := h(\delta_t(\sigma)).
\]

Properties of the flow also immediately give that the family of operators \( T(t) \), \( t \in \mathbb{R} \), defined by \( T(t)h = v(t) \) form a group on any \( L^p(\mathbb{R}) \), \( 1 \leq p \leq \infty \), and a semi-group if we restrict to \( I \).

In what follows, we consider \( A \) as operator acting of functions of \( \sigma \) with values in a Hilbert space \( \mathcal{H} \). For the application at hand, \( \mathcal{H} \) will be an exponentially weighted Sobolev space. We record the generation of the semigroup in this case and present a brief proof for clarity.

#### Proposition 3.1

Let \( \mathcal{H} \) be a Hilbert space. Let \( h \in L^2(I; \mathcal{H}) \). Then, \( \| T(t)h \| \leq e^{\kappa t/2} \| h \| \), where \( \| \cdot \| \) denotes the Hilbert space norm on \( L^2(I; \mathcal{H}) \). Moreover, \( T(t) =: e^{tA} \) is a \( c_0 \) semi-group whose generator coincides with \( A \) on \( C^1(I; \mathcal{H}) \).

**Proof.** The relation \( \| T(t)h \| \leq e^{\kappa t/2} \| h \| \) follows by a change of variables (note also that, for \( I = \mathbb{R} \), we have equality). The identity \( T(t_1)T(t_2)h = T(t_1 + t_2)h \) again follows from the flow properties. If \( h \in C^1(I; \mathcal{H}) \), then (16) gives \( \lim_{t \to 0} T(t)h \to h \). Since \( \| T(t)h \| \) is uniformly bounded for \( t \leq 1 \), this gives that \( T(t)h \to h \) as \( t \to 0 \) for all \( h \). \( \square \)
In particular, \( v \) is a strong solution of Equation (14) for \( h \in C^1(I; L^2_x(\mathbb{R})) \). If \( h \in C^1(I; H^1_x(\mathbb{R})) \), it is also a classical solution. We recall that \( \mathcal{K}_1 := H^1_x(I \times \mathbb{R}) \). The explicit formula (16) for \( v(t) \) and the fact that \( \mathcal{K}_1 \subset C^1(I; L^2_x(\mathbb{R})) \), by Sobolev embedding, implies also the following result needed later in the paper.

**Corollary 3.2.** For \( h \in \mathcal{K}_1 \), \( v \) is a strong solution of Equation (14), \( e^{tA}(\mathcal{K}_1) \subset \mathcal{K}_1 \), and \( e^{tA} \) defines a \( c_0 \) semi-group on \( \mathcal{K}_1 \).

We now turn to the study of the operator \( L_0 \). It seems difficult to apply the Lumer-Phillips Theorem to a degenerate operator like \( L_0 \). We will therefore adopt a different strategy and directly prove that the solution operator of \( \partial_t - L_0 \), which we still denote by \( e^{tL_0} \), is a semigroup generated by \( L_0 \), justifying the notation. This strategy is accomplished by an implicit operator splitting of \( L_0 \) into multiples of \( A \) and \( B \), using that \( A \) and \( B \) almost commute.

Recall that \( \delta_t(\sigma) = \theta(1 - e^{-\kappa t}) + \sigma e^{-\kappa t} \).

**Lemma 3.3.** Let \( g : I \to [0, \infty) \) be a continuous function. Assume that either \( g \) is bounded or that the parameter \( \lambda = 0 \) in the definition of the weight \( w(x) = e^{\lambda x} \). Then \( e^{tA}g B = e^{(g \circ \delta_t)B}e^{tA} \).

**Proof.** The result follows from \( e^{(g \circ \delta_t)B} \xi \circ \delta_t = (e^B \xi) \circ \delta_t = e^{tA}e^B \xi \).

The formula for the solution operator of \( \partial_t - L_0 \) will be conveniently expressed through the use of the following auxiliary function:

\[
(17) \quad \mathcal{D}_\kappa(t) = \mathcal{D}_\kappa(t, \sigma) := \frac{(\theta - \sigma)^2}{4\kappa}(1 - e^{-2\kappa t}) - \frac{\theta(\theta - \sigma)}{\kappa}(1 - e^{-\kappa t}) + \frac{1}{2}\theta^2 t.
\]

**Proposition 3.4.** The function \( \mathcal{D}(t, \sigma) \) given in Equation (17) is analytic in \((\kappa, t, \sigma) \in \mathbb{R}^3 \) and satisfies \( \mathcal{D}(0, \sigma) = 0 \) and \( \mathcal{D}(t, \sigma) > 0 \) for any \( t > 0 \) and any \( \sigma \in \mathbb{R} \).

**Proof.** The function \( \mathcal{D}(t, \sigma) \) is analytic since the singularity at zero is removable. We shall regard \( \mathcal{D}(t, \sigma) \) as a second order polynomial in \( \sigma \) with coefficients that are functions of the parameters \( t \) and \( \kappa \). We have that the leading coefficient \( \frac{1}{4\kappa} (e^{2\kappa t} - 1) \) is always positive as \( t > 0 \), so we only need to show that the discriminant of \( \mathcal{D}(t, \sigma) \) is non negative. We let \( f(t) \) be the discriminant of \( \mathcal{D}(t, \sigma) \) (regarded as a second-order polynomial in \( \sigma \), as mentioned above), so that

\[
(18) \quad f(t) = \frac{\theta^2}{2\kappa^2} [(2 + \kappa t)e^{-2\kappa t} - 4e^{-\kappa t} + 2 - \kappa t].
\]

We then have:

\[
\begin{align*}
 f'(t) &= \frac{\theta^2}{2\kappa^2} [3 - 2\kappa t]e^{2\kappa t} - 4e^{\kappa t} + 1, \\
 f''(t) &= 2\theta^2 [(1 - \kappa t)e^{2\kappa t} - e^{\kappa t}] = 2\theta^2 e^{2\kappa t} [1 - \kappa t - e^{-\kappa t}] < 0 \quad \text{for } t \neq 0.
\end{align*}
\]

It follows that \( f'(t) \) is decreasing, and hence \( f'(t) < f'(0) = 0 \) for \( t > 0 \). Consequently, \( f(t) \) is also decreasing, which gives \( f(t) < f(0) = 0 \) for positive \( t \).

This lemma allows us to define \( e^{\mathcal{D}(t)B} \), with \( \mathcal{D} \) as in Equation (17), if \( I \) is bounded or if \( \lambda = 0 \). We let then

\[
(19) \quad S(t) := e^{\mathcal{D}(t)B} e^{tA}.
\]
Then \( S(t) \) is a bounded operator, since it is the composition of bounded operators.
We will establish that \( S(t) \) is a \( c_0 \) semigroup generated by \( L_0 \) by splitting the proof in a few lemmas, for convenience. Recall that \( K_1 = H^2_0(I \times \mathbb{R}) \).

**Lemma 3.5.** For all \( t, s \geq 0 \), the family of operators \( S(t) \), defined in Equation (19), satisfies:

1. \( S(t)S(s) = S(t+s) \);
2. \( S(t)K_1 \subset K_1 \).

**Proof.** We first notice that \( \mathcal{D}(t) + \mathcal{D}(s) \circ \delta_t = \mathcal{D}(t+s) \), which is easy to check by direct calculation. By definition, using also Lemma 3.3, we have

\[
S(t)S(s) = e^{\mathcal{D}(t)B}e^{tA}e^{\mathcal{D}(s)B}e^{sA} = e^{\mathcal{D}(t)B}e^{(\mathcal{D}(s)\circ \delta_t)B}e^{tA}e^{sA} = e^{(\mathcal{D}(t)+\mathcal{D}(s)\circ \delta_t)B}e^{(t+s)A} = e^{\mathcal{D}(t+s)B}e^{(t+s)A} = S(t+s).
\]

This calculation completes the proof of the first part. The last part follows from Corollaries 3.2 and 2.19. \( \square \)

We recall that we assume \( \sigma \) is in a bounded interval \( I \subset (0, \infty) \).

**Lemma 3.6.** We have that for all \( j \geq 0 \),

\[
\| \partial^j_t \mathcal{D}(t)/t - \sigma^2/2 \|_{L^\infty(t)} \to 0 \quad \text{as } t \to 0, \quad t > 0.
\]

**Proof.** We observe that the function \( \partial^j_t \mathcal{D}(t)/t \), defined on \( I \times (0,1] \), extends to a continuous function on \( I \times [0,1] \). Since \( I \) is a bounded interval, this fact is enough to provide the result. \( \square \)

**Lemma 3.7.** The following limits in \( L^2_{\lambda} \) hold for the operators \( S(t) \) introduced in Equation (19):

(i) \( \lim_{t \to 0} S(t)\xi = \xi \) for all \( \xi \in L^2_{\lambda} \), and, similarly,
(ii) \( \lim_{t \to 0} t^{-1}(S(t)\xi - \xi) = L_0\xi \) for all \( \xi \in K_1 \).

**Proof.** By the semigroup property, the operators \( e^{tB} \) and \( e^{tA} \) are uniformly bounded if \( 0 \leq t \leq \epsilon \), for any fixed \( \epsilon > 0 \). Since \( I \) is a bounded interval, the functions \( \mathcal{D}(t) \) are uniformly bounded for \( t \leq \epsilon \). Moreover, \( \| \mathcal{D}(t) \|_{L^\infty(t)} \to 0 \) as \( t \searrow 0 \). By the definition of \( S(t) \), the first part of the lemma follows.

The second part of the lemma is proved in a similar fashion. Indeed, the relations \( S(t)K_1 \subset K_1 \) (see Lemma 3.5), \( \mathcal{D}(0) = \sigma^2/2 \) (see Lemma 3.6), the fact that \( e^{tA} \) is a \( c_0 \) semi-group that leaves \( K_1 \) invariant (Corollary 3.2), and Lemma 2.16 give that

\[
\partial_t (T(t)\xi)|_{t=0} = \partial_t (e^{\mathcal{E}(t)B}e^{tA}\xi)|_{t=0} = \lim_{t \to 0} t^{-1}(e^{\mathcal{E}(t)B}e^{tA}\xi - \xi) = \lim_{t \to 0} t^{-1}(e^{\mathcal{E}(t)B}e^{tA}\xi - \xi) + \lim_{t \to 0} t^{-1}(e^{tA}\xi - \xi) = \frac{\partial \mathcal{E}}{\partial t}(0)B\xi + A\xi = L_0\xi,
\]

whenever \( \xi \in K_1 \). \( \square \)

We have the following similar result for \( K_1 \), using also Corollary 2.19.

**Lemma 3.8.** The following limits in \( K_1 \) hold:

(i) \( \lim_{t \to 0} S(t)\xi = \xi \) for all \( \xi \in K_1 \), and, similarly,
(ii) \( \lim_{t \to 0} t^{-1}(S(t)\xi - \xi) = L_0\xi \) for all \( \xi \in K_1 \) such that \( L_0\xi \in K_1 \).

In particular, the second limit is valid if \( \xi \in H^4(I \times \mathbb{R}) \).
Lemmas 3.5, 3.7, and Lemma 3.8, finally imply the generation of the semi-group \( S(t) \) on both \( L^2 \) and \( \mathcal{K}_1 \).

**Theorem 3.9.** Let \( \kappa > 0 \) and \( I = (\alpha, \beta) \), with \( 0 < \alpha < \theta < \beta < \infty \), as before. Then, \( S(t) := e^{\Theta(t)B} e^{tA} \) defines a \( \mathcal{C}_0 \) semi-group \( e^{tL_0} \) on \( L^2 \), the generator of which coincides with \( L_0 \) on \( \mathcal{K}_1 \). Moreover, \( S(t) \) defines a \( \mathcal{C}_0 \) semi-group on \( \mathcal{K}_1 \).

**Proof.** The first part is an immediate consequence of Lemmas 3.5 and 3.7. The second part uses Lemma 3.8 instead. \( \square \)

We are now in the position to obtain an explicit formula for the kernel of the semi-group \( S(t) \) using formula (19). Obtaining explicit formulas is important in practice because it allows for very fast methods. This is one of the reasons Heston’s method [51] is so popular. Explicit formulas lead also to faster methods in solving the inverse problem of determining the implied volatility from option prices (see [12], for instance), and generally in model calibration using inference methods.

**Corollary 3.10.** Under the assumptions of Theorem 3.9, let \( h = h(\sigma, x) \in L^2(I \times \mathbb{R}) = e^{\lambda(x)} L^2(I \times \mathbb{R}) \) and set \( u(t) := S(t)h \). Then, for almost all \( \sigma \in I \):

\[
(21) \quad u(t, \sigma, x) = \frac{1}{\sqrt{4\pi t}} \int e^{-\frac{(x-y)^2}{4t}} h(\delta_t(\sigma), y) \, dy
\]

and \( u \) is a mild solution of the Initial Value Problem:

\[
\partial_t v - L_0 v = 0, \quad v(0) = h.
\]

If \( h \in \mathcal{K}_1 \), then \( u \) is a strong solution, and a classical solution provided that \( h \in C^{1,2}(I \times \mathbb{R}) \cap L^2(I \times \mathbb{R}) \).

3.2. **Lie algebra identities and semi-groups.** In the previous subsection we used implicitly commutator estimates between the operators \( A \) and \( B \). Lie algebra ideas can be exploited to derive another formula for the distributional kernel of \( e^{tL_0} \), which in turn will prove useful in Section 4. We collect in this subsection results pertaining to a general class of operators with properties similar to the operators \( A \) and \( B \), which, with abuse of notation, we continue to denote by \( A \) and \( B \).

**Remark 3.11.** Let \( V \) be a finite dimensional real vector space of (possibly unbounded) operators acting on some Banach space \( X \) and let \( A \) be a closed operator on \( X \) with domain \( D(A) \). We make the following assumptions

(i) All operators in \( V \) have the same domain \( \mathcal{K} \), which is endowed with a Banach space norm such that, for any \( B \in V \), \( B : \mathcal{K} \to X \) is continuous.

(ii) The operator \( A \) generates a \( \mathcal{C}_0 \) semi-group \( e^{tA} \), \( t \geq 0 \), of operators on \( X \) that leaves \( \mathcal{K} \) invariant and induces a \( \mathcal{C}_0 \) semi-group on \( \mathcal{K} \).

(iii) The space

\[
\mathcal{W} := \{ \xi \in \mathcal{K} \cap D(A) \cap D(ABe^{tA}) \, | \, (\forall) B \in V, \, t \geq 0 \text{ and } A\xi \in \mathcal{K} \}
\]

is dense in \( \mathcal{K} \) in its Banach space norm.

(iv) If \( B \in V \), the operator \([A, B]\) with domain \( \mathcal{W} \) is the restriction to \( \mathcal{W} \) of an operator in \( V \), unique by (iii), and denoted \( \text{ad}_A(B) \).

Then, denoting by \( e^{t\text{ad}_A} : V \to V \) the exponential of the endomorphism \( \text{ad}_A : V \to V \) of the finite dimensional space \( V \), we obtain the following Hadamard type formula

\[
(22) \quad e^{tA}B\xi = e^{t\text{ad}_A(B)}e^{tA}\xi, \quad (\forall) B \in V, \, \xi \in \mathcal{K}, \, t \geq 0.
\]
By density, we then obtain the result for $\xi$

Remark

This relation can be proved by considering the function $F : [0, \infty) \to X$

$$F(t) := e^{tA}B\xi - e^{-t\text{ad}_A}(B)e^{tA}\xi, \quad B \in V \text{ and } \xi \in W.$$  

Our assumptions imply that $F(t) \in D(A)$ for all $t$, that $F(t)$ is differentiable, and that $F'(t) = AF(t)$. By the uniqueness of strong solutions to this evolution equation [5, 76], it follows that $F(t) = 0$ for all $t \geq 0$, since $F(0) = 0$. This fact proves Formula (22) for $\xi \in W$. Since $F(t) \in X$ depends continuously on $\xi \in K$, Formula (22) for $\xi \in K$ follows from the density of $W$ in $K$. By replacing $B$ with $e^{-t\text{ad}_A}(B)$ in Formula (22), we obtain

$$e^{tA}e^{-t\text{ad}_A}(B)\xi = Be^{tA}\xi, \quad (\forall) B \in V, \xi \in K, \quad t \geq 0.\quad (23)$$

Let us assume that (the closures of) $B \in V$ and $B_1 := e^{t\text{ad}_A}(B)$ generate $c_0$ semi-groups of operators on $X$ denoted $e^{sB}$ and $e^{sB_1}$, respectively. The, we also obtain the formula

$$e^{tA}e^{sB} = e^{sB_1}e^{tA}, \quad \text{where} \quad B_1 := e^{t\text{ad}_A}(B) \in V, \quad t, s \geq 0,\quad (24)$$

as bounded operators on $X$. Indeed, for $\xi \in K$, the equality $e^{tA}e^{sB}\xi = e^{sB_1}e^{tA}\xi$ is obtained by differentiating $F(s) := e^{tA}e^{sB}\xi - e^{sB_1}e^{tA}\xi$ with respect to $s$ (see Equation (24)) and using Equation (22) (which takes care also that all the terms be defined) to obtain that $F(s) \in D(B)$ and that $F'(s) = BF(s)$. Since $F(0) = 0$, we obtain, by the uniqueness of solutions of $u' = Bu$, $u(s) \in D(B)$, that $F(s) = 0$. By density, we then obtain the result for $\xi \in X$.

We shall use the above remark in the following setting.

Remark 3.12. We keep the notation of Remark 3.11. Let $X := L^2(\mathbb{I} \times \mathbb{R}, V = \mathbb{R} \partial_x$ with $\partial_x$ acting on the first variable, and with domain $K := K_1$, where, we recall, $K_1 := H^2_1(\mathbb{I} \times \mathbb{R})$. As before, we let $A := \kappa(\theta - \sigma)\partial_x$ and consider the adjoint action of $A$ on $V$. Since $H^2_1(\mathbb{I} \times \mathbb{R}) \subset W$ and

$$A\partial_x - \partial_x A = [A, \partial_x] = [\kappa(\theta - \sigma)\partial_x, \partial_x] = [\kappa(\theta - \sigma)\partial_x, \partial_x] = \kappa\partial_x \in V,$$

it follows that $e^{tA}\partial_x = e^{\kappa t\partial_x}e^{tA}$. See also [64].

In the same spirit, we have the following remark.

Remark 3.13. We keep the same notation and assumptions as in Remark 3.11, in particular, the space $W$ has the same meaning. We now present a situation for which we can compute $e^{t(A+B)}$ in terms of $A$ and $B$ and the semi-groups that they generate, using the ideas of Remark 3.11. In addition to the four assumptions of Remark 3.11, we consider the following conditions:

(v) There exists a closed cone $C_+ \subset V$ that is invariant with respect to $e^{t\text{ad}_A}$, for all $t \in \mathbb{R}$, such that the closure of every $B \in C_+$ generates a $c_0$ semi-group;

(vi) $e^{tB_1}e^{sB_2} = e^{sB_2}e^{tB_1}$ for all $B_1, B_2 \in C_+$ and $s, t \geq 0$.

(vii) The function $C_+ \ni B \to e^{B}\xi \in X$ is continuous on $C_+$ for all $\xi \in X$.

Let $B \in C_+ \subset V$. We will show that if the four conditions above are satisfied (in addition to the four assumptions of Remark 3.11), then

$$e^{t(A+B)} = e^{tA}e^{b(t)},\quad (25)$$

where $b : [0, \infty) \to C_+$ is a suitable differentiable function with $b(0) = 0$. The proof of this result will be completed in the proof of Theorem 3.15. We first comment
briefly on the assumptions above. Let $B_i \in C_+$. It is known, for instance, that $e^{tB_1}e^{tB_2} = e^{tB_1+sB_2}$, which is an instance of the Trotter’s product formula [81]. Moreover, by results of Hille [53] and Langlands [64], $B_1$ and $B_2$ commute in an obvious sense. We also have that for $\xi \in \mathbb{K}$, the function $C_+ \ni B \to e^B\xi \in X$ is differentiable. More precisely, if $B(t) \in C_+$ depends differentiably on $t$, then $(e^{B(t)}\xi)' = e^{B(t)}B'(t)\xi = B'(t)e^{B(t)}\xi$. This follows either from the results of [64] or from Trotter’s product formula already mentioned. See [54] for a comprehensive introduction to the subject. See also [1, 14, 27, 29, 57, 78] for related results.

We turn now to Equation (25). By differentiating the right hand side with respect to $t$ and evaluating at $\xi \in X$, we formally obtain:

$$(e^{tA}e^{b(t)}\xi)' = Ae^{tA}e^{b(t)}\xi + e^{tA}(b'(t))e^{b(t)}\xi = \left[A + e^{t\text{ad}_A}(b'(t))\right]e^{tA}e^{b(t)}\xi.$$  

Motivated by Equation (25), we then set $(e^{tA}e^{b(t)}\xi)' = (A + B)e^{tA}e^{b(t)}\xi$, which gives $b'(t) = e^{-t\text{ad}_A}(B) \in C_+$. This condition can be verified, at least formally, by integrating this last formula to first see that $b(t) \in C_+$ for all $t$, since $C_+$ is closed and convex. Explicitly, let $\mathcal{E}(s) := (e^s-1)/s$, which is an entire function on $\mathbb{C}$. Then $\mathcal{E}(-t\text{ad}_A)$ is defined by holomorphic functional calculus and $b(t) = t\mathcal{E}(-t\text{ad}_A)(B)$, which is, as yet, just a formal result. Here $\mathcal{E}(-t\text{ad}_A)$ is simply the power series:

$$\mathcal{E}(-t\text{ad}_A) = \sum c_n(-t\text{ad}_A)^n$$

where $a_n(n+1)! = 1$ are the Taylor coefficients of $\mathcal{E}(s) := (e^s-1)/s = \sum c_n s^n$.

Of course, this procedure has to be justified independently or one has to make sense of all the steps in its derivation. In the previous subsections, we have chosen to verify independently Formula (19) for $X = L^2(I \times \mathbb{R})$, $V$ the space $\{pB\}$, with $p$ a polynomial of order $\leq 2$, $\mathbb{K} = L^2(I; H^2_{\lambda}(\mathbb{R}))$, and $C_+$ the set of polynomials that are $\geq 0$ on $\mathbb{R}$.

It is convenient to first prove the following Lemma.

Lemma 3.14. Let us assume that conditions (i-vii) are satisfied. Let $B \in C_+$, then $B_1 := t\mathcal{E}(-t\text{ad}_A)(B)$ and $B_2 := t\mathcal{E}(t\text{ad}_A)(B)$ are in $C_+$ and $e^{tA}e^{B_1} = e^{B_2}e^{tA}$.

Proof. We have already seen that $B_1 \in C_+$, since $B_1 = \int_0^t e^{-s\text{ad}_A}(B) \, ds$ is a closed, convex cone invariant for $e^{-s\text{ad}_A}$. Similarly, $B_2 = \int_0^t e^{s\text{ad}_A}(B) \, ds$ is in $C_+$ in view of formula (24), it is enough to prove that $e^{t\text{ad}_A}(B_1) = B_2$. Indeed, in view of the properties of the functional calculus, it is enough to check that $e^{tz}(e^{-tz} - 1)(-tz)^{-1} = t(e^{tz} - 1)(tz)^{-1}$, which is obviously true.

Let us summarize the above discussion in a formal result. Recall the function $\mathcal{E}(x) := (e^x - 1)/x$.

Theorem 3.15. Let us use the notation and the assumptions of Remarks 3.11 and 3.12. If $B \in C_+$, then $e^{t(A+B)} = e^{tA}e^{b(t)}$, for $b(t) := t\mathcal{E}(-t\text{ad}_A)(B)$.

Proof. We have that $b(t) \in C_+$ by Lemma 3.14. Let $S(t) := e^{tA}e^{b(t)}$. Then

$$S(t)S(s) = e^{tA}e^{b(t)}e^{sA}e^{b(s)} = e^{tA}e^{sA}e^{b_1(t)+b_2(s)} = e^{(t+s)A}e^{b_1(t)+b_2(s)}$$

where $b_1(t) = e^{-s\text{ad}_A}(b(t))$, by formula (24) and by Trotter’s product formula. We then compute

$$b_1(t) + b(s) = e^{-s\text{ad}_A}(t\mathcal{E}(-t\text{ad}_A)(B)) + s\mathcal{E}(-s\text{ad}_A)(B) = (t+s)\mathcal{E}(-(t+s)\text{ad}_A)(B) = b(s+t),$$

and evaluating at $\xi \in X$,

$$\left(\frac{d}{dt}e^{tA}e^{b(t)}\xi\right)|_{t=0} = A e^{tA}e^{b(t)}\xi + e^{tA}b'(t)e^{b(t)}\xi = \left[A + e^{t\text{ad}_A}(b'(t))\right]e^{tA}e^{b(t)}\xi.$$
by the properties of the functional calculus, since
\[
e^{-sz}tE(-tz) + sE(-sz) = e^{-sz}(e^{-tz} - 1)(-tz)^{-1} + s(e^{-sz} - 1)(-sz)^{-1}
\]
\[
= (-z)^{-1}\left[e^{-sz}(e^{-tz} - 1) + e^{-sz} - 1\right] = (s + t)(e^{-(s+t)z} - 1)\left(- (s + t)z\right)^{-1}
\]
\[
= (s + t)E(- (s + t)z).
\]
Therefore \(S(t)S(s) = S(t + s)\), for all \(t, s \geq 0\). To prove that \(S(t) = e^{t(A+B)}\), it is enough to check that \(S(t)\xi\) is differentiable for \(\xi \in W\) and that \((S(t)\xi)' = S(t)(A + B)\xi\), since \(S(t)\) is a semi-group consisting of uniformly bounded operators on compact subsets. Indeed, the relation \((S(t)\xi)' = S(t)(A + B)\xi\) would prove that \(S(t)\xi\) is continuous for \(\xi \in W\) (and hence everywhere, since \(S(t)\) consists of uniformly bounded operators on compact subsets and \(W\) is dense in \(X\)) and that the generator of \(S(t)\) is \(A + B\), by setting \(t = 0\). Now, for \(\xi \in W\), using the notation of Lemma 3.14, we have
\[
(S(t)\xi)' = (e^{b(t)}(tA)\xi)' = e^{b(t)}(tA)\xi + (e^{b(t)}u(t)e^{tA})\xi
\]
\[
= S(t)[A + e^{-t\text{ad}A}(u_t(t))]\xi = S(t)[A + e^{-t\text{ad}A}((e^{t\text{ad}A}(B)))]\xi.
\]
This completes the proof. \(\square\)

**Remark 3.16.** Let us use the notation of Theorem 3.15. Let us also assume that \(V = \sum_{a \in \mathbb{R}} V_a\), where
\[
[A, B_a] := AB_a - B_aA = aB_a, \quad \text{for any} \quad B_a \in V_a, \quad a \in \mathbb{R}.
\]
We can simplify the formula \(e^{t(A+B)} = e^{tA}e^{b(t)}(t) = tE(-t\text{ad}A)(B)\), even further, as follows. Let us write \(B = \sum_{a \in \mathbb{R}} B_a\), with \(B_a \in V_a\). (Of course, \(V_a = 0\), except for finitely many values \(a \in \mathbb{R}\), since \(V\) is assumed finite dimensional, so the sum \(\sum_{a \in \mathbb{R}} B_a\) is actually a finite sum). Then \(b(t) = tE(-t\text{ad}A)(B) = \sum_a f_a(t)B_a\), where \(f_a(t) = (1 - e^{-at})/a = tE(-at)\). Hence, this procedure gives the result
\[
(26) \quad e^{t(A+B)} = e^{tA}E_{\sum_a tE(-at)B_a} = e^{\sum_a tE(\text{ad}B)B_a} = e^{tA}e^{tE(\text{ad}B)}.
\]
We close by using the results just proved to derive an equivalent formula for \(S(T)\), which, by the smoothing properties of \(e^{tB}\), \(t > 0\), in \(x\), can also be used to show that \(u(t) = S(t)\xi\) defines a classical solution of \(\partial_t u - Lu = 0\) for \(t > 0\), when \(\xi \in C^1(I; L^2_0(\mathbb{R}))\). For this purpose, we introduce the function:
\[
(27) \quad \mathcal{E}(t, \sigma) := \frac{(\theta - \sigma)^2}{4\kappa} (e^{2\kappa t} - 1) - \frac{\theta(\theta - \sigma)}{\kappa} (e^{\kappa t} - 1) + \frac{1}{2} \theta^2 t.
\]
We notice that \(\mathcal{E}(t)\) is obtained from \(\mathcal{D}(t)\) (where \(\mathcal{D}\) is introduced in Equation (17)) by replacing \(\kappa\) with \(-\kappa\), therefore it retains its positivity (see Proposition 3.4 or Lemma 3.14). Applying the reasoning in the previous remark, we obtain the following alternative expression for \(S(t)\):
\[
(28) \quad S(t) := e^{\mathcal{D}(t)B}e^{tA} = e^{tA}e^{\mathcal{E}(t)B}.
\]

4. Mapping properties, asymptotic expansion, and error estimates

In this section, we prove mapping properties between weighted spaces for the semi-groups we constructed. We then use these results to compare the semi-groups \(e^{tL_0}\) and \(e^{tL}\). We continue to assume that \(I = (\alpha, \beta), 0 < \alpha < \theta < \beta < \infty\), and that \(\kappa > 0\).
4.1. **Mapping properties.** We shall need certain mapping properties for the semi-groups \( e^{tL} \) and \( e^{tL_0} \). Most of these results are consequences of the properties of analytic semi-groups. We begin with a preliminary lemma.

**Lemma 4.1.** Assume that \( I := (\alpha, \beta) \) is bounded and that \( \alpha > 0 \). Then there exists \( \epsilon > 0 \) such that \( \mathcal{D}(t, \sigma) \geq ct \) for \( \sigma \in I \) and \( t \in [0, 1] \).

**Proof.** We consider the function \( h(t, \sigma) := \mathcal{D}(t, \sigma)/t \) for \( \sigma \in [\alpha, \beta] \) and \( t \in (0, 1] \). By Proposition 3.4, \( h \) extends to a continuous function on \( [\alpha, \beta] \times [0, 1] \). By the assumption that \( \alpha > 0 \) and by Proposition 3.4 again, we have that \( h > 0 \) on \( [\alpha, \beta] \times [0, 1] \). Therefore \( \epsilon := \inf h > 0 \). \( \square \)

We recall also the following general fact.

**Remark 4.2.** If \( T \) generates a \( c_0 \) semi-group \( e^{tT} \) on a Banach space \( X \), then \( (e^{tT})^* \) will also be a semi-group (but the strong continuity property may fail). However, if \( X \) is reflexive, then \( (e^{tT})^* \) is strongly continuous and, in fact, \( (e^{tT})^* \) is a \( c_0 \) semi-group with generator \( T^* \) (see Corollary 1.10.6 in [76]). In other words, \( (e^{tT})^* = e^{tT^*} \), if \( X \) is reflexive. Moreover, if \( e^{tT} \) is an analytic semi-group, then \( (e^{tT})^* \) is also analytic since the function \( e^{tT^*} \) is holomorphic in a sector \( \Delta_{\delta} \), \( \delta > 0 \).

We first discuss mapping properties of \( e^{tL_0} \). From (28) and the analyticity of \( e^{tB} \), one expect \( e^{tL_0} \) to be smoothing in \( x \). The spaces \( H^s_{\lambda} (I \times \mathbb{R}) := H^s(I, H^s_{\lambda} (\mathbb{R})) \), used below, are discussed in more detail in (42).

**Lemma 4.3.** Let \( s \geq 0 \). There exists \( C_s > 0 \) such that, for all \( h \in L^2_s (I \times \mathbb{R}) \),

\[
\| e^{D(t)B} h \|_{H^s_{\lambda} (I \times \mathbb{R})} \leq C_s t^{-s/2} \| h \|_{L^2_{\lambda} (I \times \mathbb{R})}, \quad \text{for } t \in (0, 1].
\]

Consequently, \( \| \partial_x^k e^{tL_0} \| \leq Ct^{-k/2} \), where \( t \in (0, 1] \) and \( C \) is independent of \( t \). Moreover, \( \partial_x^k e^{tL_0} \) is continuous in \( t \).

Whenever not explicitly noted, all the norms \( \| \| \) below refer to the norm of vectors in \( L^2_{\lambda} (I \times \mathbb{R}) \) or of bounded operators on that space.

**Proof.** Let us assume first \( s = 2n \), for some positive integer \( n \). The norm \( \| g \|_{H^{2n}_{\lambda} (I \times \mathbb{R})} \) is equivalent to the norm \( \| g \| + \| B^n g \| \) (Corollary 2.6). It is therefore enough to show that there exists \( C'_s \) such that

\[
\| e^{D(t)B} h \| + \| B^n e^{D(t)B} h \| \leq C'_s t^{-n} \| h \|.
\]

since then the desired relation follows with \( C_s = CC'_s \). Lemma 4.1 gives

\[
\| e^{D(t)B} h \| + \| B^n e^{D(t)B} h \| = \| e^{D(t)B} h \| + \| e^{D(t)(-t)} B^n e^{tB} h \|
\]

\[
\leq C \left( \| h \| + \| B^n e^{tB} h \| \right) \leq C(e^t)^{-n} \| h \|,
\]

since \( e^{tB} \) is bounded on \( L^2_{\lambda} (I \times \mathbb{R}) \), if \( g \geq 0 \) is bounded measurable, and \( t^n B^n e^{tB} \) is also bounded on the same space (by Equation (35) for \( T = B \)). Here, we have used the assumption that \( I \) is bounded. This argument establishes the result for \( s = 2n \). For general \( s \geq 0 \), the result follows by complex interpolation.

To prove the last part, we write

\[
\partial_x^{2k} e^{tL_0} = \partial_x^{2k} (\mu_0 - B)^{-k} (\mu_0 - B)^k e^{D(t)B} e^{tA},
\]

where \( \mu_0 \) is large. We have that \( \partial_x^{2k} (\mu_0 - B)^{-k} \) is bounded (Theorem 2.5). Remark A.9, Lemmas 2.16 and 4.1 show that \( (\mu_0 - B)^k e^{D(t)B} \) depends smoothly on \( t \).
Then, $\partial^k_x e^{tL_0}$ depends continuously on $t$, as $e^{tA}$ does. Remark A.9 also gives that 
\[\| (\mu_0 - \hat{B})^k e^{B(t)B} \| \leq C t^{-k}.\] This implies that 
\[\| \partial^k_x e^{tL_0} \| \leq C t^{-k},\] and the desired estimate for all $k > 0$ follows by interpolation. \hfill \Box

In the same way, we obtain the following result.

**Lemma 4.4.** If \( h \in L^2(I \times \mathbb{R}) \), then
\[\| e^{tL} h \|_{H^2(I \times \mathbb{R})} \leq C t^{-s/2} \| h \|_{L^2(I \times \mathbb{R})}.\]

If \( P \) is a differential operator of order \( k \) with totally bounded coefficients on \( I \times \mathbb{R} \), then \( P e^{tL} \) and \( e^{tL} P \) extend to bounded operators on \( L^2(I \times \mathbb{R}) \) of norm \( \leq C t^{-k/2} \) that depend smoothly on \( t > 0 \).

**Proof.** The first part of the Lemma follows from (35), using that \((L - \mu) - \mu \) : \( L^2(I \times \mathbb{R}) \to H^2(I \times \mathbb{R}) \) continuously for \( \mu \) large enough, using interpolation, and using the analyticity of \( e^{tL} \).

Let \( P \) now be as in the statement of the lemma. Then \( P : H^2(I \times \mathbb{R}) \to L^2(I \times \mathbb{R}) \) is bounded. This implies the result for \( P e^{tL} \). The result for \( e^{tL} P \) is obtained by taking adjoints, since \( L^* \) is uniformly strongly elliptic with totally bounded coefficients and generates an analytic semi-group. \hfill \Box

In what follows, we will need the following result. All norms of operators are on \( L^2(I \times \mathbb{R}) \).

**Lemma 4.5.** The operator \( F(s) := e^{(t-s)L} \partial_s e^{sL} \) extends, for each \( s \in [0, t] \), to a bounded operator on \( L^2(I \times \mathbb{R}) \), and the resulting function is continuous in \( s \in [0, t] \) and differentiable for \( s \in (0, t) \).

Its derivative is the function
\[F'(s) = e^{(t-s)L}[\partial_s, L] e^{sL},\]
which satisfies \( \|F'(s)\| \leq C t^{-1} \), with \( C \) independent of \( 0 < s < t < 1 \).

**Proof.** Lemma 4.4 gives that both functions \( e^{(t-s)L} \) and \( \partial_s e^{sL} \) are continuous on \( (0, T] \) and infinitely many times differentiable on \( (0, t] \) as functions with values in the space of bounded operators; therefore, \( F(s) \) is continuous on \( [0, t] \). The formula for the derivative follow from the standard formula \((e^{sL})' = L e^{sL}\), which we note to be valid in norm, since \( L \) generates an analytic semi-group and \( s > 0 \). The continuity on \( [0, t] \) follows in the same way by considering \( e^{(t-s)L} \partial_s \) and \( e^{sL} \).

If \( s \leq t/2 \), since \([\partial_s, L]\) is a second order differential operator, Lemma 4.4 implies that \( e^{(t-s)L}[\partial_s, L] \) is bounded with norm \( \leq C(t-s)^{-1} \leq 2Ct^{-1} \). Hence, \( \|F'(s)\| \leq C t^{-1} \). The case \( s \geq t/2 \) is completely analogous using the bounds for \([\partial_s, L]\) provided by Lemma 4.4. \hfill \Box

### 4.2. A comparison of \( e^{tL} \) and \( e^{tL_0} \)

In this last section, we compare the semi-groups \( e^{tL_0} \) and \( e^{tL} \), by regarding \( L \) as a perturbation of \( L_0 \) for \( \nu \) sufficiently small. The motivation for this approach is that, while \( e^{tL} \) is better behaved as a semi-group, we lack an explicit formula for its distributional kernel.

We recall that we set \( L = L_0 + V \), where \( V = \nu L_1 + \nu \sigma \partial_x \partial_\sigma + \frac{\nu^2 \sigma^2}{2} \partial^2_\sigma \). We also recall that \( \mathcal{K}_1 = H^2(I \times \mathbb{R}) \) and \( \mathcal{K}_0 = H^2(I \times \mathbb{R}) \cap \{ u(\alpha, x) = u(\beta, x) = 0 \} \), where \( I = (\alpha, \beta) \) is a fixed bounded interval containing \( \theta \).

**Lemma 4.6.** Let \( \xi \in \mathcal{K}_1 \). Then \( F(s) := e^{(t-s)L} e^{tL_0} \xi \) is continuous on \([0, t]\) and differentiable on \((0, t)\) with values in \( L^2(I \times \mathbb{R}) \), with \( F'(s) = -e^{(t-s)L} V e^{sL_0} \xi \).
Proof. Since $\xi$ is in the domain of $L_0$ (which contains $K_1$, by Theorem 3.9), the function $\zeta(s) := e^{sL_0}\xi$ is differentiable for $s \geq 0$. But $e^{tL}$ is a $C_0$ semi-group, therefore Lemma 2.16 gives that $F(s) = e^{(t-s)L}\zeta(s)$ is continuous on $[0, t]$. Since $e^{tL}$ is an analytic semi-group, it follows in addition that $F(s)$ is differentiable for $s \in (0, t)$, by Lemma 2.18, and its derivative is $F'(s) = -e^{(t-s)L}Ve^{sL_0}\xi$.

We continue to assume that $\| \cdot \|$ refers to the norm in $L^2(I \times \mathbb{R})$ or the operator norm of bounded operators on this space.

**Lemma 4.7.** Let $\xi \in K_1$, then $e^{(t-s)L}L_1e^{sL_0}\xi$ depends continuously on $s$ and

$$\rho \nu^{-1}\|e^{(t-s)L}L_1e^{sL_0}\xi\| = \|e^{(t-s)L}\sigma^2\partial_\sigma \partial_\xi e^{sL_0}\xi\| \leq C(t-s)^{-1/2}s^{-1/2}\|\xi\|.$$ 
Consequently,

$$\left\| \int_0^t e^{(t-s)L}L_1e^{sL_0} \, ds \right\| \leq C\rho\nu.$$ 

**Proof.** Lemmas 4.3 and 4.4 show that $e^{(t-s)L}\sigma^2\partial_\sigma$ and $\partial_\sigma e^{s(L_0-\kappa)}\xi$ satisfy the assumptions of Lemma 2.16, so $e^{(t-s)L}\sigma^2\partial_\sigma \partial_\xi e^{s(L_0-\kappa)}\xi$ is continuous in $s$. Similarly, Lemmas 4.3 and 4.4 give

$$\|e^{(t-s)L}\sigma^2\partial_\sigma \partial_\xi e^{s(L_0-\kappa)}\xi\| \leq \|e^{(t-s)L}\sigma^2\partial_\sigma \| \|\partial_\sigma e^{s(L_0-\kappa)}\xi\| \leq C(t-s)^{-1/2}s^{-1/2}\|\xi\|.$$ 

The integral can be estimated by splitting the interval $[0, t]$ in two halves. □

To estimate the terms involving $L_2$, we exploit the next result.

**Lemma 4.8.** Let $\xi \in K_1$, then $\partial_\sigma e^{tL_0}\xi = e^{(t-L_0-\kappa)}\partial_\sigma \xi + \frac{\partial D(t, \sigma)}{\partial \sigma}Be^{tL_0}\xi$.

**Proof.** The main calculation is contained in Remark 3.12. More precisely, this is a direct calculation using Equation (19), together with Lemma 2.18, with Hadamard’s theorem (see Remarks 3.11 and 3.12), and with the fact that $\text{ad}_{L_0}(\partial_\sigma)\text{ad}_{A}(\partial_\sigma) = \kappa\partial_\sigma$. □

However, the terms in $L_2$ present some additional challenges, since $L_0$ is not elliptic.

**Lemma 4.9.** Let $\xi \in K_1$, then $e^{(t-s)L}L_2e^{sL_0}\xi$ depends continuously on $s$ and the following estimate holds:

$$\frac{2}{\rho^2}\|e^{(t-s)L}L_2e^{sL_0}\xi\| = \|e^{(t-s)L}\sigma^2\partial_\sigma^2 e^{sL_0}\xi\| \leq C(t-s)^{-1/2}\left(\|\partial_\sigma \xi\| + \|\xi\|\right).$$ 

Consequently,

$$\left\| \int_0^t e^{(t-s)L}L_2e^{sL_0} \, ds \right\| \leq C\sqrt{t}\left(\|\partial_\sigma \xi\| + \|\xi\|\right).$$ 

**Proof.** Lemma 4.8 gives

$$e^{(t-s)L}\sigma^2\partial_\sigma^2 e^{sL_0}\xi = e^{(t-s)L}\sigma^2\partial_\sigma \left(e^{s(L_0-\kappa)}\partial_\sigma \xi + \frac{\partial D(s, \sigma)}{\partial \sigma}Be^{sL_0}\xi\right).$$

As in the proof of Lemma 4.7, Lemmas 4.4 and 4.3 give that both $e^{(t-s)L}\sigma^2\partial_\sigma e^{sL_0}$ and $e^{(t-s)L}\sigma^2\partial_\sigma \frac{\partial D}{\partial \sigma}Be^{sL_0}$ define bounded operators that depend continuously on $s \in (0, t)$ in the strong operator topology. We estimate separately the norm of each of them. Again from Lemma 4.4, we obtain

$$\|e^{(t-s)L}\sigma^2\partial_\sigma e^{s(L_0-\kappa)}\| \leq \|e^{(t-s)L}\sigma^2\partial_\sigma\| \|e^{s(L_0-\kappa)}\| \leq C(t-s)^{-1/2}.$$
For the estimate of the second term, we first notice that \( \| \frac{\partial D(t, \sigma)}{\partial \sigma} \|_{L^\infty(I)} \leq C t \), since the function \( \frac{\partial D(t, \sigma)}{\partial \sigma} \) extends to a continuous function on \( \overline{T} \times [0, 1] \). Hence, \( \| \frac{\partial D(s, \sigma)}{\partial \sigma} Be^{sL_0} \| \leq \| s Be^{sL_0} \| \leq C \) by Lemma 4.3, and
\[
\left\| e^{(t-s)\lambda_2^2} \frac{\partial D(s, \sigma)}{\partial \sigma} Be^{sL_0} \right\| \leq \left\| e^{(t-s)\lambda_2^2} \partial_\sigma \right\| \| \frac{\partial D(s, \sigma)}{\partial \sigma} Be^{sL_0} \| \leq C (t-s)^{-1/2}.
\]
The last two displayed equations and Equation (30) then combine to give the first part of the statement. The last relation in the statement follows directly by integrating the first one. \( \square \)

Combining the previous two lemmas we obtain the following corollary.

**Corollary 4.10.** The family \( G(s) := e^{(t-s)L}Ve^{sL_0} \) consists of bounded operators on \( L^2_\lambda \). Moreover, for any \( \xi \in K_1 \), \( G(s)\xi \) is continuous and integrable in \( s \in (0, t) \) and we have:
\[
\left\| \int_0^t G(s)\xi ds \right\| := \left\| \int_0^t e^{(t-s)L}Ve^{sL_0}\xi ds \right\| \leq C \left( \rho \nu \|\xi\| + \nu^2 \sqrt{t} (\|\partial_\sigma \xi\| + \|\xi\|) \right).
\]

Lemma 4.6 and Corollary 4.10 then give:
\[
e^{tL}\xi - e^{tL_0}\xi = F(0) - F(t) = \int_0^t e^{(t-s)L}Ve^{sL_0}\xi ds.
\]
The final estimate is for \( \xi \in H^1(I, L^2_\lambda(\mathbb{R})) := \{ \xi \in L^2_\lambda(I \times \mathbb{R}), \partial_\sigma \xi \in L^2_\lambda(I \times \mathbb{R}) \} \).

**Theorem 4.11.** There is \( C > 0 \) such that
\[
\|e^{tL}\xi - e^{tL_0}\xi\| \leq C \nu \left( \|\xi\| + \nu \|\partial_\sigma \xi\| \right),
\]
for \( \xi \in H^1(I, L^2_\lambda(\mathbb{R})) \) and \( 0 \leq t \leq T \). The bound \( C \) depends on \( T \), but not on \( \xi \).

**Proof.** The statement was proved for \( \xi \in K_1 \). For general \( \xi \), it follows from the density of \( K_1 := H^2_\lambda(I \times \mathbb{R}) \) in \( H^1(I, L^2_\lambda(\mathbb{R})) \) and the continuity on \( H^1(I, L^2_\lambda(\mathbb{R})) \) of all the operators appearing on the left and right sides of the inequality. \( \square \)

The approach presented in this subsection can be iterated to derive higher-order approximate solutions in the parameter \( \nu \) by applying Duhamel’s formula repeatedly, provided the data is sufficiently smooth. In fact, it is possible to derive a regular perturbation expansion of the solutions in powers of \( \nu \), the terms of which are algorithmically computable. These are the focus of current work by the authors.

**Theorem 4.12 ([83]).** For any integer \( k \geq 2 \), denote
\[
E_k := e^{tL} - e^{tL_0} - \int_0^t e^{(t-s)L_0}Ve^{sL_0}ds - \ldots
- \int_0^t \ldots \int_0^{s_k-1} e^{(t-s_1)L_0}Ve^{(s_1-s_2)L_0}V \ldots e^{s_kL_0}ds_1 \ldots ds_k.
\]
Then, there exists \( C > 0 \), independent of \( \nu \), such that
\[
\|E_k\xi\| \leq C \nu^{k+1} (\nu^{k+1} \|\partial_\sigma^{2k+1}\xi\| + \ldots + \nu \|\partial_\sigma^{k+1}\xi\| + \|\partial_\sigma^k\xi\| + \ldots + \|\xi\|),
\]
for \( \xi \in H^{2,0}_\lambda(\mathbb{R} \times I) \cap H^{0,2k+2}_\lambda(\mathbb{R} \times I) \) and \( 0 \leq t \leq T \).

We close by observing that similar commutator estimates were obtained in [17, 18, 22, 44]. The main difficulty addressed in this work is that \( L_0 \) is not an elliptic operator.
APPENDIX A. SEMI-GROUPS AND SOLUTIONS OF EVOLUTION EQUATIONS

This section is devoted to briefly review known facts about abstract evolution equations and semi-groups of operators needed for the analysis. We also review facts about the function spaces we employ, in particular exponentially weighted Sobolev spaces. As remarked in the Introduction, these spaces are needed to handle initial conditions of the form $h(\sigma, x) := |e^\sigma - K|_{+}, (\sigma, x) \in (0, \infty) \times \mathbb{R}$, which arise in applications. We follow primarily, [5, 69, 76] (see also [9]).

A.1. Unbounded operators and $c_0$ semi-groups. We begin by recalling the notion of a semi-group generated by a linear operator. Throughout, $L(x)$ will denote the space of bounded linear operators on a Banach space $X$.

**Definition A.1.** Let $X$ be a Banach space. A strongly continuous or $c_0$ semi-group of operators on $X$ is a family of bounded operators $S(t) : X \to X$, $t \geq 0$, satisfying:

(i) $S(t_1 + t_2) = S(t_1)S(t_2)$, for all $t_i \geq 0$,
(ii) $S(0) = I$, where $I$ represent the identity operator on $X$,
(iii) $\lim_{t \to 0} S(t)x = x$, for all $x \in X$, where the limit is taken with respect to the topology of $X$.

It follows from (iii) that $S(t)$ is strongly continuous in $t$, that is, the map $S(\cdot)x : [0, \infty) \to X$ is continuous for every $x \in X$, hence the name.

We will need also the notion of analytic semi-groups. To this end, for a given $\delta > 0$, we let $\Delta_\delta$ denote the sector:

$$\Delta_\delta := \{ z = re^{i\theta}, \quad -\delta < \theta < \delta, \quad r > 0 \}.$$ 

Also, for any Banach space $X$, let $\mathcal{L}(X)$ denote the Banach algebra of bounded operators on $X$.

**Definition A.2.** Let $X$ be a Banach space. An analytic semi-group of operators on $X$ is a function $S : \Delta_\delta \cup \{0\} \to \mathcal{L}(X)$, $\delta > 0$, with the properties

(i) $S$ is analytic in $\Delta_\delta$;
(ii) $S(z_1 + z_2) = S(z_1)S(z_2)$, if $z_i \in \Delta_\delta \cup \{0\}$;
(iii) $S(0) = I$, the identity operator on $X$;
(iv) $\lim_{z \to 0} S(z)x = x$, for all $x \in X$.

The limit $\lim_{z \to 0} S(z)x$ is computed for $z \in \Delta_\delta$. An analytic semi-group is, in particular, a $c_0$ semi-group.

**Definition A.3.** The generator $T$ of a $c_0$ semi-group $S(t)$ on $X$ is the unbounded operator $T$ defined by:

$$T\xi := \lim_{t \to 0} t^{-1}(S(t)\xi - \xi),$$

for every $\xi \in X$ for which the limit exists. The collection of such vectors forms the domain of the operator.

It is known that the generator of a $c_0$ semi-group is closed and densely defined. We next review criteria for an unbounded operator $T$ to generate a $c_0$ semi-group $S(t)$. When this is the case, then $u(t) := S(t)h$ is a (suitable) solution of $u' - Tu = 0$, $u(0) = h$. A useful criterion for $T$ to generate a $c_0$ semi-group is provided by the Lumer-Phillips theorem, which we discuss next. Since two $c_0$ semi-groups with the same generator coincide (see e.g. [5, 76]), we shall write $S(t) = e^{tT}$ for the semi-group generated by $T$, if such a semi-group exists.
A.2. Dissipativity. In the following, \( \Re(z) = \Re z \) will denote the real part of \( z \in \mathbb{C} \). Let \( X \) be a Banach space and let \( X^* \) denote its dual. If \( x \in X \), the Hahn-Banach theorem implies, in particular, that the set
\[
\mathcal{F}(x) := \{ f \in X^*, f(x) = \|x\|^2 = \|f\|^2 \}
\]
is not empty.

**Definition A.4.** A (possibly unbounded) operator \( T \) on a Banach space \( X \) is called quasi-dissipative if there exists \( \mu \geq 0 \) such that, for every \( x \in D(T) \), there exists an \( f \in \mathcal{F}(x) \subset X^* \) with the property that and \( \Re(f(Tx - \mu x)) \leq 0 \).

This definition is simply saying that for some \( \mu > 0 \), the operator \( Tx - \mu x \) is dissipative. The numerical range \( \mathcal{R}(T) \), denoted \( \mathcal{R}(T) \), is the set
\[
\mathcal{R}(T) := \{ f(Tx), \|x\| = 1, f \in \mathcal{F}(x) \}.
\]
A quasi-dissipative operator \( T \) is thus one that has the property that
\[
\mathcal{R}(T) \subset \{ z \in \mathbb{C}, \, \Re(z) \leq \mu \} = \mu + \Delta_{\vartheta/2}
\]
with \( \Delta_{\vartheta} \) defined in Equation (32) and \( \Delta_{\vartheta} := \mathbb{C} \setminus \Delta_{\vartheta} \) its complement.

Quasi-dissipativity, together with some mild conditions on the operator \( T \) stated below, is sufficient for the generation of a \( c_0 \) semigroup, by the celebrated Lumer-Phillips theorem, which we now recall, in the form that we are going to use, for the benefit of the reader [5, 76].

**Theorem A.5 (Lumer-Phillips).** Let \( X \) be a Banach space and let \( T \) be a densely defined, quasi-dissipative operator on \( X \) such that \( T - \lambda \) is invertible for \( \lambda \) large. Then \( T \) generates a \( c_0 \) semi-group on \( X \).

By strengthening condition (34), we obtain the following similar theorem that yields generators of analytic semi-groups. The proof of this theorem is contained in the proof of Theorem 7.2.7 in [76].

**Theorem A.6.** Let \( X \) be a Banach space and let \( T \) be a densely defined operator on \( X \) such that \( \mathcal{R}(T) \subset \mu + \Delta_{\vartheta} \) for some \( \mu \in \mathbb{R} \) and some \( \vartheta > \pi/2 \). Assume also that \( T - \lambda \) is invertible for \( \lambda \) large. Then \( T \) generates an analytic semi-group.

We note that the assumption that \( T - \lambda \) be invertible in Theorems A.5 and A.6 implies that \( T \) is closed. The theorem above is especially useful when \( T \) is a uniformly strongly elliptic operator (see Definition 2.4) in view of the following Lemma, the proof of which is again contained in the proof of Theorem 7.2.7 in [76]. See also [66, 59].

**Lemma A.7.** Let \( P \) be an order \( 2m \) differential operator on some domain \( \Omega \subset \mathbb{R}^n \), regarded as an unbounded operator on \( L^2(\Omega) \) with domain \( D(P) \subset H^{2m}(\Omega) \). We assume that there exists \( C > 0 \) such that
\[
\Re(Pv, v) \leq -C^{-1}\|v\|^2_{H^{2m}(\Omega)} \quad \text{and} \quad |(Pv, v)| \leq C\|v\|_{H^{2m}(\Omega)}, \quad (\forall) \, v \in D(P).
\]
Then \( \mathcal{R}(P) \subset \Delta_{\vartheta} \) for some \( \vartheta > \pi/2 \).

From Theorem A.6 and Lemma A.7, we get the following corollary.

**Corollary A.8.** Let \( P \) be as in Lemma A.7 and assume that \( D(P) \) is dense in \( L^2(\Omega) \) and that \( P - \lambda \) is invertible for \( \lambda \) large. Then \( P \) generates an analytic semi-group on \( X \).
Remark A.9. We recall that, if $T$ is the generator of an analytic semi-group $e^{tT}$ on a Banach space $X$, then $T^n e^{tT}$ extends to a bounded operator on $X$ and there exists $C > 0$ such that

$$
\|T^n e^{tT}\| \leq C t^{-n}, \quad \text{for all } t \in (0, 1].
$$

A.3. Classical and other types of solutions. We consider the initial-value problem for abstract parabolic equations of the form (7) (that is $\partial_t u - Pu = F$, $u(0) = h \in X$) where $P$ is a (usually unbounded) operator on a Banach space $X$ and with domain $D(P)$. In our applications, $X$ will be a space of functions on $\Omega$, but it is convenient to consider this equation also abstractly, from the point of view of semi-groups of operators.

Definition A.10. A function $u : [0, T] \to X$ is a **strong solution** of the initial value problem (7) for $F \in C([0, T]; X)$ if

(i) $u$ is continuous for the norm topology on $X$ and $u(0) = h$;
(ii) $\partial_t u = u'$ is defined and continuous as a function $(0, T] \to X$;
(iii) $u(t) \in D(P)$ for $t \in (0, T]$; and
(iv) $u$ satisfies the equation $\partial_t u(t) - Pu(t) = F(t) \in X$, for $t \in (0, T]$.

We shall also need the following weaker form of a solution.

Definition A.11. A function $u : [0, T] \to X$ is called a **mild solution** of the initial-value problem (7) if $h \in X$, $F \in L^1([0, T], X)$, and

$$
\begin{align*}
  u(t) &= e^{tP} h + \int_0^t e^{(t-\tau)P} F(\tau) \, d\tau,
\end{align*}
$$

with equality as elements of $X$ pointwise in time $t \in (0, T]$.

The following remark recalls the connection between semi-groups and the various types of solutions of the Initial Value Problem (7).

Remark A.12. For the applications of interest in this work, we can reduce to homogeneous equations, that is $F(0) = 0$, as we assume now. We also assume that the operator $P$ generates a $c_0$ semi-group $e^{tP}$ on $X$. Then $u(t) := e^{tP} h$ is a mild solution for any $h \in X$. If, moreover, $h \in D(P)$ or if $P$ generates an analytic semi-group, then $u(t) := e^{tP} h$ is also a strong solution of Equation (7) (see [5, 69, 76], for instance).

We specialize to the case when $P$ is a $m$-th order partial differential operator defined on a domain $\Omega \subset \mathbb{R}^d$:

$$
P := \sum_{|\alpha| \leq m} a_\alpha \partial^\alpha,
$$

with coefficients $a_\alpha \in C^\infty(\Omega)$, and assume that $X$ is a space of functions on $X$, that is, $X \subset L^1_{\text{loc}}(\Omega)$. We also assume that the domain of $P$ contains the space of smooth functions with compact support in $\Omega$, and hence the same is satisfied by its adjoint.

We next recall the notion of classical and weak solutions. We use the convenient notation:

$$
u(t)(q) := u(t, q), \quad t \geq 0 \text{ and } q \in \Omega,
$$

which is in agreement with (7).
Definition A.13. A function \( u : [0, T] \times \Omega \to \mathbb{C} \) is a classical solution of the initial value problem (7) if

(i) \( u \) is continuous on \([0, T] \times \Omega\) and \( u(0, q) = h(q) \), for all \( q \in \Omega \);

(ii) \( \partial_t u = u' \) and \( \partial^\alpha u, \, |\alpha| \leq m \), are defined and continuous on \((0, T] \times \Omega\); and

(iii) \( u \) satisfies the equation \( \partial_t u - Pu = F \) pointwise in \((0, T] \times \Omega\).

If boundary conditions for \( u \) on \( \partial\Omega \) are given, we require them to be satisfied as equalities of continuous functions.

We note that in the abstract setting, strong solutions are often referred to as classical or strict solutions (see e.g. [69, 76]).

The following lemma follows from known results (see [69, Section 4.3, Chapter 5]).

Lemma A.14. Assume that there exists \( n \geq 0 \) such that \( D(P^n) \ni f \to \partial^\alpha f \in C(\overline{\Omega}) \) is continuous for all \( |\alpha| \leq m \). In addition, assume that \( P \) generates a \( c_0 \) semi-group on \( X \) and that \( F = 0 \). Then \( u(t) := e^{tP}h \) is a classical solution of Equation (7) for all \( h \in D(P^{n+1}) \).

We denote by

\[
P^t v := \sum_{|\alpha| \leq m} (-1)^{|\alpha|} \partial^\alpha (a_\alpha v)
\]

be the transpose of \( P \) (so that \( \int_\Omega (Pu)v dx = \int_\Omega u(P^t v) dx \) whenever \( u \) and \( v \) are compactly supported in \( \Omega \)).

Definition A.15. A function \( u : [0, T) \times \Omega \to \mathbb{C} \) is a weak solution of the initial value problem (7) if \( u, F \in L^1_{loc}([0, T) \times \Omega) \) and, for all \( \phi \in C^\infty_c([0, T) \times \Omega) \),

\[
\int_\Omega \left[ \phi(0, x)h(x) + \int_0^T (\partial_t \phi + P^t \phi) u dt + \int_0^T \phi F dt \right] dx = 0.
\]

If, moreover, \( u \) is also a classical solution on \([\delta, T]\) for all \( \delta > 0 \), \( [T < R] \), we shall say that \( v \) is a classical solution on \((0, R)\).

Again, the following lemma is well-known (see e.g. [69, 76]).

Lemma A.16. Assume that \( P \) generates a \( c_0 \) semi-group on \( X \). Then \( u(t) := e^{tP}h \) is a weak solution of the homogeneous Initial-Value Problem (7) with \( F = 0 \) for all \( h \in X \).

Combining the two lemmas above we obtain.

Proposition A.17. Assume that \( D(P^n) \ni f \to \partial^\alpha f \in C(\overline{\Omega}) \) is continuous for all \( |\alpha| \leq m \), for some \( n \geq 0 \). Assume in addition that \( P \) generates an analytic semi-group on \( X \) and that \( F = 0 \). Then, for all \( h \in X \), \( \bar{u}(t) := e^{tP}h \) is a classical solution on \((0, \infty)\) of the IVP (7).

A.4. Function spaces. We introduce here the weighted Sobolev spaces that we need in this paper and recall some of their main properties. Let \( \Omega \subset \mathbb{R}^d \) be an open subset, as in the previous subsection, and let \( w \in L^1_{loc}(\Omega) \) satisfy \( w \geq 0 \). If \( X \) is any Banach space of functions on \( \Omega \) with norm \( \| \cdot \|_X \), we define

\[
wX := \{ w\xi, \, \xi \in X \},
\]
with the norm $\|w\xi\|_{wX} := \|\xi\|_X$. Thus, if $p < \infty$, if $X = L^p(\Omega, d\mu)$, and if $w > 0$ almost everywhere with respect to $\mu$, $\mu \geq 0$, then $wX = L^p(\Omega, w^{-1/p} d\mu)$. Of course, for any linear operator $T$ we have

$$T : wX \to wX$$

is bounded if, and only if $w^{-1}Tw : X \to X$ is bounded.

In fact, these two operators are unitarily equivalent.

In this work, we choose weights of the form $w := e^{\lambda(x)}$, where $\langle \cdot \rangle$ denotes the Japanese bracket:

$$\langle x \rangle := \sqrt{1 + x^2},$$

and $\lambda \in \mathbb{R}$ is a parameter. This choice of the weight function $w$ is justified by the specific form of the initial data $h(x) := |e^x - K|_+$ for the Cauchy problem for the $\lambda$SABR model (1). For simplicity, we usually write

$$H^m(\mathbb{R}) := e^{\lambda(x)}H^m(\mathbb{R}) = \{ f : \mathbb{R} \to \mathbb{C}, e^{-\lambda(x)}f \in H^m(\mathbb{R}) \} = \{ f : \mathbb{R} \to \mathbb{C}, e^{-\lambda(x)}\partial^i f \in L^2(\mathbb{R}), i \leq m \},$$

where the last equality is valid due to the fact that the weight $w(x) = e^{\lambda(x)}$ has the property that $w^{-1}\partial^i w$ forms a bounded family as operators on $H^m(\mathbb{R})$ (by writing $f = wg$, with $g \in H^m(\mathbb{R})$). We also let $L^2_\lambda = H^0_\lambda$.

Let $I$ be a closed interval in $\mathbb{R}$. We consider, similarly, the spaces

$$H^{i,j}_\lambda(I \times \mathbb{R}) := wH^i(I; H^j(\mathbb{R})) = \{ u, \partial^\alpha_\sigma \partial^\beta x u \in L^2_\lambda(I \times \mathbb{R}), \alpha \leq i, \beta \leq j \}

= \{ u, \partial^\alpha_\sigma \partial^\beta_x (e^{-\lambda(x)} u) \in L^2(I \times \mathbb{R}), \alpha \leq i, \beta \leq j \} = H^i(I; H^j_\lambda(\mathbb{R})),

where $w$ is viewed as a function of $x$ and $\sigma$, constant in $\sigma$. When $i = j$, we will simply write $H^i_\lambda(I \times \mathbb{R})$ for $H^{i,i}_\lambda(I \times \mathbb{R})$.

References


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