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# On symmetric differential delay equations with cubic nonlinearity

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**ABSTRACT:** The paper deals with special symmetric periodic solutions of the differential delay equation  $\dot{x}(t) = \alpha x(t-1)[-1 + \beta x^2(t) + \gamma x^2(t-1)]$ ,  $\alpha > 0, \beta, \gamma \in \mathbb{R}$ . They are slowly oscillating solutions satisfying the symmetry property  $x(t+2) = -x(t)$ ,  $\forall t \in \mathbb{R}$ . The problems of existence, uniqueness, bifurcation, and stability of SSPSs are discussed and new results derived. They complement recent results obtained in (Ivanov and Lani-Wayda, 2006).

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## 1 INTRODUCTION

The differential delay equation (DDE)

$$\dot{x}(t) = \alpha f(x(t), x(t-1)), \alpha \in \mathbb{R}_+ \quad (1)$$

is called *symmetric* if function  $f$  is even in the first variable and odd in the second one, that is,

$$f(-x, y) = f(x, y) = -f(x, -y) \quad \text{for all } (x, y) \in \mathcal{U}, \quad (2)$$

where  $\mathcal{U}$  is the entire plane  $\mathbb{R}^2$  or a part of it. A periodic solution  $x = p(t)$  of equation (1) is called symmetric if

$$p(t + \omega) = -p(t) \quad \text{for all } t \in \mathbb{R} \quad \text{and some } \omega > 0.$$

Of particular interest are the so-called special symmetric periodic solutions (SSPSs for brief), when  $\omega = 2$ . The SSPSs are slowly oscillating periodic solutions with period 4. In order for the slowly oscillating solutions to exist one needs to assume the negative feedback property on  $f$ :

$$y \cdot f(x, y) < 0 \quad \forall (x, y) \in \mathcal{U}, y \neq 0. \quad (3)$$

The symmetry condition (2) and the negative feedback condition (3) are the standard assumptions in the studies of SSPSs of equation (1). For the existence of solutions of DDE (1) the Lipschitz continuity of  $f$  is normally required. Here, for the sake of simplicity and in view of the cubic nonlinearity (4) considered we assume that  $f$  has continuous partial derivatives  $f_x$  and  $f_y$  on any bounded symmetric part of  $\mathbb{R}^2$  in the domain  $\mathcal{U}$ .

Note that the symmetric equation  $\dot{x}(t) = f(x(t), x(t-\tau))$  with the general constant delay  $\tau > 0$  can be transformed, via the change  $t = \tau s$  of the independent variable, to the parametric form (1) with  $\alpha = \tau$ .

We discuss the problems of existence, uniqueness, bifurcation, stability/instability of SSPSs of equation (1) with the cubic nonlinearity  $f$

$$f(x, y) = y(-1 + \beta x^2 + \gamma y^2), \quad \beta, \gamma \in \mathbb{R}. \quad (4)$$

The form of the cubic nonlinearity  $f$  given by (4) is the most general one satisfying the symmetry (2) and the negative feedback (3) assumptions. Some of the results for SSPSs with the cubic  $f$ , including the stability and uniqueness, were recently obtained in (Ivanov and Lani-Wayda, 2006). Here we emphasize the qualitative analysis of the corresponding ODE system (5) and give the precise conditions for the existence of SSPSs in terms of the parameters. The new results of this paper should be viewed as complimentary to those given by Ivanov and Lani-Wayda.

## 2 PRELIMINARIES AND BACKGROUND

This section contains some well known results for symmetric DDEs necessary for the subsequent exposition and analysis in Section 3. They are given for the general equation (1) and can be found, for example, in references [2, 4].

### 2.1 Existence

Associated with DDE (1) is the following system of ordinary differential equations (ODE) in the plane

$$\dot{x} = \alpha f(x, y), \quad \dot{y} = -\alpha f(y, x). \quad (5)$$

The phase portrait of dynamical system (5) is symmetric with respect to  $x = 0, y = 0, y = x$ , and  $y = -x$ . The vector field at symmetric points is asymmetric: the symmetric image of the vector field at the symmetric point equals in the absolute value but is of the opposite direction to the vector field at a given point; Some neighborhood  $\mathcal{U}$  of the steady state  $(x, y) = (0, 0)$  consists of a one-parameter family of closed trajectories: the trajectory  $(x, y)$  through  $(x_0, 0), 0 < x_0 < \delta$  is a closed one and fulfills the symmetry  $x(t + \frac{\omega}{2}) = -x(t), \forall t \in \mathbb{R}$ , where  $\omega$  is the period of  $(x, y)$ . The principal relationship between DDE (1) and ODE system (5) is given by the following statement

**THEOREM 1** DDE (1) has a SSPS if and only if ODE (5) has a closed symmetric trajectory of minimal period 4.

## 2.2 Primary Branch and Bifurcation

Any SSPS  $x = p(t)$  of equation (1) can be normalized so that  $p(0) = 0$  and  $\dot{p}(0) > 0$ . In this case the quantity  $z := p(1)$  is called the *amplitude* of the SSPS.

Under the above assumptions of the continuity of partial derivatives  $f_x$  and  $f_y$  DDE (1) possesses the so-called *primary branch*, PB, of SSPSs. The nonlinearity  $f$  can be normalized so that  $f_v(0, 0) = -1$  with  $\alpha > 0$  being the parameter (this is the case for cubic  $f$ ). The primary branch is defined by

$$\text{PB} := \{(\alpha, z) : \text{DDE}(f) \text{ has a SSPS with amplitude } z\}.$$

Its existence is given by the following result from (Dormayer and Ivanov, 1999).

**THEOREM 2** There are continuous maps  $\alpha(z) : \mathbb{R}^+ \rightarrow \mathbb{R}$  and  $x : \mathbb{R} \times \mathbb{R}^+ \rightarrow \mathbb{R}$  such that for  $z > 0$ ,  $x(t, z)$  is a SSPS of equation (1) with the amplitude  $z$ .

For the cubic equation (1) the PB is a smooth curve in the  $(\alpha, z)$  plane. It can be viewed as a function  $\alpha$  of the independent variable  $z$ . The curve bifurcates off at  $\alpha = \frac{\pi}{2}$  with  $\frac{d\alpha}{dz}(0) = 0$ . The conditions for the direction of the bifurcation (forward or backward) are known (Dormayer and Ivanov, 1999): it is forward bifurcation if  $f_{yyy}(0, 0) + f_{xxy}(0, 0) > 0$  and backward one if  $f_{yyy}(0, 0) + f_{xxy}(0, 0) < 0$ . The forward bifurcation implies the stability of SSPSs with small amplitudes around the bifurcation point on the primary branch, while the backward bifurcation means the small SSPSs are unstable. This follows from the following result in (Dormayer and Ivanov, 1999).

**THEOREM 3** There exist a  $C^1$ -map  $s : \mathbb{R} \rightarrow \mathbb{R}$  and  $\varepsilon > 0$  such that for every SSPS of equation (1) with amplitude  $z \in (0, \varepsilon)$   $s(\alpha)$  is a nontrivial Floquet multiplier with greatest norm. If additionally  $f_{yyy}(0, 0) + f_{xxy}(0, 0) \neq 0$  then

$$s(\alpha^*) = 1, \quad s'(\alpha^*) = \frac{16\pi}{4 + \pi^2} \cdot f_v(0, 0).$$

## 2.3 Uniqueness

A SSPS of DDE (1) is unique if the corresponding system (5) has a unique symmetric periodic trajectory of minimal period 4. We shall use here the considerations and a uniqueness statement from (Ivanov, 1999).

Consider the function

$$\varphi(u, v) := \frac{f(u, v)}{v}, \quad v \neq 0,$$

where  $f$  is from the right hand side of equation (1). Note that  $\varphi(u, v)$  is an even function in both variables in its domain.

One has the standard ordering in the plane with respect to the positive cone  $\mathbb{R}_+^2 := \{(x, y) \in \mathbb{R}^2 \mid x > 0, y > 0\}$ . It is said that  $(u_1, v_1) < (u_2, v_2)$  if one of the following (a), (b), or (c) holds: (a)  $u_1 < u_2$  and  $v_1 < v_2$ ; (b)  $u_1 = u_2$  and  $v_1 < v_2$ ; (c)  $u_1 < u_2$  and  $v_1 = v_2$ . As usual,  $\varphi(u, v)$  is said to be increasing (decreasing) in  $\mathcal{U}$  if for all  $(u_1, v_1) < (u_2, v_2)$  in  $\mathcal{U}$  one has  $\varphi(u_1, v_1) < \varphi(u_2, v_2)$  (or  $\varphi(u_1, v_1) > \varphi(u_2, v_2)$ ).

Let  $T_u$  be the period of the periodic trajectory of system (5) through the initial point  $(u, 0)$ ,  $0 < u < u_0$ .

**THEOREM 4** Suppose  $\varphi(u, v)$  is increasing (decreasing) in  $\mathcal{U} \cap \mathbb{R}_+^2$ . Then  $T_u$  is increasing (decreasing) in  $u$ .

## 2.4 Stability

Stability of SSPSs is determined by the location of their Floquet multipliers in the complex plane. Floquet multipliers are the eigenvalues of the monodromy operator  $V$ , the shift by the period 4 along the solutions of the variational equation

$$\begin{aligned} \dot{v}(t) &= f_x(p(t), p(t-1))v(t) + f_y(p(t), p(t-1))v(t-1) = \\ &:= a(t)v(t) + b(t)v(t-1). \end{aligned} \quad (6)$$

Due to the symmetry assumptions (2) the coefficients  $a(t)$  and  $b(t)$  of equation (6) are periodic with period 2. Therefore, for the shift operator  $W$  by time 2 along solutions of equation (6) one has  $V = W \circ W$ . If  $\mu$  is an eigenvalue of  $W$  then  $\lambda = \mu^2$  is an eigenvalue of  $V$ . Eigenvalues of  $W$  are called *semi-Floquet* multipliers. The semi-Floquet multipliers can be completely characterized in terms of the so-called *characteristic function*. The characteristic function was first introduced and studied in (Walther, 1983).

Set  $c(t) = a(t+1)$ ,  $d(t) = b(t)$ ,  $t \in \mathbb{R}$ , and consider the following 2-periodic system of ODEs:

$$\frac{d}{dt} \begin{pmatrix} \eta \\ \xi \end{pmatrix} = \begin{pmatrix} a(t) & b(t) \\ \frac{1}{\mu}d(t) & c(t) \end{pmatrix} \cdot \begin{pmatrix} \eta \\ \xi \end{pmatrix} := A(t, \mu) \cdot \begin{pmatrix} \eta \\ \xi \end{pmatrix}. \quad (7)$$

Let  $S(t, \mu)$  be the fundamental matrix solution of system (7). The characteristic function  $r(\mu)$  is given by the following expression

$$r(\mu) := \det \left[ S(1, \mu) - \begin{pmatrix} 0 & \mu \\ 1 & 0 \end{pmatrix} \right].$$

The relationship between the Floquet multipliers  $\lambda = \mu^2$  and the solutions of the characteristic equation  $r(\mu) = 0$  is given by the following statement.

**THEOREM 5**  $\mu$  is a semi-Floquet multiplier if and only if it is a zero of the characteristic function  $r$ .

### 3 ANALYSIS OF ODE SYSTEM IN PLANE

#### 3.1 Equilibria and phase portrait

The equilibria of dynamical system (5) in the plane with the cubic nonlinearity (4) are found from the algebraic system

$$\alpha y(-1 + \beta x^2 + \gamma y^2) = 0, \quad -\alpha x(-1 + \beta y^2 + \gamma x^2) = 0. \quad (8)$$

System (8) always has the central (trivial) equilibrium  $(x_c, y_c) = (0, 0)$ . If  $\beta \leq 0$  and  $\gamma \leq 0$  it is the only equilibrium of the system. In this case, the phase portrait of system (5) is a one-parameter family of symmetric closed trajectories filling up the entire plane.

We look next for the equilibria located on the coordinate axes. They are found from the assumption  $x \neq 0$  and  $y = 0$  (or  $y \neq 0$  and  $x = 0$ , due to the symmetry of the phase portrait). By solving system (8) for this case, one necessarily has  $\gamma > 0$ , and the four equilibria are given by

$$\left(\frac{1}{\sqrt{\gamma}}, 0\right), \quad \left(-\frac{1}{\sqrt{\gamma}}, 0\right), \quad \left(0, \frac{1}{\sqrt{\gamma}}\right), \quad \left(0, -\frac{1}{\sqrt{\gamma}}\right). \quad (9)$$

Next step is to look for the equilibria located on the diagonals  $y = x$  and  $y = -x$ . They are found by solving system (8) with the assumption  $|x| = |y|$ . One necessarily has  $\beta + \gamma > 0$ , and the equilibria are given by

$$\left(\frac{1}{\sqrt{\beta + \gamma}}, \frac{1}{\sqrt{\beta + \gamma}}\right), \quad \left(\frac{1}{\sqrt{\beta + \gamma}}, -\frac{1}{\sqrt{\beta + \gamma}}\right), \quad \left(-\frac{1}{\sqrt{\beta + \gamma}}, \frac{1}{\sqrt{\beta + \gamma}}\right), \quad \left(-\frac{1}{\sqrt{\beta + \gamma}}, -\frac{1}{\sqrt{\beta + \gamma}}\right). \quad (10)$$

Thus, for all  $\alpha > 0$  the parametric plane  $(\beta, \gamma)$  can be divided into four regions depending on the number and type of the equilibria the system (5) has (see Fig. 1).

**Domain 1:**  $\gamma \leq 0$  and  $\beta + \gamma \leq 0$ . System (5) has the only equilibrium  $(0, 0)$ . The entire phase plane  $(x, y)$  is filled up with closed symmetric trajectories encompassing the steady state with the counterclockwise direction of the motion along the trajectories.

**Domain 2:**  $\gamma \leq 0$  and  $\beta + \gamma > 0$ . In addition to the trivial one system (5) has exactly four more equilibria given by (10). They all are the saddle points. There are four symmetric separatrices connecting the equilibria with the counterclockwise motion along them and between the equilibria. The central part about  $(0, 0)$  of the phase plane, formed and bounded by the separatrices, is filled up with closed symmetric trajectories with the counterclockwise motion along them.

A typical representative of systems falling within Domain 2 comes from the well known Jones' equation

$$\dot{x}(t) = -\alpha x(t-1)[1 - x^2(t)], \quad \alpha > 0,$$

which is the partial case of (4) with  $\beta = 1$  and  $\gamma = 0$ . The corresponding system (5) is given by

$$\dot{x} = -\alpha y(1 - x^2), \quad \dot{y} = \alpha x(1 - y^2).$$

It has five equilibria,  $(0, 0)$ ,  $(1, 1)$ ,  $(1, -1)$ ,  $(-1, 1)$ ,  $(-1, -1)$ , and four separatrices,  $x = 1$ ,  $y = 1$ ,  $x = -1$ ,  $y = -1$ . The central square  $|x| < 1$ ,  $|y| < 1$  is filled up with the closed symmetric trajectories. The motion along the trajectories is counterclockwise there (see Fig. 2).

**Domain 3:**  $\gamma > 0$  and  $\beta + \gamma \leq 0$ . In addition to the trivial one system (5) has exactly four more equilibria given by (9). They all are the saddle points. There are four symmetric separatrices connecting the equilibria with the counterclockwise motion along them. The central part of the phase space, formed and bounded by the separatrices, is filled up with closed symmetric trajectories with the counterclockwise motion along them.

An example of the system from domain 3 is given by the parametric values  $\beta = -2$ ,  $\gamma = 1$ . The corresponding system (5) has four equilibria  $(1, 0)$ ,  $(0, 1)$ ,  $(-1, 0)$ ,  $(0, -1)$ . They all are the saddle points connected by symmetric separatrices. The central part formed by the separatrices contains a one-parameter family of closed trajectories. The direction of motion along these trajectories is counterclockwise (see Fig. 3).

**Domain 4:**  $\gamma > 0$  and  $\beta + \gamma > 0$ . System (5) has precisely nine equilibria: the central one  $(0, 0)$ , four given by (9), and four given by (10). As in the case of Domain 3, the steady states (9) all are the saddle points. There are four symmetric separatrices connecting the equilibria with the counterclockwise motion along them. The central part of the phase space, formed and bounded by the separatrices, is filled up with closed symmetric trajectories with the counterclockwise motion along them. The remaining four equilibria given by (10) are all centers. They are located outside the central region formed by the separatrices of the first four equilibria.

A typical representative of systems with parameters in Domain 4 is given by the equation

$$\dot{x}(t) = -\alpha x(t-1)[1 - x^2(t-1)], \quad \alpha > 0,$$

which is a partial case of (4) with  $\beta = 0$  and  $\gamma = 1$ . The corresponding system (5) in the plane

$$\dot{x} = -\alpha y(1 - y^2), \quad \dot{y} = \alpha x(1 - x^2)$$

has nine equilibria

$$(0, 0), (1, 0), (0, 1), (-1, 0), (0, -1), (1, 1), (1, -1), (-1, 1), (-1, -1).$$

The equilibria  $(1, 0)$ ,  $(0, 1)$ ,  $(-1, 0)$ ,  $(0, -1)$  are all saddle points with the connecting separatrices  $x + y = 1$ ,  $x + y = -1$ ,  $x - y = 1$ ,  $x - y = -1$ . The central square  $|x + y| < 1$ ,  $|x - y| < 1$  is filled up with the symmetric closed trajectories. The remaining four equilibria  $(1, 1)$ ,  $(1, -1)$ ,  $(-1, 1)$ ,  $(-1, -1)$  all are the centers located in one-sided infinite stripes bounded by three segments of the separatrices. See Fig. 4 for more details.

Finally note that system (5) with the cubic nonlinearity (4) is a Hamiltonian one with the Hamilton function

$$H(x, y) = -2(x^2 + y^2) + 2\beta x^2 y^2 + \gamma(x^4 + y^4).$$

### 3.2 Existence, uniqueness, bifurcation, and stability

In this subsection we discuss the questions of the existence, uniqueness, bifurcation, and stability of SSPSs of DDE (1) with the cubic nonlinearity (4). We note that some of these questions are addressed and discussed in (Ivanov and Lani-Wayda, 2006). The results given here should be viewed as complimentary ones.

#### 3.2.1 Existence

The existence of a SSPS of DDE (1) follows from the existence of the closed symmetric periodic trajectories of system (5) of period 4 (see Theorem 1). We shall use the intermediate value theorem for the period  $T_a$  of the symmetric periodic trajectory passing through the initial point  $(a, 0)$ ,  $a > 0$

We consider the case when system (5) has an equilibrium different from the central one  $(0, 0)$ . It is the case of parametric Domains 2, 3, and 4, when there are symmetric regions in the phase space of system (5) containing  $(0, 0)$  and bounded by the separatrices of the saddle equilibria. The regions are filled up by the symmetric periodic trajectories. In all three cases there exists a finite positive  $a_* > 0$  such that for every  $0 < a < a_*$  the trajectory through  $(a, 0)$  is the symmetric one, and  $\lim_{a \rightarrow a_*} T_a = +\infty$ . The period  $T_a$  can also be estimated around the origin  $(0, 0)$  following the approach in (Ivanov, 1999). One has  $\lim_{a \rightarrow 0} T_a = \frac{2\pi}{\alpha}$ . Therefore, the existence of a SSPS of DDE (1) follows for all  $\alpha > \frac{\pi}{2}$ .

In this case the primary branch PB of SSPSs lies in the strip  $\alpha > 0$ ,  $0 \leq z < a_*$ , for some  $a_* > 0$ , and extends to  $\alpha > \frac{\pi}{2}$  with  $\lim_{z \rightarrow a_*} \alpha(z) = +\infty$ .

#### 3.2.2 Bifurcation of Primary Branch

As it was mentioned above, the direction of bifurcation of the Primary Branch PB at  $(\frac{\pi}{2}, 0)$  is known together with the stability of SSPSs with the small amplitude (Dormayer and Ivanov, 1999). If  $\beta + 3\gamma > 0$  one has the forward Hopf bifurcation with the stable SSPSs; while  $\beta + 3\gamma < 0$  yields the backward Hopf bifurcation with unstable SSPSs. In addition, if  $\beta < 0$  and  $\gamma < 0$  the SSPS is unique. Note, however, that the uniqueness does not imply the stability, which is not known in general away from the bifurcations point and beyond the conditions given in (Ivanov and Lani-Wayda, 2006).

#### 3.2.3 Uniqueness

As it was stated in the introductory section the uniqueness of SSPSs is tied to the monotonicity of the function

$$\varphi(x, y) := \frac{f(x, y)}{y} = -1 + \beta x^2 + \gamma y^2, \quad (x, y) \in \mathbb{R}_+^2 \cap \mathcal{U}.$$

It is easily seen that  $\varphi$  is increasing if  $\beta > 0$  and  $\gamma > 0$ , and it is decreasing if  $\beta < 0$  and  $\gamma < 0$ . Therefore, in each of these cases the SSPS is unique provided it exists.

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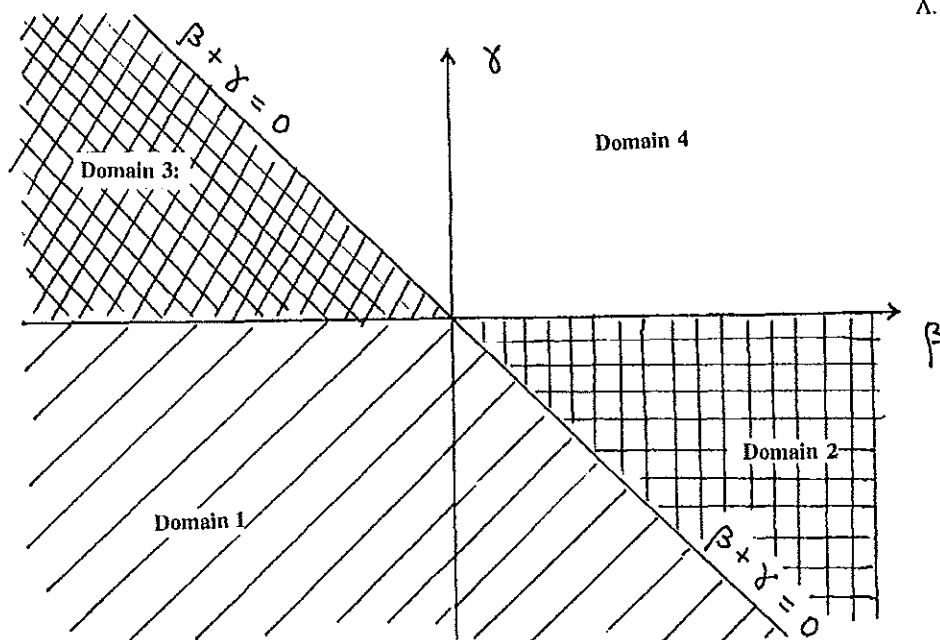


Fig. 1  $(\beta, \gamma)$  parametric domains for equilibria and their types.

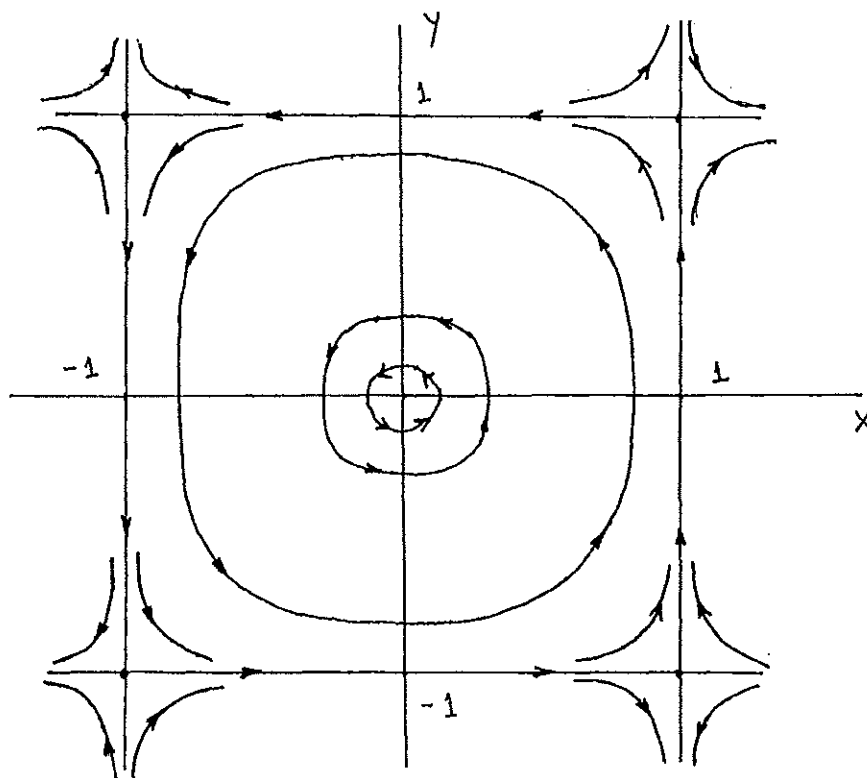
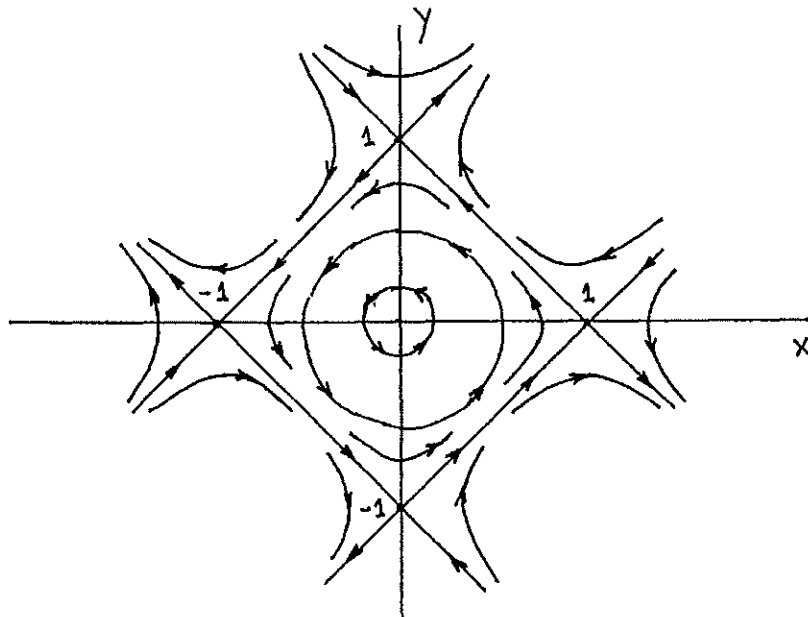
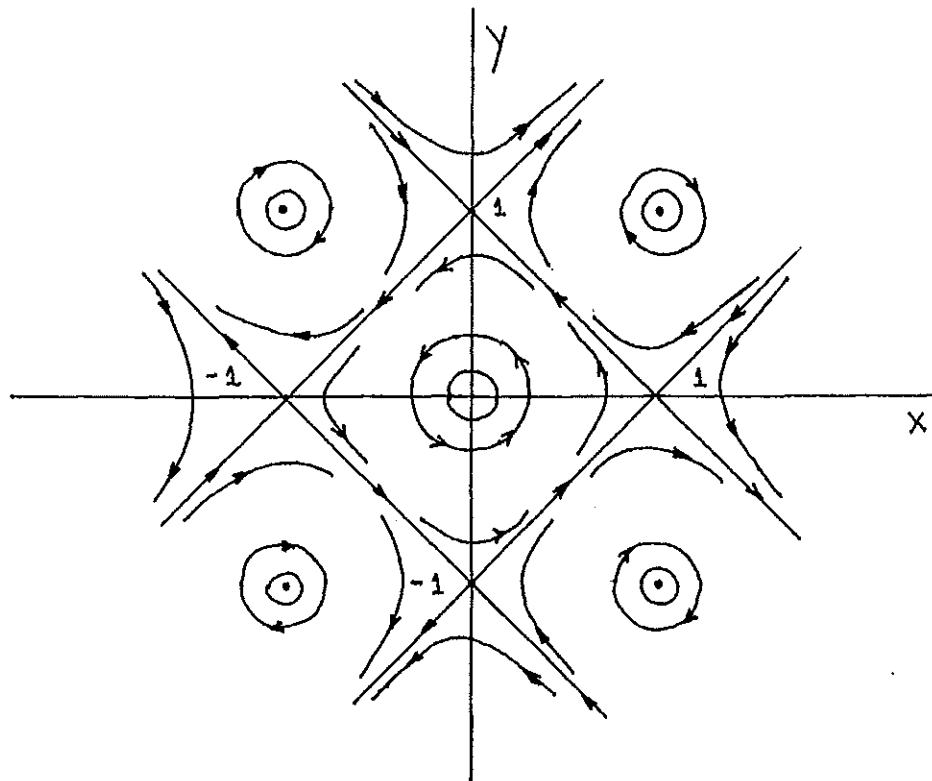


Fig. 2 Phase portrait of ODE corresponding to Jones' equation  $\dot{x}(t) = -\alpha x(t-1)[1-x^2(t)]$ .



**Fig. 3** Phase portrait of ODE corresponding to DDE  $\dot{x}(t) = -\alpha x(t-1)[1+2x^2(t)-x^2(t-1)]$  ( $\beta = -2, \gamma = 1$ ).



**Fig. 4** Phase portrait of ODE corresponding to DDE  $\dot{x}(t) = -\alpha x(t-1)[1-x^2(t-1)]$  ( $\beta = 0, \gamma = 1$ ).