

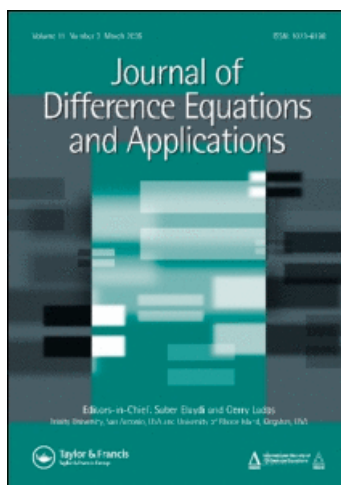
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Periodic solutions of a discretized differential delay equation

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Dedicated to Peter Kloeden on the Occasion of his 60th Birthday

Several aspects of dynamics are addressed for the differential-difference equation $\epsilon \dot{x}(t) + x(t) = f(x([t - k + 1])), 0 < \epsilon \ll 1$, where $[\cdot]$ is the integer part function, k is a positive integer. The equation can be viewed as a special discretization (discrete version) of the singularly perturbed differential delay equation $\epsilon \dot{x}(t) + x(t) = f(x(t - k))$. Sufficient conditions for the invariance, existence, stability and shape of periodic solutions are derived. The principal analysis is based on reduction to special multi-dimensional maps whose relevant properties follow from those of 1D map f .

Keywords: differential delay and difference equations; discretizations; singular perturbations; periodic solutions and their stability; reduction to finite dimensional maps; interval maps

AMS Subject Classification: 34K13; 34K26; 37E05

1. Introduction

This paper deals with several aspects of comparative dynamics among three equations with delay. The first equation is the pure difference equation

$$x(t) = f(x(t - k)), \quad f: \mathbb{R} \rightarrow \mathbb{R} \quad (1)$$

with continuous f , which can be viewed either as an equation with continuous argument $t \in \mathbb{R}_+ := \{t | t \geq 0\}$ or as an equation with discrete argument $t \in \mathbb{Z}_+ := \mathbb{N} \cup \{0\}$. The second equation is a singular perturbation of equation (1), the differential delay equation of the form

$$\epsilon \dot{x}(t) + x(t) = f(x(t - k)), \quad \epsilon > 0. \quad (2)$$

The third equation is a version of equation (2), a differential equation with piece wise constant argument

$$\epsilon \dot{x}(t) + x(t) = f(x([t - k + 1])), \quad (3)$$

where $[\cdot]$ is the integer part function. The latter can be viewed as a special discretization of equation (2) (in accordance with the procedure described below). It also falls within a special class of differential delay equations, the so-called equations with piece wise constant argument (EPCA), which are the object of studies in a significant number of papers in recent years (see, e.g. Refs. [3,8] and further references therein). One of the

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emphases of this paper is on the comparison of the dynamics of equations (3) and (2) with those of equation (1) and the interval map f .

All three equations with delay define in general infinite dimensional dynamical systems. The common underlying basis for all three is, however, a much simpler 1D dynamical system defined by the interval map f . Most of the dynamical properties of equation (1) are determined by those of the map f [20]. Equation (2) was originally proposed as small singular perturbation of equation (1) and approached with the expectation that many of its principal dynamical properties would follow from those of equation (1) and the interval map f [19,20,22]. However, the relationship between the dynamics of the two has appeared to be much more complicated. There are two interesting phenomena observed here, the complication and the simplification of the dynamics, which are indicative of the complex dependence of the dynamics between the two equations. We describe the phenomena in more details later in Section 1.

We propose the differential-difference equation (3) as both a singular perturbation of the difference equation (1) and as a special discretization of the differential delay equation (2). The new results of this paper concern equation (3) (which are presented in Section 2). Based on them, it appears that equation (3) mimics more closely the dynamics of the interval map f than the original singular equation (2) does. In particular, equation (3) always possesses stable periodic solutions which are ‘close’ to the attracting cycles of the map f (which is not always true for equation (2)). In this sense, equation (3) can be viewed as a better mathematical model than equation (2) as far as the following the dynamics of the interval map f is concerned.

Equation (3) appears as a special discretization of the differential delay equation (2) originating in applied situations. In order to solve an initial value problem for a differential delay equation, one usually needs an initial function. For one of the simplest equations,

$$\dot{x}(t) = f(x(t), x(t - \tau)), \quad x, t \in \mathbb{R}, \quad (4)$$

with delay $\tau > 0$ and continuous function $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ the set of initial functions is normally chosen to be $\mathcal{C} := C^0([-\tau, 0], \mathbb{R})$. For every initial function $\phi \in \mathcal{C}$, the corresponding solution $x = x_\phi(t)$ is obtained for $t \geq 0$ by successive integration.

In many of applied problems described by differential delay equations, the initial data are given or found as a discrete set. This leads to initial functions $\phi(t)$ defined only on a finite subset of the initial interval $[-\tau, 0]$: $\phi(t_i) = \phi_i \in \mathbb{R}$, $-\tau \leq t_N < t_{N-1} < \dots < t_1 < t_0 \leq 0$. One of the natural ways to extend the initial data to an initial function that would produce a solution for $t \geq 0$ is to make it a piece-wise constant function by setting $\phi(t) := \phi_i$ for all $t \in [t_i, t_{i-1})$. Equation (4) reduces then to an ordinary differential equation, $\dot{x} = f(x, \phi_i)$, on each of the intervals $[t_i + 1, t_{i-1} + 1)$, $j = 1, \dots, N$. As in the case of a continuous initial function, the corresponding solution $x_\phi(t)$ can be found for all $t \in [0, \tau]$ by successive step-by-step integration of the set of ordinary differential equations.

The simplest case of such initial value problems for differential delay equation (4) is the equi-distributed values of $t_i: t_{i+1} - t_i = \tau/N := h, \forall i$. Equation (4) can be rewritten then as

$$\dot{x}(t) = f\left(x(t), x\left(\left[\frac{t - \tau}{h}\right]h\right)\right), \quad t \in [0, \tau], \quad (5)$$

where $[\cdot]$ is the integer part function. Since

$$f\left(x(t), x\left(\left[\frac{t-\tau}{h}\right]h\right)\right) = f\left(x(t), x\left(\left[\frac{t}{h}\right]h - \tau\right)\right),$$

the change of variables $y(s) := x(hs)$, $t := hs$ transforms (5) into

$$y'(s) = h^{-1}f(y(s), y([s - N])). \quad (6)$$

One can use then equation (6) as a discrete version (or discretization) of equation (4) for all $s \geq 0$ (and not only for $t = hs \in [0, \tau]$). Equation (6) is called a differential delay equation with piece-wise constant argument [3,8]. It represents a special discretization of differential delay equation (4). See Ref. [14] for an alternative derivation of equation (6). We also refer the reader to papers [1,3,4] for additional related aspects of discretizations and numerical approximations of this and more general differential delay equations.

A particular case of equation (4) of special interest is the singularly perturbed differential delay equation

$$\epsilon \dot{x}(t) + x(t) = f(x(t-1)) \quad (7)$$

with $\tau = 1$ (normalization). Equation (2) can be transformed into equation (7) by the time rescaling $t = k \cdot s$. In addition to numerous other applications, which partial list can be found in Refs. [6,7,9,11,12,18], equation (7) appears as an exact reduction of nonlinear boundary value problems for hyperbolic partial differential equations (PDEs) [5,15,19,20,22]. Those PDEs often describe wave phenomena of real life processes such as string oscillations [22] or electronic circuits with strongly nonlinear elements [19] as well as several others [15,20]. When the viscosity or capacity of the oscillating media is disregarded, the boundary value problems are exactly reduced (in simplest cases) to the continuous time difference equation

$$x(t) = f(x(t-1)), \quad t \in \mathbb{R}^+. \quad (8)$$

The dynamics of solutions of equation (8) are well understood. They are essentially based on the dynamics of the interval map f . By now the theory and applications of difference equation (8) is a well developed area of modern nonlinear dynamics, which is summed up in the monograph [20].

The more accurate models, however, are those where the (small) viscosity/capacity effects are taken into account. Mathematically, in simplest cases, it leads to singular perturbations of difference equation (8) in the form of differential delay equation (7). As the nonlinearity f reflects actual physical properties of the underlying real life processes, the dynamics of solutions of difference equation (8) follow many essential properties of the interval map f . It has been a natural expectation that many of the dynamical properties of the differential delay equation (7) would also follow those of equation (8) and of the interval map f . The correspondence in the dynamics between the two has appeared to be much more involved.

There is a straightforward correspondence for the *global stability* property. When map f has a globally attracting fixed point x_* the corresponding constant solution $x_\phi(t) \equiv x_*$ of equation (7) is also globally attracting for all $\epsilon > 0$. The situation becomes sharply different already for the next dynamical level of the map f , when it has a globally attracting cycle of period two: $f(a_1) = a_2, f(a_2) = a_1$, and for every initial value $x \in \mathbb{R}$ the sequence $f^n(x)$ converges to the cycle $\{a_1, a_2\}$ as $n \rightarrow \infty$. Equation (7) is known to possess in this

case periodic solutions $x = p_\epsilon(t)$ of period $T = 2 + \mathcal{O}(\epsilon)$ [17], which converge as $\epsilon \rightarrow 0+$ to the two-periodic solution of difference equation (8) given by

$$p_0(t) = a_1 \quad \text{for } t \in [0, 1); \quad p_0(t) = a_2 \quad \text{for } t \in [1, 2); \quad p(t+2) \equiv p(t) \quad \forall t \in \mathbb{R}.$$

Note that no stability results have been established for such periodic solutions.

Two interesting phenomena are observed for the dynamics of differential delay equation (7) in relation to the dynamics of the interval map f . We shall call them the *complication* and the *simplification* of the dynamics. The complication phenomenon reflects situations when the dynamics of interval map f are relatively simple while the dynamics of differential delay equation (7) are quite complicated (for all sufficiently small $0 < \epsilon < \epsilon_0$). In the simplification situations, the opposite happens: the dynamics of interval map f are complex while the dynamics of the corresponding equation (7) are rather simple (for all $\epsilon > 0$).

1.1 Complication

Examples of functions f with globally attracting two-cycles are constructed when the dynamics of the corresponding differential delay equation (7) can be quite complicated, including multiple stable and unstable periodic solutions of arbitrarily large periods and chaotic behaviours [10,16]. A typical example is provided by the following function f

$$f(x) = \begin{cases} 1 & \text{if } x \leq -\theta_1, \\ A & \text{if } x \in (-\theta_1, 0), \\ -B & \text{if } x \in [0, \theta_2), \\ -1 & \text{if } x \geq \theta_2, \end{cases} \quad (9)$$

where $A, B > 1$ and $0 < \theta_1, \theta_2 < 1$. As it is easily seen the corresponding 1D map f has globally attracting cycle of period 2, $\{-1, 1\}$. For the differential delay equation (7), a special parameter dependent subset C_* , with $\lambda \in \mathbb{R}$ being the parameter, of the initial set $\mathcal{C} := C^0([-1, 0], \mathbb{R})$ can be identified with the following property. For every $\varphi_\lambda \in C_*$, there exists a time $t' \geq 1$ such that the segment of the solution $x(t) = x_\varphi^\epsilon(t)$, considered at the time t' , is an element of the subset C_* characterized by a parameter λ' , that it $x_{t'} \in C_*$ and $x_{t'} := \varphi_{\lambda'}$ for some $\lambda' \in \mathbb{R}$. This correspondence induces a 1D map $F : \lambda \rightarrow \lambda'$ which dynamics completely describe those of equation (7) on the subset C_* . One of the choices for the set C_* has been a subset of initial functions $\phi \in \mathcal{C}$ normalized to $\phi(0) = 0$, $\phi(-1) = \lambda > 0$ and $\phi(s) > 0$ for all $s \in [-1, 0)$. Another choice for C_* has been a subset of initial functions $\psi \in \mathcal{C}$ such that $\psi(0) = \phi(-\lambda) = \theta_2$, where $0 < \lambda < 1$, and $\psi(s) > \theta_2$ for $s \in (-\lambda, 0)$, $0 < \psi(s) < \theta_2$ for $s \in [-1, -\lambda)$. Both choices lead to induced 1D maps which are dynamically equivalent (see the review paper [10] for these and additional details).

The map F can be calculated in the explicit form, and in most cases it is a piece-wise linear-fractional (Möbius) map (see Refs. [10,16] for more details and derivation of the map F). In general, the dynamics of piece-wise linear-fractional maps can be as complicated as dynamics of arbitrary continuous maps [21]. In particular, they can possess cycles of arbitrarily large periods, including attracting ones. They can exhibit chaotic behaviour, including the existence of absolutely continuous invariant measures. Parameter dependent Möbius maps can go through the same bifurcation patterns as other well-known

families (e.g. $G_\lambda(x) = \lambda x(1 - x)$), including period doubling of attracting cycles and eventual transition to chaos. The complicated dynamics persist when map (9) is replaced by a close to it continuous or smooth one (when a ‘smoothing’ is done in a small neighbourhood of the discontinuity points $\theta_1, 0, \theta_2$). For more details on these and additional properties of Mobius maps, see Ref. [21]. We note that for small $\epsilon > 0$ all the complicated dynamics of equation (7) on the subset \mathcal{C}_* happens in a small neighbourhood, defined by the Hausdorff metric for graphs in \mathbb{R}^2 , of the generalized two-periodic solution $P_0(t)$ of difference equation (8) given by

$$\begin{aligned} P_0(t) &= 1 \quad \text{for } t \in (0, 1); \quad P_0(t) = -1 \quad \text{for } t \in (1, 2); \\ P_0(t) &= [-B, A] \quad \text{for } t = 0 \quad \text{and } t = 1; \end{aligned}$$

Moreover, the size of the neighbourhood goes to zero as $\epsilon \rightarrow 0+$.

1.2 Simplification

On the other hand, there are examples of maps f with complicated dynamics (such as, e.g. chaos) while the dynamics of corresponding differential delay equation (7) are relatively simple (e.g. it only has a globally attracting steady state for all $\epsilon > 0$). In general, the simplification can happen when map f has an attracting interval I_0 with the domain J of immediate attraction, $J \supset I_0$, such that the set $I \setminus J$ is relatively small compared with the domain J . Here I is an invariant interval, the entire phase space for map f . If the set $I \setminus J$ does not contain fixed points then for every initial function $\varphi \in \mathcal{C}$ such that $\varphi(t) \in I, \forall t \in [-1, 0]$, the corresponding solution $x(t) = x_\varphi^\epsilon(t)$ has the property: $x(t) \in J$ for all sufficiently large $t, t \geq t_1$, for some $t_1 \geq 0$ and every $\epsilon > 0$. This means, due to the attractivity of interval I_0 , that $x_\varphi^\epsilon(t) \rightarrow I_0$ as $t \rightarrow +\infty$ for all solutions of equation (7). Therefore, in this case the dynamics of map f on the set $I \setminus J$ do not determine any part of the asymptotic dynamics of solutions of equation (7) as $t \rightarrow +\infty$. If I_0 is a fixed point $x = x_*$ then the dynamics of all solutions of equation (7) are simple: they all are asymptotically constant, being attracted by the steady state $x(t) \equiv x_*$. At the same time, the dynamics of map f on the set $I \setminus J$ can be arbitrarily complicated. However, this part of dynamics of f does not affect the asymptotic behaviour of all solutions of equation (7). Note that within finite time intervals the dynamics of f on $I \setminus J$ can be followed arbitrarily close (for sufficiently small $\epsilon > 0$) by solutions of equation (7), due to the so-called continuous dependence on the parameter ϵ . See the review paper [10] for more details, examples, generic cases of the ‘simplification phenomenon’ and proofs.

A partial case of the simplification phenomenon can be provided when the nonlinearity f given by (9) has the following specifications: $A = B = 0$ and $\theta_1 = \theta_2 = h$ (this example is considered in full details in Ref. [10]). The corresponding interval map, considered on the invariant interval $[-1, 1]$, has the attracting fixed point $x = 0$ with the domain of immediate attraction $|x| < h$ and the attracting two-cycle $\{-1, 1\}$ with the domain of immediate attraction $1 > |x| > h$. When $h > 1/2$, all solutions of equation (7) satisfy $\lim_{t \rightarrow \infty} x_\varphi^\epsilon(t) = 0$ (for every $\epsilon > 0$ and any initial function $\phi \in \mathcal{C}$) [10]. This implies that the existence of the attracting two-cycle of the map f has no effect on the asymptotic behaviour of solutions of equation (7), since the dynamic of the attracting fixed point is dominant (its domain is simply larger than the domain of the two-cycle in this case). We refer the reader to paper [10] for additional details of this example. See also a generalization of this example in Subsection 2.3.

This paper studies several aspects of global dynamics of the scalar differential-difference equation (3) where $f : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function, $[\cdot]$ is the integer part function, k is a fixed positive integer, and $\epsilon > 0$ is a small parameter. In particular, we are interested in the existence of periodic solutions of equation (3). As described earlier equation (3) is a special discretization (discrete version) of the differential delay equation (2) in the sense of the above equation with the piece-wise constant argument (6). The limiting case of equation (2), when $\epsilon = 0$, is the difference equation (1), which is equation (8) when $k = 1$. Remarkably, as it will be shown below, (1) can also be considered as a limiting case of (3). Observe here that, in contrast with the autonomous equations (1) and (2), model (3) is one-periodic scalar differential-difference equation. Indeed, (3) can be written in the standard form

$$\epsilon \dot{x}(t) + x(t) = \tilde{f}(t, x_t),$$

where $\tilde{f} : \mathbb{R} \times C([-k, 0], \mathbb{R}) \rightarrow \mathbb{R}$ is defined by $\tilde{f}(t, \phi) := f(\phi(-k + 1 - \{t\}))$, with $\{t\}$ being the fractional part of t . It is easy to see that $\tilde{f}(t + 1, \phi) = \tilde{f}(t, \phi)$, $t \in \mathbb{R}$. Because equation (3) is not autonomous, we will distinguish two periodic solutions $p_1(t) \neq p_2(t)$ even if $p_1(t) = p_2(t + r)$ for some integer r . As it is well known, the periodic problem can be reduced to the study of fixed points of iterations of the return map (or the Poincaré map) $\mathcal{R} : \mathcal{C} \rightarrow \mathcal{C}$, where $\mathcal{C} := C([-k, 0], \mathbb{R})$. Recall that, by definition, the operator \mathcal{R} returns an initial function shifted on k units of time along the corresponding solution. Fortunately, in case of equation (3) the map $\mathcal{R} : \mathcal{C} \rightarrow \mathcal{C}$, can be represented by a simpler map $T_\epsilon : \mathbb{R}^k \rightarrow \mathbb{R}^k$ without loss of the information regarding dynamics of (3) (see Proposition 2.2 below). In this way, unlike equations (1) and (2), the correspondence in dynamical properties between the interval map f (difference equation (1)) and its singular perturbation in the form of equation (3) is much simpler and more coherent. We show in this paper that equation (3) possesses a multitude of periodic solutions which are close to attracting cycles of map f for all sufficiently small $\epsilon > 0$. The multiple periodic solutions primarily result from the multiple options to choose initial data in various combinations around points of the attracting cycles of f . We show then that each of those periodic solutions is locally asymptotically stable with the asymptotic phase provided the corresponding attracting cycle of the map f is hyperbolic. Our results suggest in part that equation (3) can be viewed as better differential delay model than equation (2), at least as far as following the dynamics of attracting cycles of the map f is concerned. In this regard, equation (3) may prove to be useful for numerical simulations of equation (2) when it is used in place of or as a supplement to other standard numerical schemes.

2. Existence and stability of periodic solutions

2.1 Preliminaries

We recall first some basic definitions and notions from the interval map theory necessary for the exposition in this paper. For proofs and additional related material on the interval maps, we refer the reader to monographs [2, 21].

As usual, given continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$, a point $b_1 \in \mathbb{R}$ is called periodic with (minimal) period m if $f^m(b_1) = b_1$ and $f^i(b_1) \neq b_1$ for any $i < m$. Here f^i stands for the i th iteration of the map f , i.e. $f^0(x) \equiv x$, $f^{i+1}(x) = (f \circ f^i)(x) := f(f^i(x))$, $i = 0, 1, 2, \dots$. When $m = 1$, b_1 is a fixed point. The points $b_k := f^k(b_1)$, $0 \leq k < m$, are said to form a cycle, $\mathcal{B} := \{b_1, b_2, \dots, b_m\}$ of period m . Clearly, each point of the cycle is periodic with period m . It is also a fixed point for the map f^m . We use the notation $b_i < b_{i+1}$

if $b_{i+1} = f(b_i)$ and consider m -cycle $\mathcal{B} = \{b_1 < b_2 < \dots < b_m\}$ as an ordered set. Therefore, we distinguish between two orbitally equivalent cycles $\mathcal{B} = \{b_1 < b_2 < \dots < b_m\}$ and $\mathcal{B}_1 = \{b_2 < b_3 < \dots < b_m < b_1\}$. Given two m -cycles \mathcal{B} and $\mathcal{C} = \{c_1 < c_2 < \dots < c_m\}$, they are said to be δ -close if there exists $\delta > 0$ such that $|c_j - b_j| < \delta, j = 1, \dots, m$. Notice that m does not have to be the minimal period: every time when m is minimal, we will indicate it explicitly.

Fixed point b_1 is called (locally) attracting (under map f) if there exists a neighbourhood $\mathcal{U}_1 \ni b_1$ such that $\lim_{j \rightarrow \infty} f^j(x) = b_1$ for every $x \in \mathcal{U}_1$. The maximal connected interval \mathcal{U}_1 with the above property is called the *domain of immediate attraction* of the fixed point b_1 . \mathcal{U}_1 is an open interval (relative to the topology of the phase space) which is mapped into itself, $f(\mathcal{U}_1) \subseteq \mathcal{U}_1$. The cycle $\mathcal{B} := \{b_1, b_2, \dots, b_m\}$ is called attracting if its every point $b_k, 1 \leq k \leq m$, is an attracting fixed point under the map f^m . Given cycle \mathcal{B} let $\mathcal{U}_k, 1 \leq k \leq m$, be the domain of immediate attraction of the fixed point b_k under the map f^m . Then the set $\mathcal{U} = \mathcal{U}_1 \cup \mathcal{U}_2 \cup \dots \cup \mathcal{U}_m$, a union of m disjoint open intervals, is called the domain of immediate attraction of the cycle \mathcal{B} .

An interval $I \subseteq \mathbb{R}$ is called invariant under map f if $f(I) \subseteq I$. The standing assumption throughout the paper will be that the map f possesses a maximal invariant bounded interval I with all the cycles and other dynamics happening within I . We say that the invariant interval I_0 is attracting if there exists its neighbourhood $\mathcal{U} \supset I_0$ such that for every $x \in \mathcal{U}$ the sequence $f^n(x)$ converges to I_0 as $n \rightarrow \infty$. The convergence here is meant in the sense of the distance between the set $\{I_0\}$ and the point $f^n(x)$. The maximal open connected interval \mathcal{U} with this property is called the domain of immediate attraction of I_0 .

The above mentioned terminology and definitions for the interval maps are also applied to the difference equation $x_{n+1} = f(x_n)$, which dynamics are defined by and are equivalent to those of the map f . The difference equation approach is used in Subsection 2.2.

In order to obtain a solution of equation (3) for $t \geq 0$, one needs an initial function, $\phi(t), t \in [-k, 0]$. Since the corresponding solution $x = x_\phi^\epsilon(t), t \geq 0$ is continuous we consider $\phi(t)$ to be continuous too, $\phi \in \mathcal{C} := C[-k, 0]$. Let I be an invariant interval of the map f . Introduce the subset \mathcal{C}_I of the set \mathcal{C} by

$$\mathcal{C}_I := \{\phi \in \mathcal{C} \mid \phi(t) \in I \text{ for all } t \in [-k, 0]\}.$$

For any initial function $\phi \in \mathcal{C}$, the solution $x_\phi^\epsilon(t)$ is found by integration the initial value problem $\epsilon \dot{x}(t) + x(t) = f(x(-k+1)), x(0) = x_0$ and is given by

$$x_\phi^\epsilon(t) = f(x(-k+1)) + [x(0) - f(x(-k+1))]\exp(-t/\epsilon), \quad t \in [0, 1]. \quad (10)$$

When $x_0 = \phi(0)$, the solution is continuous at $t = 0$, and therefore for all $-k+1 \leq t \leq 1$. The solution $x_\phi^\epsilon(t)$ is found for all $t \geq 0$ by successive integration on the intervals $[i, i+1], i \in \mathbb{N} \cup \{0\}$.

In some cases, we will need the uniform boundedness of solutions of equation (3) for all (sufficiently small) $\epsilon > 0$. For the delay differential equation (2), such uniform boundedness is in place for all $\epsilon > 0$ when the interval map f has a closed bounded invariant interval $I : f(I) \subseteq I$, in view of the so-called *invariance property* (see Ref. [10] for more details). Similar to equation (2), the invariance property is also valid for equation (3).

PROPOSITION 2.1 (INVARIANCE). *Suppose $\phi \in \mathcal{C}_I$. Then the corresponding solution $x_\phi^\epsilon(t)$ of differential-difference equation (3) satisfies $x_\phi^\epsilon(t) \in I$ for all $t \geq 0$ and every $\epsilon > 0$.*

Proof. It can be derived the same way as for equation (7) in paper [10]. Another way to see the invariance property holding for equation (3) is from the form of its solutions given by (10). Since $x_\phi^\epsilon(t)$ is monotone for $t \in [0, 1]$ with its range contained in the interval formed by the endpoints $x(0)$ and $f(x(-k+1))$ ($[f(x(-k+1)), x(0)]$ or $[x(0), f(x(-k+1))]$) $\subset I$, its range remains within I for all $t \in [0, 1]$. The validity of the claim for all $t \geq 0$ follows by induction. \square

For the remainder of the paper, we will be assuming that map f has an invariant interval I which contains any of the cycles \mathcal{B} under consideration. This will allow us to restrict considerations to the subset \mathcal{C}_I , which guarantees, in view of the invariance property (Proposition 2.1), the uniform boundedness of solutions for all $t \geq 0$.

Note, however, that the existence of an invariant interval I for the map f is not a necessary assumption for the validity of the main results of this paper. In the general case of $f \in C(\mathbb{R}, \mathbb{R})$ and the existence of an attracting cycle \mathcal{B} of the map f it can also be shown that all solutions of differential-difference equation (3) remain within a closed interval J for all $t \geq 0$ and all sufficiently small ϵ , $0 < \epsilon < \epsilon_0$, provided initial functions ϕ are in \mathcal{C}_J (for some J and ϵ_0 depending on the nonlinearity f and the cycle \mathcal{B}).

PROPOSITION 2.2. *Let $S(t)$ be the shift operator by time t along solutions of equation (3) given by*

$$S(t) : \mathcal{C} \rightarrow \mathcal{C}, \quad (S\phi)(t) := x_\phi^\epsilon(t+s), \quad s \in [-k, 0], \quad \phi \in \mathcal{C}.$$

The map $S(1)$ is equivalent to the k -dimensional map T_ϵ given by

$$T_\epsilon(x_{-k}, \dots, x_{-1}) = (x_{-k+1}, \dots, x_{-1}, f(x_{-k}) + \epsilon g(x_{-1}, \dots, x_{-k})), \quad (11)$$

where $g(x_{-1}, \dots, x_{-k}) := x_{-1} - x_{-k}$, $\epsilon := e^{-1/\epsilon}$.

Proof. With an initial function $\phi \in \mathcal{C}$, the solution $x = x_\phi^\epsilon(t)$ of equation (3) is given by formula (10)

$$x_\phi^\epsilon(t) = f(\phi(-k+1)) + [\phi(0) - \phi(-k+1)]e^{-t/\epsilon}, \quad t \in [0, 1].$$

Set $(x_{-k}, \dots, x_{-1}) := (\phi(-k+1), \dots, \phi(0))$ and consider the solution $x_\phi^\epsilon(t)$ at $t = 1$ as an element ψ of the phase space \mathcal{C} , that is $\psi(s) := x_\phi^\epsilon(1+s)$, $s \in [-k, 0]$. If one defines $(x'_{-k}, \dots, x'_{-1}) := (\psi(-k+1), \dots, \psi(0))$, then the x'_{-k}, \dots, x'_{-1} are easily found as

$$\begin{aligned} x'_{-1} &:= x_\phi^\epsilon(1) = f(\phi(-k+1)) + [\phi(0) - \phi(-k+1)]e^{-1/\epsilon}, \quad \text{and} \\ x'_{j-2} &:= x_\phi^\epsilon(j) = \phi(j), \quad j = -k+1, \dots, 0. \end{aligned}$$

Therefore, the map $S(1)$ is equivalent to the discrete map T_ϵ as defined by (11). \square

Proposition 2.2 implies that for arbitrary $\phi \in \mathcal{C}_I$ the corresponding solution $x_\phi^\epsilon(t)$ is entirely defined by the discrete values $\phi(-k+1), \dots, \phi(-1), \phi(0)$. Its asymptotic behaviour as $t \rightarrow +\infty$ is determined by successive iterations of the map T_ϵ , T_ϵ^n as $n \rightarrow \infty$, applied to the initial discrete set.

2.2 Main results

In this subsection, and for the remainder of the paper, we use the notations $\mathcal{P} := \{1, \dots, m\}^k$, $\prod_{s=k}^1 A_s = A_k \times \dots \times A_1$, and make the following standing hypothesis

(G): The interval map f (difference equation $x_{n+1} = f(x_n)$) possesses an attracting cycle $\mathcal{B} = \{b_1 < b_2 < \dots < b_m\}$ with minimal period m and domain of immediate attraction $\mathcal{U} = \cup_{j=1}^m (\alpha_j, \beta_j)$. Our convention is that $b_j = (\alpha_j, \beta_j) \cap \mathcal{B}$.

Consider the scalar difference equation with delay

$$x(n) = f(x(n-k)). \quad (12)$$

LEMMA 2.3. Assume (G). Then equation (12) has m^k different km -cycles

$$\Gamma_\sigma = \{b_{i_1} < b_{i_2} < \dots < b_{i_k} < \dots < b_{i_{km}}\}, \quad \sigma = (i_1, i_2, \dots, i_k) \in \mathcal{P}.$$

Each of them is completely determined by the initial segment $b_{i_1} < b_{i_2} < \dots < b_{i_k}$.

Proof. Equation (12) is equivalent to the map $T : \mathbb{R}^k \rightarrow \mathbb{R}^k$ defined by

$$T(x_{-k}, \dots, x_{-1}) = (x_{-k+1}, \dots, x_{-1}, f(x_{-k})).$$

Set

$$F(x_{-k}, \dots, x_{-1}) := T^k(x_{-k}, \dots, x_{-1}) = (f(x_{-k}), \dots, f(x_{-1})). \quad (13)$$

It is clear that every fixed point of F^m generates km -cycle of T , and, in this way, generates mk -periodic solution of (12). \square

Assume hypothesis (G) to hold and let $\delta \in \Delta := (0, 0.5 \min\{\beta_s - b_s, b_s - \alpha_s\})$. Set

$$\omega(\delta) = \delta + \max_{s,j=1,\dots,m} \text{diam} f^s([b_j - 0.5\delta, b_j + 0.5\delta]),$$

where $\text{diam } A$ denotes the length of interval A . It is easy to see that $\omega(\delta)$ is increasing on Δ and $\omega(0+) = 0$.

LEMMA 2.4. Assume (G) and let $g : \mathbb{R}^k \rightarrow \mathbb{R}$ be a continuous function. Then, for every positive δ there is $\varepsilon_0 > 0$ such that, for each $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$,

(A) the perturbed difference equation

$$x(n) = f(x(n-k)) + \varepsilon g(x(n-1), \dots, x(n-k)) \quad (14)$$

has m^k different km -cycles $\tilde{\Gamma}_\sigma$ which are $\omega(\delta)$ -close to the corresponding km -cycles Γ_σ of (12);

(B) every solution $\{x(i)\}_{i \geq 0}$ of (14) with initial values $x(s) \in \cup_{j=1}^m (\alpha_j + \delta, \beta_j - \delta)$, $s = -k, \dots, -1$, satisfies

$$|x(i) - b_{t(i)}| \leq \omega(\delta), \quad i \geq i_0, \quad (15)$$

for sufficiently large i_0 and some km -periodic function $t(i) : \mathbb{Z} \rightarrow \{1, \dots, m\}$.

Proof. By the assumption, F^m has m^k different fixed points $b_\sigma := (b_{i_1}, b_{i_2}, \dots, b_{i_k})$, $\sigma \in \mathcal{P}$, which coincide with the initial segments of cycles Γ_σ . Since each b_s is locally attracting fixed point of f^m with the domain of immediate attraction (α_s, β_s) , we obtain that, for some $\delta_s, \delta'_s \in (0, 0.25 \min\{\beta_s - b_s, b_s - \alpha_s\})$,

$$b_s \in f^m([\alpha_s + \delta_s, \beta_s - \delta'_s]) \subset (\alpha_s + \delta_s, \beta_s - \delta'_s). \quad (16)$$

Similarly, for every small $\delta > 0$ there is sufficiently large positive integer $r = mr'$, $r' \in \mathbb{N}$, such that

$$b_s \in f^r([\alpha_s + \delta, \beta_s - \delta]) \subset (b_s - 0.25\delta, b_s + 0.25\delta).$$

As a consequence of (16), we obtain that for each $\sigma = (i_1, i_2, \dots, i_k) \in \mathcal{P}$, the k -dimensional convex closed box $B_\sigma = \prod_{s=k}^1 [\alpha_{i_s} + \delta_s, \beta_{i_s} - \delta'_s]$ is transformed by F^m into a subset of its interior:

$$F^m(B_\sigma) \subset \text{Int } B_\sigma, \quad \sigma \in \mathcal{P}.$$

Define $B_\sigma^*(\delta) := \prod_{s=k}^1 [b_{i_s} - 0.5\delta, b_{i_s} + 0.5\delta] \subset \text{Int } B_\sigma$ and let $T_\varepsilon : \mathbb{R}^k \rightarrow \mathbb{R}^k$ be as in (11). Since $F_\varepsilon(x_{-k}, \dots, x_{-1}) := T_\varepsilon^k(x_{-k}, \dots, x_{-1})$ depends continuously on $(x_{-k}, \dots, x_{-1}, \varepsilon)$, we obtain that there exists $\varepsilon_0 = \varepsilon_0(\delta) > 0$ such that

$$F_\varepsilon^m(B_\sigma^*(\delta)) \subset F_\varepsilon^m(B_\sigma) \subset \text{Int } B_\sigma, \quad F_\varepsilon^r(B_\sigma) \subset \text{Int } B_\sigma^*(\delta), \quad (17)$$

$$T_\varepsilon^q(B_\sigma^*(\delta)) \subset \mathcal{O}_\delta[T^q(B_\sigma^*(\delta))] \quad \text{for all } q = 1, \dots, rk, \quad \sigma \in \mathcal{P} \text{ and } |\varepsilon| < \varepsilon_0(\delta). \quad (18)$$

Here $\mathcal{O}_\delta[A]$ stands for δ -neighbourhood of a subset A of the metric space \mathbb{R}^k equipped with the max-norm. For example, $\mathcal{O}_\delta[A] = \cup_{a \in A} \mathcal{O}_\delta[a]$. In virtue of the Brouwer fixed point theorem, inclusions (17) imply that, for each $\sigma \in \mathcal{P}$, F_ε^m has a fixed point $\tilde{b}_\sigma := (\tilde{b}_{i_k}, \dots, \tilde{b}_{i_1}) \in B_\sigma^*(\delta)$. Furthermore, due to (18) every mk -cycle $\tilde{\Gamma}_\sigma$ of (14), starting at the initial point \tilde{b}_σ , is $\omega(\delta)$ -close to the corresponding mk -cycle Γ_σ of (12) with the initial point b_σ .

Similarly, since $F_\varepsilon^r(B_\sigma^*(\delta)) \subset F_\varepsilon^r(B_\sigma) \subset B_\sigma^*(\delta)$, relations (18) assure that (15) holds with $i_0 = rk$. \square

LEMMA 2.5. Assume (G), and let f and $g : \mathbb{R}^k \rightarrow \mathbb{R}$ be C^1 -continuous functions. Suppose also that the multiplier $\mu(B)$ of the cycle B satisfies $|\mu(B)| < 1$. Then, for every positive δ there is $\varepsilon_0 > 0$ such that, for each $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$,

(C) equation (14) has m^k different attracting km -cycles $\Gamma_\sigma(\varepsilon)$,

$$\Gamma_\sigma(\varepsilon) = \{b_{i_1}(\varepsilon) < b_{i_2}(\varepsilon) < \dots < b_{i_k}(\varepsilon) < \dots < b_{i_{km}}(\varepsilon)\},$$

whose components $b_{i_s}(\varepsilon)$ depend C^1 -continuously on ε . Moreover, $\Gamma_\sigma(0)$ coincide with the generating km -cycles Γ_σ of (12), $\Gamma_\sigma(0) = \Gamma_\sigma$;

(D) every solution $\{x(i)\}_{i \geq 0}$ of (14) with initial values $x(s) \in \cup_{j=1}^m (\alpha_j + \delta, \beta_j - \delta)$, $s = -k, \dots, -1$, converges to some $\Gamma_\sigma(\varepsilon)$ as $i \rightarrow +\infty$.

Proof. (C) Arguing as in Lemma 2.4, we can find $\varepsilon_0 > 0$ such that $F_\varepsilon^m(B_\sigma) \subset \text{Int } B_\sigma$ for all $|\varepsilon| \leq \varepsilon_0$ and $\sigma \in \mathcal{P}$. For $u \in B_\sigma$, consider the equation

$$F_\varepsilon^m(u) - u = 0. \quad (19)$$

Due to our assumptions, $F_\varepsilon^m(u)$ is C^1 -continuous as function of ε, u . We also have that $F_0^m(b_\sigma) - b_\sigma = 0$ for the initial segment b_σ of the generating cycle Γ_σ , and

$$\left. \frac{\partial F_\varepsilon^m(u)}{\partial u} \right|_{u=b_\sigma, \varepsilon=0} = \mu(\mathcal{B})I,$$

where I is the identity matrix (we use (13) and (11) to obtain the latter formula). Thus, we apply the implicit function theorem to conclude the existence of a C^1 -continuous family of solutions $u = b_\sigma(\varepsilon)$, $b_\sigma(0) = b_\sigma$ of (19). These $b_\sigma(\varepsilon) = (b_{i_1}(\varepsilon), \dots, b_{i_k}(\varepsilon))$ are the initial segments of the cycles $\Gamma_\sigma(\varepsilon)$. Observe that each of $\Gamma_\sigma(\varepsilon)$ is locally attracting since, for all sufficiently small ε ,

$$\left\| \frac{\partial F_\varepsilon^m(u)}{\partial u} \right\|_{u=b_\sigma(\varepsilon)} < 0.5(1 + \mu(\mathcal{B})) < 1.$$

(D) Let $\delta, \varepsilon_1 > 0$ be such that

$$\left\| \frac{\partial F_\varepsilon^m(u)}{\partial u} \right\| < 0.5(1 + \mu(\mathcal{B})) < 1,$$

for all $|\varepsilon| \leq \varepsilon_1, |u - b_\sigma| \leq \omega(\delta)$, $\sigma \in \mathcal{P}$. Without restricting the generality, we can assume that $|b_\sigma(\varepsilon) - b_\sigma| \leq \delta$ for $|\varepsilon| \leq \varepsilon_1, \sigma \in \mathcal{P}$. Fix $\delta > 0$ and let $\varepsilon_0 < \varepsilon_1$ be as in Lemma 2.4. Set $u' = (x_{-k}, \dots, x_{-1})$, where $x_s := x(s)$ are as in Lemma 2.4, (B). If u satisfies $|u - b_\sigma| \leq \omega(\delta)$, we have

$$|F_\varepsilon^m(u) - F_\varepsilon^m(b_\sigma(\varepsilon))| \leq \sup_{|u - b_\sigma| \leq \omega(\delta)} \left\| \frac{\partial F_\varepsilon^m(u)}{\partial u} \right\| |u - b_\sigma(\varepsilon)| \leq \frac{1 + \mu(\mathcal{B})}{2} |u - b_\sigma(\varepsilon)|.$$

Lemma 2.4(B) assures that

$$|F_\varepsilon^r(u') - F^r(b_\sigma)| \leq \omega(\delta), \quad |F_\varepsilon^r(b_\sigma(\varepsilon)) - F^r(b_\sigma)| \leq \omega(\delta),$$

for all large $r \geq r'm$. Therefore, when $j \geq r'm$ then

$$|F_\varepsilon^j(u') - F_\varepsilon^j(b_\sigma(\varepsilon))| \leq 2\omega(\delta) \left(\frac{1 + \mu(\mathcal{B})}{2} \right)^{[j/m] - r'} \rightarrow 0, \quad \text{as } j \rightarrow +\infty. \quad \square$$

By definition, the square wave $sw(t) = sw(t, b_{i-km}, \dots, b_{i-1})$ is a piece-wise constant km -periodic function such that $sw(t) = b_{i-j}$ for $t \in (-j-1, -j]$, $j \in \mathbb{Z}$.

THEOREM 2.6. Assume (G). Then for every $\delta > 0$ there exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ differential-difference equation (3) has at least m^k different periodic solutions $x = p_\varepsilon(t, b_{i-km}, \dots, b_{i-1})$ of period mk (not necessarily minimal). The periodic solutions satisfy

$$|p_\varepsilon(t, b_{i-km}, \dots, b_{i-1}) - sw(t, b_{i-km}, \dots, b_{i-1})| < \delta + \omega(\delta), \quad t \in \mathbb{R} \setminus \bigcup_j [j, j + \delta).$$

Moreover, if $x(t, \phi)$ solves the initial value problem $x(s, \phi) = \phi(s)$, $s \in [-k, 0]$ with ϕ satisfying the following conditions:

$$\phi(j) \in \cup_{j=1}^m (\alpha_j + \delta, \beta_j - \delta), \quad j = -k+1, \dots, 0, \quad (20)$$

then

$$|x(t, \phi) - sw(t, b_{i-km}, \dots, b_{i-1})| < \delta + \omega(\delta), \quad t \in [t_1, +\infty) \setminus \cup_j [j, j + \delta).$$

for sufficiently large t_1 and some appropriate $(b_{i-km}, \dots, b_{i-1})$.

Proof. With $\varepsilon := \exp(-1/\epsilon)$, it is clear that $\varepsilon \rightarrow 0$ if and only if $\epsilon \rightarrow 0 +$. If $x(t)$ solves the initial value problem $x(s) = \phi(s)$, $s \in [-k, 0]$ for (3), then the direct integration of (3) yields

$$\begin{aligned} x(n) &= f(x(n-k)) + \varepsilon(x(n-1) - f(x(n-k))), \quad n \in \mathbb{N} \cup \{0\}, \\ x(t) &= f(x(n-k)) + (x(n-1) - f(x(n-k)))e^{-(t-n+1)/\epsilon}, \quad t \in [n-1, n]. \end{aligned}$$

Therefore $x(t)$ is a mk -periodic solution of (3) if and only if the sequence $x(n)$ is a mk -periodic solution of (14) where $r(x(n-1), \dots, x(n-k)) = x(n-1) - f(x(n-k))$. (See also Proposition 2.2).

Fix $\delta > 0$ and take ε_0 as in Lemma 2.4. Let ϕ satisfy conditions (20) and let $x(t) = x(t, \phi)$ be the solution of the initial value problem $x(s, \phi) = \phi(s)$, $s \in [-k, 0]$. Then part (B) of Lemma 2.4 (or Proposition 2.1) implies that

$$|x(n-1) - f(x(n-k))| \leq \theta := \max_j \beta_j - \min_j \alpha_j.$$

In consequence,

$$\begin{aligned} |x(t) - x(n)| &\leq \theta(e^{-(t-n+1)/\epsilon} - e^{-1/\epsilon}), \quad t \in [n-1, n], \\ |x(t) - x(n)| &\leq \delta, \quad t \in \left(n-1 - \epsilon \ln \left[\frac{\delta}{\theta} + e^{-1/\epsilon} \right], n \right]. \end{aligned}$$

Hence, Theorem 2.6 follows from Lemma 2.4 if $t_1 = i_0$ and ϵ_0 is chosen to satisfy $\epsilon_0 \leq -1/\ln \varepsilon_0$, $\delta/\theta + e^{-1/\epsilon_0} > e^{-\delta/\epsilon_0}$. \square

THEOREM 2.7. Assume (G), and let f be C^1 -continuous with the multiplier μ of \mathcal{B} satisfying $|\mu(\mathcal{B})| < 1$. Then differential-difference equation (3) has at least m^k different periodic solutions $x = p_\epsilon(t, b_{i-km}, \dots, b_{i-1})$ of period (not necessarily minimal) mk such that

$$\lim_{\epsilon \rightarrow 0+} p_\epsilon(t, b_{i-km}, \dots, b_{i-1}) = sw(t, b_{i-km}, \dots, b_{i-1}), \quad t \notin \mathbb{Z},$$

where the convergence is uniform on $\mathbb{R} \setminus \cup_j [j, j + \delta)$ for every fixed positive δ .

Moreover, for every $\delta > 0$ there exists $\epsilon_0 > 0$ such that if $x(t, \phi, \epsilon)$, $\epsilon \in (0, \epsilon_0)$, solves the initial value problem $x(s, \phi, \epsilon) = \phi(s)$, $s \in [-k, 0]$ for equation (3) with ϕ satisfying (20) then $\lim_{t \rightarrow \infty} [x(t, \phi) - p_\epsilon(t, b_{i-km}, \dots, b_{i-1})] = 0$, for some appropriate $(b_{i-km}, \dots, b_{i-1})$.

Proof. Theorem 2.7 is proved similarly to Theorem 2.6 if one uses Lemma 2.5 instead of Lemma 2.4. We leave these technical details to the reader. \square

Remark 1. Some of the m^k periodic solutions of equation (3) are orbitally equivalent in the sense that they coincide after appropriate shifts of argument t . For example, it is easy to see that when $k = 2, m = 4$ there are only two orbitally different solutions in the set of possible 16 periodic solutions (all of them have the minimal period 8). Liang [13] calculated the number of orbitally different solutions in the case of arbitrary m, k .

2.3 Example

To show the range of applicability of our new results and to further demonstrate the interplay between the dynamics of the three equations, (3), (2), and (1), we consider a generic example in this subsection. This example generalizes the partial case of the simplification phenomenon described in the Section 1 for the nonlinearity f given by formula (9).

Let $f(x)$ be a continuous decreasing function with the unique fixed point $a_0 = 0$. Let, in addition, $f(x)$ has a pair of points a_1 and a_2 , ($a_1 < 0 < a_2$) such that $f(a_1) = a_2, f(a_2) = a_1$ and $f(f(x)) < x$ for all $x \in (0, a_2)$. An easy example of such f can be provided by $f(x) = -x + \gamma \sin x, 0 < \gamma < 1$ with $a_1 = -\pi, a_2 = \pi$. The function $f(x)$ will be considered fixed on the interval $[a_1, a_2] := I_1$ for the remainder of this paper.

Assume next that $f(x)$ has another pair of points, a_3 and a_4 ($a_3 < a_1, a_4 > a_2$), such that $f(a_3) = a_4, f(a_4) = a_3$ and $f(f(x)) > x$ for all $x \in (a_2, a_4)$. Set $I_2 := [a_3, a_4]$.

The dynamics of the map f (discrete difference equation (1)) on the invariant interval I_2 is simple. It has the unique fixed point $a_0 = 0$ and two cycles of period 2, $\{a_1, a_2\}$ and $\{a_3, a_4\}$. The cycle $\{a_3, a_4\}$ is attracting with the domain of (immediate) attraction $U_2 = [a_3, a_1] \cup (a_2, a_4]$. The fixed point $a_0 = 0$ is attracting with the domain of (immediate) attraction $U_1 = (a_1, a_2)$. The other two-cycle, $\{a_1, a_2\}$, is repelling. It separates the two domains U_1 and U_2 .

The set of initial functions $C_{I_2} := \{\phi \in \mathcal{C} | \phi(s) \in I_2 \forall s \in [-k, 0]\}$ is invariant under the semi-flows defined by both equations (3) and (2) (due to the Invariance Property). The attracting fixed point $a_0 = 0$ of the map f generates an asymptotically stable steady state $x(t) \equiv 0, t \geq 0$ for both equations with the domains of attraction containing at least the set $C_{I_1} := \{\phi \in \mathcal{C} | \phi(s) \in I_1 \forall s \in [-k, 0]\}$. The latter is due to the Global Stability Property (see Ref. [10] for more details).

As the following proposition from Ref. [10] shows the steady state $x(t) \equiv 0$ is the global attractor (on the set C_{I_2}) for equation (2) provided the domain U_2 of attraction for the cycle $\{a_3, a_4\}$ is small compared with the domain U_1 of attraction of the fixed point $a_0 = 0$.

PROPOSITION 2.8. *Given f on the interval $[a_1, a_2]$ there exists $\delta_0 > 0$ such that if $\max\{|a_1 - a_3|, |a_2 - a_4|\} := \Delta \leq \delta_0$ then all solutions of equation (2) satisfy*

$$\lim_{t \rightarrow \infty} x_\phi^\epsilon(t) = 0,$$

for every initial function $\phi \in C_{I_2}$ and every $\epsilon > 0$.

This proposition also shows that the existence of the attracting two-cycle $\{a_3, a_4\}$ of the map f has no implication on the asymptotic dynamics of solutions of equation (2). Its all

solutions are asymptotically constant with $x(t) \equiv 0$ being the global attractor. This example is a special case of the more general phenomenon that we called the *simplification* in the Section 1 (see Ref. [10] for additional details).

On the other hand, regardless of the size of the Δ , the differential-difference equation (3) always has a variety of asymptotically stable periodic solutions, generated by the attracting cycle $\{a_3, a_4\}$, for all sufficiently small $\epsilon > 0$. This is guaranteed by Theorems 2.6 and 2.7. Thus, this part of the dynamics of the interval map f is reflected in the differential-difference equation (3), while it is totally missing from the dynamics of differential delay equation (2). Therefore, in this sense, equation (3) is a better model than equation (2) regarding the property of following the dynamics of the map f .

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