

PERIODIC SOLUTIONS FOR THREE-DIMENSIONAL NON-MONOTONE CYCLIC SYSTEMS WITH TIME DELAYS

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ABSTRACT. We study a model for three cyclically coupled neurons with eventually negative delayed feedback, and without symmetry or monotonicity properties. Periodic solutions are obtained from the Schauder fixed point theorem. It turns out that, contrary to lower dimensional cases, instability at zero does not exclude monotonously decaying solutions.

1. Introduction. Consider a cyclically coupled system of differential equations of the form

$$\begin{cases} \dot{y}_1(t) + \mu_1 y_1(t) = f_1(y_2(t - \tau_2)) \\ \dot{y}_2(t) + \mu_2 y_2(t) = f_2(y_3(t - \tau_3)) \\ \dots \\ \dot{y}_N(t) + \mu_N y_N(t) = f_N(y_1(t - \tau_1)), \end{cases}$$

with delays $\tau_j \geq 0$ and with decay coefficients $\mu_j > 0$, $j = 1, \dots, N$, and C^1 feedback functions $f_j : \mathbb{R} \rightarrow \mathbb{R}$, $j = 1, \dots, N$. Such systems appear in biological applications, e.g., as models for protein synthesis or for neural networks with a cyclical architecture. See for example [13], [9] or [19].

The theory of such systems in the case of monotone coupling is established in [15], the main result being that a Poincaré-Bendixson-type theorem holds. In particular, if the ω -limit set of a solution does not contain equilibria, it must be a nonconstant periodic solution.

We shall study the existence of periodic solutions to the above system without monotonicity conditions, but with the assumption that each f_j has either negative or positive feedback with respect to zero, and that the feedback is eventually negative. That is, for $x \in \mathbb{R} \setminus \{0\}$ and $j = 1, \dots, N$ one has

$$\text{sign}[f_j(x) \cdot x] = \sigma_j \in \{-1, +1\}, \text{ and } \sigma_1 \cdot \sigma_2 \cdot \dots \cdot \sigma_N = -1. \quad (\text{H1})$$

(In particular, $f_j(0) = 0$, $j = 1, \dots, N$.) Together with conditions on the linearization at zero, negative feedback is the essential prerequisite for oscillatory behavior of solutions. Some systems may not have this form at first sight, but achieve it after a transformation of the form $y = y^* + x$, where $y^* \in \mathbb{R}^N$ is an equilibrium

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of the equation; compare [9], pp. 42-43. In the nonmonotone case with negative delayed feedback, even for $N = 1$ the dynamics can be complicated (see [12], [11], and the related result for ordinary differential equations from [5]).

For the cases $N = 1$ or $N = 2$, periodic solutions were obtained from the Browder ejective fixed point theorem in, e.g., [7], in [16], and in [1]. A similar approach yielded periodic solutions of a two-dimensional system with two delays in [17], and for arbitrary N in the singularly perturbed case in [8].

In the present paper, we focus on the case $N = 3$. Thus, the increase in dimension is only a modest one, but we think that our treatment exhibits the difficulties as well as the possibly successful techniques for the case of general $N > 3$. Note also that the system with some N is not a special case of the system with larger N . As a technical variant, we decided to apply the Schauder fixed point theorem instead of the ejective fixed point theorem.

We now sketch how the above system (with $N = 3$) is transformed to a more convenient standard form (compare [1]). Setting $\tau := \tau_1 + \tau_2 + \tau_3$ and $z_1 := y_1$, $z_2 := y_2(t - \tau_2)$, $z_3 := y_3(t - \tau_2 - \tau_3)$, the system transforms into a system with the single delay τ that appears only in the third equation. Then, setting $q_j(t) := q_j(\tau t)$, and writing μ_j and f_j instead of $\tau\mu_j$ and τf_j , one obtains the following equations, where the delay is normalized to 1.

$$\begin{cases} \dot{q}_1(t) + \mu_1 q_1(t) = f_1(q_2(t)) \\ \dot{q}_2(t) + \mu_2 q_2(t) = f_2(q_3(t)) \\ \dot{q}_3(t) + \mu_3 q_3(t) = f_3(q_1(t-1)) \end{cases}$$

By further transformations of the form $x_j(t) = -q_j(t)$ and $g_j(x) := f_j(-x)$, we can achieve that for the transformed system

$$(S) \begin{cases} \dot{x}_1(t) + \mu_1 x_1(t) = g_1(x_2(t)) \\ \dot{x}_2(t) + \mu_2 x_2(t) = g_2(x_3(t)) \\ \dot{x}_3(t) + \mu_3 x_3(t) = g_3(x_1(t-1)) \end{cases}$$

one has

$$xg_j(x) > 0 \text{ for } x \neq 0 \text{ and } j = 1, 2, \text{ and } xg_3(x) < 0 \text{ for } x \neq 0. \quad (H2)$$

We consider system (S) (where the g_j are C^1 functions) from now on. We assume that

$$k_j := g'_j(0) \neq 0, \quad (j = 1, 2, 3), \quad (H3)$$

which together with (H2) implies $k_1, k_2 > 0$ and $k_3 < 0$, and we set

$$K := -k_1 k_2 k_3 > 0.$$

By a solution of system (S) we mean a triple (x_1, x_2, x_3) , where $x_1 : [-1, \infty) \rightarrow \mathbb{R}$ is continuous and has a differentiable restriction to $[0, \infty)$, and $x_2, x_3 : [0, \infty) \rightarrow \mathbb{R}$ are differentiable, and the equations in system (S) hold for all $t \geq 0$. Set $C := C^0([-1, 0], \mathbb{R})$. As state space for system (S) we take the space $\mathcal{C} := C \times \mathbb{R} \times \mathbb{R}$. It is seen by the method of steps that each initial value $\psi = (\varphi, x_2^0, x_3^0) \in \mathcal{C}$ defines a unique solution (x_1, x_2, x_3) of (S) with $x_2(0) = x_2^0, x_3(0) = x_3^0$, and such that the initial segment $x_{1,0}$ of the function x_1 coincides with φ . (As usual, the segment x_t at t of a function x is defined by $x_t(\theta) := x(t + \theta)$, $\theta \in [-1, 0]$, if x is defined at least on $[t-1, t]$.)

Our main result is the following.

Theorem 1.1. *a) There exist numbers $K_c > 0$ and $K_u > 0$ (determined by μ_1, μ_2, μ_3) such that the linearization of (S) at zero*

- has no real characteristic values if and only if $K > K_c$;*
- has characteristic values with positive real part if and only if $K > K_u$.*

b) If $K > \max\{K_u, K_c\}$ and if $|g_3|$ is bounded then system (S) has a nonconstant periodic solution.

The boundedness assumption on g_3 could be relaxed, as well as the smoothness conditions on the g_j . We do not pursue this, in order to avoid too many technicalities.

Part a) of the theorem is proved in Section 2, where we analyze the characteristic equation of the system linearized at zero. We also obtain detailed information on the stability border K_u , and we show that both cases $K_u > K_c$ and $K_c < K_u$ can occur. This is in contrast to the one- and two-dimensional cases, where instability at zero automatically implies the absence of real eigenvalues. The condition $K > K_c$ excludes solutions going to zero monotonously and thus enforces oscillation of solutions, at first for the linearized system.

We define a suitable cone and a return map P_S for system (S) in Section 3. The construction of P_S requires the extension of oscillation properties to solutions of the full nonlinear system, which is based on the fact that a non-oscillating solution would converge to zero and asymptotically satisfy the linearized system. For the details of this argument, the nonexistence of super-exponential decay of solutions is important. It is sufficient for our purposes to prove this for solutions starting in the cone. A general result for a class of systems including (S) seems to be unavailable at present.

Part b) of the theorem is proved in Section 4, using the Schauder fixed point theorem. We prove a growth property for solutions close to zero starting in the cone. With this result, the construction of an invariant convex, compact set away from zero for an iterate of P_S is not difficult.

The proof of the growth property (ejectivity of zero) proceeds, as in previous works (e.g., by [1], [7]), via the Laplace transform of solutions with small amplitude. We use an argument expressing the domination of the linear over the nonlinear part. It is in particular in this argument, and in the detailed analysis of the characteristic equation, where our approach is presently limited to dimension three.

We briefly mention two possible alternative approaches, which are not realized at this time:

1) The Morse decomposition result for scalar delay equations from [14] shows, in particular, that nonconstant periodic solutions exist in each level set of a zero-counting Liapunov functional. A corresponding result for systems of more than one equation would include our main result, but is to our knowledge presently not proved.

2) In the spirit of the geometric description of subsets of the global attractor in [18] (for negative monotone feedback) and [10] (for positive monotone feedback), one might conjecture the following: If the linearization at zero has a conjugate pair of eigenvalues in the right half plane then (under additional, presently unknown conditions) the global continuation of the local unstable manifold at zero contains a nonconstant periodic orbit in its closure. Again, such a result would include ours, but does not exist presently.

Our treatment of the three-dimensional case provides a systematic arrangement of techniques which may also work in higher dimensions, and certainly makes the structure of earlier proofs for $N = 1, 2$ more transparent. It further gives a view on some seemingly rather unexplored areas, which are of interest for future research. Such are, e.g., a systematic analysis of characteristic equations in higher dimensions, or the exclusion of super-exponential decay for general systems.

2. The characteristic equation. The linearization of system (S) at the zero solution is given by

$$(L) \quad \begin{cases} \dot{x}_1(t) + \mu_1 x_1(t) = k_1 x_2(t) \\ \dot{x}_2(t) + \mu_2 x_2(t) = k_2 x_3(t) \\ \dot{x}_3(t) + \mu_3 x_3(t) = k_3 x_1(t-1) \end{cases}$$

The exponential Ansatz $x(t) = e^{zt}\xi$ with $\xi \in \mathbb{C}^3$ for complex-valued solutions of (L) leads to the following characteristic equation for $z \in \mathbb{C}$: $(z + \mu_1)(z + \mu_2)(z + \mu_3) + Ke^{-z} = 0$, with $K = -k_1 k_2 k_3 > 0$ as above. This equation determines the stability of the zero solution as well as the oscillation properties of solutions close to zero. Setting

$p(z) := (z + \mu_1)(z + \mu_2)(z + \mu_3)$ for $z \in \mathbb{C}$, we therefore study the equation

$$f_K(z) := p(z) + Ke^{-z} = 0, \quad (\text{ch})$$

for $\mu_1, \mu_2, \mu_3 > 0$ and $K \geq 0$. For these K , we define

$$\Sigma_K := \left\{ z \in \mathbb{C} \mid f_K(z) = 0 \right\}$$

(the dependence on the μ_j is not expressed). Let ν_1, ν_2, ν_3 denote the numbers $-\mu_1, -\mu_2, -\mu_3$ in increasing order, i.e.,

$$\nu_1 \leq \nu_2 \leq \nu_3 < 0 \text{ and } \{\nu_1, \nu_2, \nu_3\} = \{-\mu_1, -\mu_2, -\mu_3\}.$$

Proposition 2.1. *Let $K > 0$.*

- (i) $\Sigma_K \cap [0, \infty) = \emptyset$.
- (ii) $\forall l \in \mathbb{N}_0 : \Sigma_K \cap \left\{ r + (2l+1)\pi i \mid r \in [0, \infty) \right\} = \emptyset$.
- (iii) *If $z \in \Sigma_K \setminus \mathbb{R}$ then z is a simple zero of f_K .*
- (iv) $\forall z \in \Sigma_K : \operatorname{Re}(z) \leq \max\{0, \log \frac{K}{|\nu_3|^3}\}$.

Proof. Ad (i): For $z \in [0, \infty)$, one has $f_K(z) \in \mathbb{R}$ and $f_K(z) \geq \mu_1 \mu_2 \mu_3 + Ke^{-z} > 0$.

Ad (ii): If $z = r + (2l+1)\pi i$, $r \geq 0$ and $f_K(z) = 0$ then $p(z) = -Ke^{-r}e^{-i\pi} = Ke^{-r} \in (0, \infty)$. But we have $(z - \nu_j) = r_j e^{i\varphi_j}$, $j = 1, 2, 3$, with $\varphi_j \in (0, \pi/2)$ and $r_j > 0$, so $p(z) = r_1 r_2 r_3 e^{i(\varphi_1 + \varphi_2 + \varphi_3)}$ and $\varphi_1 + \varphi_2 + \varphi_3 \in (0, 3\pi/2)$ imply $p(z) \notin (0, \infty)$. This contradiction proves the assertion.

Ad (iii): Define $q(z) := p(z) + p'(z)$ ($z \in \mathbb{C}$).

Claim: $q^{-1}(\{0\}) \subset (-\infty, 0)$.

Proof: We have $\lim_{z \rightarrow \pm\infty} q(z) = \lim_{z \rightarrow \pm\infty} p(z) = \pm\infty$. There exist $z_{\max} \in [\nu_1, \nu_2]$ and $z_{\min} \in [\nu_2, \nu_3]$ with $p(z_{\max}) \geq 0$, $p(z_{\min}) \leq 0$, $p'(z_{\min}) = p'(z_{\max}) = 0$. Note that $p'(z) > 0$ for $z > z_{\min}$ and $p(z) \geq 0$ for $z \geq 0$ imply $q(z) > 0$ for $z \geq 0$.

If $\nu_1 < \nu_2 < \nu_3$ then $q(z_{\max}) > 0 > q(z_{\min})$, and it follows that q has zeroes in $(-\infty, z_{\max})$, in (z_{\max}, z_{\min}) and in $(z_{\min}, 0)$, which implies the assertion, since q is of degree three.

If $\nu_1 = \nu_2 < \nu_3$ then ν_1 is a simple zero of q , and $q(z) > 0$ for $z < \nu_1$ and close to ν_1 , and $q(z) < 0$ for $z > \nu_1$ close to ν_1 . Further, as above, $q(z_{\min}) < 0$, $q(0) > 0$. Hence q has zeroes in $(-\infty, \nu_1)$ and in $(z_{\min}, 0)$, and $q(\nu_1) = 0$.

In case $\nu_1 < \nu_2 = \nu_3$, the argument is analogous.

Finally, if $\nu_1 = \nu_2 = \nu_3$ then $q(z) = 3(z - \nu_1)^2 + (z - \nu_1)^3 = (z - \nu_1)^2 \cdot (z - \nu_1 + 3)$, which has the double zero ν_1 and the simple zero $\nu_1 - 3$. The claim is proved.

Assume now that $z \in \mathbb{C}$, $K > 0$ and $f_K(z) = f'_K(z) = 0$. Then $p(z) + Ke^{-z} = 0 = p'(z) - Ke^{-z}$, so $q(z) = p(z) + p'(z) = 0$. The above claim shows $z \in \mathbb{R}$. Hence, $z \in \Sigma_K \setminus \mathbb{R}$ implies $f'_K(z) \neq 0$.

Ad (iv): Let $z \in \mathbb{C}$. If $\operatorname{Re}(z) > \max\{0, \log \frac{K}{|\nu_3|^3}\}$ then $|p(z)| \geq |\operatorname{Re}(z - \nu_1) \operatorname{Re}(z - \nu_2) \operatorname{Re}(z - \nu_3)| \geq |\nu_3|^3$ and $|Ke^{-z}| < |\nu_3|^3 \leq |p(z)|$, so $f_K(z) \neq 0$. \square

Define now $\alpha : (0, \infty) \rightarrow \mathbb{R}$, $\alpha(\omega) := \sum_{j=1}^3 \arctan(\omega/\mu_j)$.

Proposition 2.2. For $l \in \mathbb{N}_0$, the equation

$$-\omega + (2l + 1)\pi = \alpha(\omega) \quad (2.2.1)$$

has exactly one real solution ω_l^* , and the following properties hold:

- (i) $\omega_0^* \in (0, \pi)$.
- (ii) $(2l - \frac{1}{2})\pi < \omega_l^* < (2l + 1)\pi$ and $\omega_l^* < \omega_{l+1}^*$ ($l \in \mathbb{N}_0$).
- (iii) $\lim_{l \rightarrow \infty} \omega_l^* - (2l - \frac{1}{2})\pi = 0$.
- (iv) The following equivalence is true.

$$K > 0, \omega > 0 \text{ and } f_K(i\omega) = 0$$

$$\iff \omega = \omega_l^* \text{ for some } l \in \mathbb{N}_0,$$

$$\text{and } K = K_l^* := \sqrt{[(\omega_l^*)^2 + \mu_1^2][(\omega_l^*)^2 + \mu_2^2][(\omega_l^*)^2 + \mu_3^2]}.$$

- (v) For $\mu_j \rightarrow 0$ ($j = 1, 2, 3$), the corresponding values of K_0^* satisfy $K_0^* \rightarrow 0$.

Proof. Existence and uniqueness of ω_l^* for $l \in \mathbb{N}_0$ follow from the facts that $\alpha' > 0$ and $\alpha(\omega) \rightarrow 0$ ($\omega \rightarrow 0$).

Ad (i): For $l = 0$, the left hand side of (2.2.1) decreases from π to 0 on $(0, \pi)$, while the right hand side increases from 0 to $\arctan \pi$ on the same interval. It follows that $\omega_0^* \in (0, \pi)$.

Ad (ii): The first two inequalities follow from $\alpha(\omega) \in (0, 3\pi/2)$ for $\omega > 0$. The fact that the function $\omega \mapsto \omega + \alpha(\omega)$ is strictly increasing together with the equality $\omega_l^* + \alpha(\omega_l^*) = (2l + 1)\pi$ implies $\omega_l^* < \omega_{l+1}^*$.

Ad (iii): From (ii) we see that $\omega_l^* \rightarrow \infty$ ($l \rightarrow \infty$). Now using that $\alpha(\omega) \rightarrow 3\pi/2$ for $\omega \rightarrow \infty$ and eq. (2.2.1) one obtains the assertion.

Ad (iv): If $f_K(i\omega) = 0$ then, for $j = 1, 2, 3$, there exist unique $r_j > 0$ and $\varphi_j \in (0, \pi/2)$ with $i\omega - (-\mu_j) = r_j e^{i\varphi_j}$, $j = 1, 2, 3$. Namely, $\varphi_j = \arctan(\omega/\mu_j)$ and $r_j = \sqrt{\omega^2 + \mu_j^2}$. We have

$$\begin{aligned} 0 &= p(i\omega) + Ke^{i\omega} = r_1 r_2 r_3 e^{i(\varphi_2 + \varphi_3)} + Ke^{-i\omega} \\ &= \sqrt{(\omega^2 + \mu_1^2)(\omega^2 + \mu_2^2)(\omega^2 + \mu_3^2)} e^{i\alpha(\omega)} + Ke^{-i\omega}. \end{aligned}$$

Hence, we must have $\alpha(\omega) = -\omega + (2l+1)\pi$ for some $l \in \mathbb{Z}$, or $\omega + \alpha(\omega) = (2l+1)\pi$. Now $\omega > 0$ and $\alpha(\omega) > 0$ imply that $l \in \mathbb{N}_0$, so $\omega = \omega_l^*$ for this l . It follows that $K = K_l^*$.

Ad (v): Let $\varepsilon > 0$. There exists $\delta > 0$ such that $\mu_j < \delta$ ($j = 1, 2, 3$) implies $\alpha(\omega) > \pi$ for $\omega \in [\varepsilon, \infty)$. It follows from eq. (2.2.1) (for the case $l = 0$) that $\omega_0^* < \varepsilon$ for such values of the μ_j . Hence we have $\omega_0^* \rightarrow 0$ as $\mu_j \rightarrow 0$ ($j = 1, 2, 3$). The assertion now follows from the formula for K_0^* in (iv). \square

Define the region $R := \{z \in \mathbb{C} \mid \operatorname{Re}(z) > 0, \operatorname{Im}(z) \in (0, \pi)\}$.

The following lemma contains part a) of Theorem 1.1 (set $K_u := K_0^*$).

Lemma 2.3. *a) For fixed $\mu_1, \mu_2, \mu_3 > 0$, there exists $K_c > 0$ such that with K_0^* as in Proposition 2.2, (iv), the following equivalences hold for $K > 0$.*

- (i) f_K has no real zero $\iff K > K_c$.
- (ii) f_K has zeroes with positive real part $\iff K > K_0^*$
 $\iff f_K$ has a zero in the region R .

b) Depending on the values of μ_1, μ_2, μ_3 , both cases $K_c > K_0^*$ and $K_c < K_0^*$ can occur.

Proof. Ad (i): We know from Proposition 2.1, (i) that for $K > 0$ the function f_K has no zeroes in $[0, \infty)$, so real zeroes of f_K can occur only in $(-\infty, 0)$.

Recall the notation ν_j for the numbers $-\mu_k$ in increasing order, and set $\xi := \nu_1 - 1$. Then $p(\xi) < 0$. There exists $K_0 > 0$ with $Ke^{-\xi} < |p(\xi)|$ for $K \in (0, K_0]$, so $f_K(\xi) < 0$ for these K . Now $f_K(x) \rightarrow \infty$ ($x \rightarrow -\infty$), so f_K has a zero in $(-\infty, \xi)$ for $K \in (0, K_0]$. Hence, the following set contains positive numbers:

$$\mathcal{K} := \{K \geq 0 \mid f_K \text{ has a zero in } (-\infty, 0)\}.$$

Claim: If $b \in \mathcal{K}$ then $[0, b] \subset \mathcal{K}$.

Proof: The assertion is clear for $b = 0$. Assume $b \in \mathcal{K}$, $b > 0$, and $c \in [0, b) \setminus \mathcal{K}$. Then $c > 0$ necessarily, and f_c has no zero in \mathbb{R} , so $f_c(x) > 0$ for all $x \in \mathbb{R}$, since $\lim_{x \rightarrow -\infty} f_c(x) = \infty$. Monotonous dependence of f_K on K implies that also $f_b(x) > 0$ for all $x \in \mathbb{R}$, contradicting $b \in \mathcal{K}$. The claim is proved.

There exists $\eta \in (-\infty, 0)$ such that $\forall x \in (-\infty, \eta] : e^{-x} > |p(x)|$, and there exists $K_1 \geq 1$ such that $\forall x \in [\eta, 0] : |p(x)| < K_1$. For $K \geq K_1$ and $x \in (-\infty, 0)$ we obtain in case $x \in (-\infty, \eta)$ that

$$|f_K(x)| \geq K_1 e^{-x} - |p(x)| \geq e^{-x} - |p(x)| > 0,$$

and in case $x \in [\eta, 0)$ we have $|f_K(x)| \geq K_1 - |p(x)| > 0$. Thus, f_K has no zero in $(-\infty, 0)$ for $K \geq K_1$, so \mathcal{K} is bounded. Setting

$$K_c := \sup \mathcal{K},$$

we have $K_c > 0$. The above claim shows that $\mathcal{K} \supset [0, K_c)$. It follows from $f_{K_c} \rightarrow \infty$ ($x \rightarrow -\infty$) and from continuous dependence of f_K on K that f_{K_c} must also have a zero in $(-\infty, 0)$, so we conclude $\mathcal{K} = [0, K_c]$. The assertion of (i) follows.

Ad (ii): 1. We first show that existence of a zero $z = r + is$ of f_K with $r > 0$ implies $K > K_0^*$. Proposition 2.1, (i) shows that $s \neq 0$, and we can assume $s > 0$, since \bar{z} is also a zero of f_K . Using Proposition 2.1, (iii), and the implicit function

theorem, one obtains the existence of a maximal open interval $I \subset (0, \infty)$ with $K \in I$ and of a differentiable function $\zeta : I \rightarrow \mathbb{C}$ such that $\zeta(K) = z$ and

$$\forall \kappa \in I : f_\kappa(\zeta(\kappa)) = 0, f'_\kappa(\zeta(\kappa)) \neq 0.$$

Abbreviating $\zeta(\kappa)$ with ζ , we calculate (using that $p(\zeta) + \kappa e^{-\zeta} = 0$)

$$\zeta'(\kappa) = \frac{-e^{-\zeta}}{f'_\kappa(\zeta)} = \frac{p(\zeta)}{\kappa[p'(\zeta) + p(\zeta)]} \text{ for } \kappa \in I.$$

Set $a := \inf \{ \tilde{\kappa} \in I \mid \operatorname{Re}(\zeta(\kappa)) > 0 \text{ for all } \kappa \in (\tilde{\kappa}, K] \}$. Then $a < K$ and $\operatorname{Re}(\zeta(\cdot)) > 0$ on $(a, K]$. Recall that $q := p + p'$ has zeroes only in $(-\infty, 0)$ (see the claim in the proof of Proposition 2.1, (iii)). It follows that there exists $\varepsilon > 0$ with $|p'(\zeta(\kappa)) + p(\zeta(\kappa))| \geq \varepsilon$ for all $\kappa \in (a, K]$. The above formula for ζ' now shows that ζ' is bounded on $(a, K]$, which implies the existence of $z_1 := \lim_{\kappa \searrow a} \zeta(\kappa)$. Continuity implies $\operatorname{Re}(z_1) \geq 0$ and $f_a(z_1) = 0$.

If $\operatorname{Re}(z_1) > 0$ then, again, parts (i) and (iii) of Proposition 2.1 show that $f'_a(z_1) \neq 0$, so ζ is defined also for arguments below a , i.e., $a > \inf I$. But then $\operatorname{Re}(\zeta(\kappa)) > 0$ also for $\kappa \in (a - \delta, K]$ for some $\delta > 0$, contradicting the definition of a . Hence we must have $\operatorname{Re}(z_1) = 0$, and $z_1 = i\omega$ for some $\omega \in \mathbb{R}$. Using $s > 0$, continuity of ζ , and Proposition 2.1, (i), we see that $\omega \geq 0$. Since $\omega = 0$ would imply $f_a(0) = 0$ (which is impossible since $f_a(0) \geq \mu_1\mu_2\mu_3 > 0$), we have $\omega > 0$. It follows that also $a > 0$, since f_0 has only the zeroes ν_1, ν_2, ν_3 . Now Proposition 2.2, (iv) shows that $\omega = \omega_l^*$ for some $l \in \mathbb{N}_0$, and $a = K_l^*$. Hence $K > a = K_l^* \geq K_0^*$.

2. We next show that $K > K_0^*$ implies that f_K has a zero in the region R . We have $f_{K_0^*}(i\omega_0^*) = 0$, and Proposition 2.1, (iii) shows that $f'_{K_0^*}(i\omega_0^*) \neq 0$. As above, there exists a maximal open interval $J \subset (0, \infty)$ and a differentiable function $\zeta : J \rightarrow \mathbb{C}$ with $\zeta(K_0^*) = i\omega_0^*$ such that

$$\forall \kappa \in J : f_\kappa(\zeta(\kappa)) = 0, f'_\kappa(\zeta(\kappa)) \neq 0.$$

For $\kappa \in J$, we have (again writing ζ for $\zeta(\kappa)$)

$$[\operatorname{Re} \zeta]'(\kappa) = \operatorname{Re}[\zeta'(\kappa)] = \operatorname{Re} \frac{1}{\kappa} \frac{p(\zeta)}{p'(\zeta) + p(\zeta)}.$$

With $m(\kappa) := \frac{1}{\kappa |p'(\zeta(\kappa)) + p(\zeta(\kappa))|^2} > 0$ ($\kappa \in J$), we obtain

$$\operatorname{Re} \zeta'(\kappa) = m(\kappa) \operatorname{Re}\{p(\zeta)[\overline{p'(\zeta)} + \overline{p(\zeta)}]\} = m(\kappa)[|p(\zeta)|^2 + \operatorname{Re}(p(\zeta)\overline{p'(\zeta)})].$$

Claim: $\operatorname{Re}(\zeta'(K_0^*)) > 0$.

Proof: Setting $a_2 := \mu_1 + \mu_2 + \mu_3$, $a_1 := \mu_1\mu_2 + \mu_2\mu_3 + \mu_3\mu_1$, and $a_0 := \mu_1\mu_2\mu_3$, we have for $z \in \mathbb{C}$

$$p(z) = z^3 + a_2z^2 + a_1z + a_0 \text{ and } p'(z) = 3z^2 + 2a_2z + a_1.$$

We hence obtain $p(\zeta)\overline{p'(\zeta)} = (\zeta^3 + a_2\zeta^2 + a_1\zeta + a_0)(3\bar{\zeta}^2 + 2a_2\bar{\zeta} + a_1)$. Now for $\kappa = K_0^*$ we have $\zeta = \zeta(K_0^*) = i\omega_0^*$. Writing ω instead of ω_0^* , we see that

$$p(i\omega)\overline{p'(i\omega)} = (-i\omega^3 - a_2\omega^2 + a_1i\omega + a_0)(-3\omega^2 - 2a_2i\omega + a_1), \text{ and}$$

$$\begin{aligned} \operatorname{Re}(p(i\omega)\overline{p'(i\omega)}) &= (a_0 - a_2\omega^2)(a_1 - 3\omega^2) + 2a_2\omega(a_1\omega - \omega^3) \\ &= a_2\omega^4 + (a_2a_1 - 3a_0)\omega^2 + a_0a_1. \end{aligned}$$

Now

$$\begin{aligned} a_2 a_1 - 3a_0 &= (\mu_1 + \mu_2 + \mu_3)(\mu_1 \mu_2 + \mu_2 \mu_3 + \mu_3 \mu_1) - 3\mu_1 \mu_2 \mu_3 \\ &= \mu_1^2(\mu_2 + \mu_3) + \mu_2^2(\mu_1 + \mu_3) + \mu_3^2(\mu_1 + \mu_2) > 0, \end{aligned}$$

so $\operatorname{Re}(p(i\omega)\overline{p'(i\omega)}) > 0$. With the above formula for $\operatorname{Re} \zeta'(\kappa)$, we conclude that

$$\operatorname{Re}(\zeta'(K_0^*)) = m(K_0^*)[|p(i\omega_0^*)|^2 + \operatorname{Re}(p(i\omega_0^*)\overline{p'(i\omega_0^*)})] > 0.$$

The claim is proved.

Claim: For all $\kappa \in (K_0^*, \sup J)$ one has $\zeta(\kappa) \in R$.

Proof: If the claim were not true, continuity of ζ would imply the existence of a zero of f_κ on ∂R for some $\kappa \in (K_0^*, \sup J)$. Note that

$$\partial R = \left\{ i\omega \mid \omega \in (0, \pi) \right\} \cup [0, \infty) \cup \left\{ r + i\pi \mid r \in [0, \infty) \right\}.$$

We know from Proposition 2.1, (i) and (ii) that the last two sets contain no zeroes of f_κ . Further, a zero $i\omega$ of f_κ with $\omega \in (0, \pi)$ is impossible for $\kappa > K_0^*$, since $\omega_l^* > \pi$ for $l \geq 1$ (see Proposition 2.2). Hence the claim is true.

Claim: $\sup J = \infty$.

Proof: Assume that $b := \sup J$ is finite. As in part 1 of the proof, ζ' is bounded on $[K_0^*, b)$, and $z^+ := \lim_{\kappa \nearrow b} \zeta(\kappa)$ exists. Continuity implies $f_b(z^+) = 0$ and $z^+ \in \operatorname{clos} R$. As above, we cannot have $z^+ \in \partial R$, so $z^+ \in R$. It follows from Proposition 2.1, (iii) that $f'_b(z^+) \neq 0$, and hence ζ can be continued beyond b , contradicting maximality of the interval J . The claim is proved.

The last two claims combined show that $\forall K > K_0^* : \Sigma_K \cap R \neq \emptyset$.

3. From steps 1 and 2 and the trivial implication

$$\Sigma_K \cap R \neq \emptyset \implies f_K \text{ has a zero with positive real part,}$$

we obtain the asserted equivalences. Part a) of the lemma is proved.

Proof of b): (We show first that $K_0^* < K_c$ is possible.) There exists $K_1 > 0$ such that for $K \in [0, K_1]$ the function $z \mapsto z^3 + Ke^{-z}$ is negative on $[-2, -1]$. There exists $m_1 > 0$ such that for $\mu_1, \mu_2, \mu_3 \in (0, m_1]$ and $K \in [0, K_1]$, the corresponding function f_K is also negative on $[-2, -1]$. It follows that for $K \in (0, K_1]$ and $\mu_j \in (0, m_1]$ ($j = 1, 2, 3$), there is a zero of f_K in $(-\infty, -2)$, and hence $K_c > K_1$ for these values of the μ_j . In combination with Proposition 2.2, (v), one sees that for sufficiently small values of the μ_j one has $K_0^* < K_1 < K_c$.

We show now that $K_c < K_0^*$ is possible. There exists $M \geq 1$ such that for $\mu_j = M, j = 1, 2, 3$, the corresponding functions $f_K : z \mapsto (z + M)^3 + Ke^{-z}$ have no real zeroes if $K \geq 1$. It follows that $K_c \leq 1$ for these μ_j . On the other hand, using Proposition 2.2, (iv) one obtains $K_0^* > \mu_1 \mu_2 \mu_3 = M^3 \geq 1$ for these μ_j , so $K_0^* > K_c$ in this example. \square

Remark 2.4. Both cases $K_0^* > K_c$ and $K_0^* < K_c$ are easy to observe numerically, if one searches for solutions of equation (ch) with a Newton procedure. For example, for $\mu_1 = 0.1, \mu_2 = 0.2, \mu_3 = 0.3$ one sees that $K_0^* < 0.4 < K_c$.

3. Oscillation of solutions and the return map. On the space of initial values $\mathcal{C} = C \times \mathbb{R} \times \mathbb{R}$ we use the norm defined by

$$\|\psi\| := \max\{|\varphi|, |y_2^0|, |y_3^0|\}, \text{ if } \psi = (\varphi, y_2^0, y_3^0), \text{ and } |\varphi| = \max_{\theta \in [-1, 0]} |\varphi(\theta)|.$$

(We sometimes use the notation $|\cdot|_{C^0}$ for the sup-norm on other intervals than $[-1, 0]$.) For $X = (x_1, x_2, x_3)$, where $x_1 : [-1, \infty) \rightarrow \mathbb{R}$ and $x_2, x_3 : [0, \infty) \rightarrow \mathbb{R}$ are continuous, and for $t \geq 0$, the symbol X_t denotes the triple $(x_{1,t}, x_2(t), x_3(t)) \in \mathcal{C}$.

We say that a function f has a property *eventually* if there exists $T \in \mathbb{R}$ such that f has this property on $[T, \infty)$. We abbreviate ‘eventually’ with ‘ev.’

We make some elementary observations concerning the initial value problem

$$\dot{u}(t) + \mu u(t) = a(t), \quad u(t_0) = u_0 \in \mathbb{R}, \quad \mu > 0 \quad (3.1)$$

with a continuous function a . (The equations of system (S) have this form.) The solution of (3.1) is given by the variation of constants formula,

$$u(t) = u_0 e^{-\mu(t-t_0)} + \int_{t_0}^t e^{\mu(s-t)} a(s) ds. \quad (3.2)$$

Proposition 3.1. *Let $t_1 \geq t_0$, and let u be the solution of (3.1). The following implications hold.*

- (i) $u_0 > 0$ and $a \geq 0$ on $[t_0, t_1] \implies u > 0$ on $[t_0, t_1]$.
 $u_0 < 0$ and $a \leq 0$ on $[t_0, t_1] \implies u < 0$ on $[t_0, t_1]$.

$$u_0 = 0 \text{ and } a \begin{cases} < 0 \\ > 0 \end{cases} \text{ on } (t_0, t_1] \implies u \begin{cases} < 0 \\ > 0 \end{cases} \text{ on } (t_0, t_1].$$

- (ii) $a \leq 0 < u$ on $[t_0, t_1] \implies \dot{u} < 0$ on $[t_0, t_1]$,
 $u < 0 \leq a$ on $[t_0, t_1] \implies \dot{u} > 0$ on $[t_0, t_1]$.
- (iii) If $\lim_{t \rightarrow \infty} a(t) = 0$, and a is differentiable, then
 $u > 0$ ev. and $a > 0$ ev., $\dot{a} < 0$ ev. $\implies \dot{u} < 0$ ev.
 $u < 0$ ev. and $a < 0$ ev., $\dot{a} > 0$ ev. $\implies \dot{u} > 0$ ev.

- (iv) If $a(t) \rightarrow \alpha \in \mathbb{R}$ then $u(t) \rightarrow \alpha/\mu$, as $t \rightarrow \infty$.

- (v) $u_0 = 0$ and $|a| \leq \alpha$ on $[t_0, t_1] \implies |u| \leq \alpha/\mu$ on $[t_0, t_1]$.

Proof. Ad (i): These implications are seen from formula (3.2).

Ad (ii): We have $\dot{u}(t) = -\mu u(t) + a(t)$, which is negative on $[t_0, t_1]$ in the first case and positive in the second case.

Ad (iii): (Proof of the first implication.) There exists $t_1 \geq t_0$ such that $u > 0$, $a > 0$, and $\dot{a} < 0$ on $[t_1, \infty)$. It suffices to prove the assertion under the assumptions $u > 0$ and $a > 0, \dot{a} < 0$ on $[t_0, \infty)$, which we make from now on.

Case 1: $u_0 > a(t_0)/\mu$. Assume that there exists a first point t_2 such that $u(t_2) = a(t_2)/\mu$ and $u(t) > a(t)/\mu$ for all $t \in [t_0, t_2)$. Taking difference quotients for u and for a/μ at t_2 , one sees that $\dot{u}(t_2) \leq \dot{a}(t_2)/\mu < 0$. On the other hand, $\dot{u}(t_2) = a(t_2) - \mu u(t_2) = 0$, a contradiction. Hence $u > a/\mu$ on $[t_0, \infty)$, so $\dot{u} < 0$ on $[t_0, \infty)$.
Case 2: $u_0 \leq a(t_0)/\mu$. If $u_0 < a(t_0)/\mu$ then $\dot{u}(t_0) > 0$, and $\dot{u} > 0$ as long as $u < a/\mu$. Since $u_0 \geq 0$ and $a(t)$ is decreasing to zero as $t \rightarrow \infty$, there exists $t_3 > t_0$ such that $u(t_3) = a(t_3)/\mu$. If $u_0 = a(t_0)/\mu$, set $t_3 := t_0$.

One has $\dot{u}(t_3) = 0$, and $\dot{a}(t_3) < 0$. Therefore, $u(t) > a(t)/\mu$ for all t in some interval of the form $(t_3, t_3 + \delta]$. It follows now from case 1 that $u > a/\mu$ and $\dot{u} < 0$ on $[t_3 + \delta, \infty)$.

The first implication is proved. Proof of the second implication: Set $v(t) := -u(t)$ for $t \geq t_0$. Then $\dot{v}(t) + \mu v(t) = -(\dot{u}(t) + \mu u(t)) = -a(t)$, and $-a > 0$ ev., and $-\dot{a} < 0$ ev. The first implication applied to v shows that $\dot{v} < 0$ ev. The assertion for u follows.

Ad (iv): The first term in (3.2) converges to zero, and the second term equals $\int_{\mathbb{R}} \chi_{[0, t-t_0]}(u) e^{-\mu u} a(t-u) du$, where $\chi \dots$ denotes the characteristic function. There exists $A > 0$ such that $|a|$ is bounded by A . It follows from the Lebesgue dominated convergence theorem that the second term converges to $\int_0^\infty \alpha e^{-\mu u} du = \alpha/\mu$.

Ad (v): For $t \in [t_0, t_1]$, one obtains from (3.2)

$$u(t) \leq \alpha \int_{t_0}^t e^{\mu(s-t)} ds = \alpha \int_0^{t-t_0} e^{-\mu s} ds \leq \alpha/\mu.$$

□

We define some notions of oscillation now.

Definition 3.2. a) Let z be a scalar function defined on an interval of the form $[a, \infty)$. We say that z is oscillatory or oscillates, b) Let $X = (x_1, x_2, x_3)$ be a nonzero solution of system (S). We say that X is oscillatory (oscillates), if all its components x_1, x_2, x_3 oscillate.

We shall need a statement which excludes monotonous superexponential decay of solutions. The following preparatory remark excludes, in particular, finite-time convergence to zero.

Remark 3.3. Let $X = (x_1, x_2, x_3)$ be a solution of system (S). If $x_{j,T} = 0$ ($\in C$) for some $j \in \{1, 2, 3\}$ and some $T \geq 1$, then $X_0 = 0$.

Proof. For $t \geq 1$, the following implications follow from system (S):

$$x_{1,t} = 0 \implies x_{2,t} = 0 \implies x_{3,t} = 0 \implies x_{1,t-1} = 0 \quad (3.3.1)$$

Assume now $x_{j,T} = 0$ for some $j \in \{1, 2, 3\}$ and $T \geq 1$. Then $x_{k,T-1} = 0$ for $k = 1, 2, 3$, and thus $x_k(t) = 0$ for all $t \geq T-1$, $k = 1, 2, 3$. Take $n \in [T, \infty) \cap \mathbb{N}$. Using (3.3.1) inductively, it follows from $x_{1,n} = x_{2,n} = x_{3,n} = 0$ that $x_{1,1} = x_{2,1} = x_{3,1} = 0$. In particular, $x_2(0) = x_3(0) = 0$. The last implication in (3.3.1) now shows that $x_{1,0} = 0$, so $X_0 = 0$. □

Corollary 3.4. Let $X = (x_1, x_2, x_3)$ be a solution of system (S).

- a) If one component of X converges then all components converge to zero.
- b) If one component of X oscillates then all components oscillate.
- c) If X is not zero and does not oscillate, then there exists $T > 0$ such that all components x_j have no zero on $[T, \infty)$, $j = 1, 2, 3$.

Proof. Ad a): The equations of system (S) combined with Proposition 3.1,(iv) give the following implications:

$$\begin{aligned} x_1(t) \rightarrow \lambda_1 \in \mathbb{R} &\implies x_3(t) \rightarrow g_3(\lambda_1)/\mu_1, \\ x_3(t) \rightarrow \lambda_3 \in \mathbb{R} &\implies x_2(t) \rightarrow g_2(\lambda_3)/\mu_2, \\ x_2(t) \rightarrow \lambda_2 \in \mathbb{R} &\implies x_1(t) \rightarrow g_1(\lambda_2)/\mu_1. \end{aligned}$$

Together, one sees that if one component converges, then all components converge, and the limits λ_i of x_i satisfy

$$\lambda_3 = g_3(\lambda_1)/\mu_1, \lambda_2 = g_2(\lambda_3)/\mu_2, \lambda_1 = g_1(\lambda_2)/\mu_1.$$

The feedback conditions on the functions g_i now imply that if $\lambda_1 \neq 0$ then

$$\text{sign}(\lambda_3) = -\text{sign}(\lambda_1) = -\text{sign}(\lambda_2) = -\text{sign}(\lambda_3),$$

which is impossible. Hence $\lambda_1 = 0 = \lambda_3 = \lambda_2$.

Ad b): Let $j \in \{1, 2, 3\}$. If the component x_j does not oscillate, we must have $x_j \geq 0$ ev. or $x_j \leq 0$ ev.

Claim: $x_{j-1} \geq 0$ ev. or $x_{j-1} \leq 0$ ev. (where $j-1$ is taken mod 3). Proof: The claim is clear if x_{j-1} has only finitely many zeroes. Otherwise, it follows from the $j-1$ st equation of system (S) and from the feedback condition on g_{j-1} , if we apply formula (3.2) with a sufficiently large zero t_0 of x_{j-1} . (The claim is proved.) Hence x_{j-1} does also not oscillate. It follows that either all components oscillate, or no component oscillates.

Ad c): From the definition of oscillation, we know that there exists $t_0 > 0$ such that $x_j \geq 0$ or $x_j \leq 0$ on $[t_0, \infty)$, $j = 1, 2, 3$. Assume now that for one $j \in \{1, 2, 3\}$, the component x_j has arbitrarily large zeroes, and let $z^j \in [t_0 + 1, \infty)$ be such a zero. From formula (3.2), applied to $x_j(z^j + \cdot)$, we obtain for $t \geq z_j$ that

$$x_j(t) = \int_{z^j}^t \exp[\mu_j(s-t)] g_j(x_{j+1}(s-\tau_j)) ds, \quad (3.4.1)$$

where the index $j+1$ is to be taken modulo 3, and where $\tau_1 = \tau_2 = 0, \tau_3 = 1$. Since X is nonzero, Remark 3.3 shows that the function $s \mapsto x_{j+1}(s-\tau_j)$ is not constantly zero on $[z^j, \infty)$. Further, we know that this function takes either nonnegative or nonpositive values. Now the feedback condition on g_j implies that the right-hand side of (3.4.1) is either positive or negative for all sufficiently large values of t . This contradicts the existence of arbitrarily large zeroes of x_j . The assertion follows. \square

The proof of the following remark is obvious.

Remark 3.5. Let $X = (x_1, x_2, x_3)$ be solution of system (S). Then $Y := -X$ satisfies a system (S^-) analogous to (S), where the functions g_j are replaced by $h_j := -g_j(-\cdot)$, $j = 1, 2, 3$. The functions h_j have the same smoothness and feedback properties as the g_j , and the same derivatives at zero.

The functions $\gamma_j : [0, \infty) \rightarrow \mathbb{R}$, $\gamma_j(r) := \max \left\{ |g_j(x)| \mid |x| \leq r \right\}$ do not change their values if, in their definition, g_j is replaced by h_j , $j = 1, 2, 3$.

Proposition 3.6. Let $X = (x_1, x_2, x_3)$ be a nonzero and non-oscillating solution of system (S). Then $\text{sign}(x_i(\cdot)) = -\text{sign}(\dot{x}_i(\cdot))$ eventually, and $|x_i|$ strictly decays to zero eventually ($i = 1, 2, 3$).

Proof. We know from Corollary 3.4,c) that $|x_i| > 0$ ev., $i = 1, 2, 3$. Remark 3.5 permits us to assume that $x_1 > 0$ ev.

Claim: The following implications hold:

- (i) $x_2 < 0$ ev., $x_3 > 0$ ev. $\implies \dot{x}_1 < 0$ ev., $\dot{x}_2 > 0$ ev., $\dot{x}_3 < 0$ ev.
- (ii) $x_2 < 0$ ev., $x_3 < 0$ ev. $\implies \dot{x}_1 < 0$ ev., $\dot{x}_2 > 0$ ev., $\dot{x}_3 > 0$ ev.
- (iii) $x_2 > 0$ ev., $x_3 > 0$ ev. $\implies \dot{x}_1 < 0$ ev., $\dot{x}_2 < 0$ ev., $\dot{x}_3 < 0$ ev.
- (iv) $x_2 > 0$ ev., $x_3 < 0$ ev. $\implies \dot{x}_1 < 0$ ev., $\dot{x}_2 < 0$ ev., $\dot{x}_3 > 0$ ev.

We prove two auxiliary implications first:

$$x_2 > 0 \text{ ev.}, \dot{x}_2 < 0 \text{ ev.} \implies \dot{x}_1 < 0 \text{ ev.} \quad (3.6.1)$$

$$x_3 < 0 \text{ ev.}, \dot{x}_1 < 0 \text{ ev.} \implies \dot{x}_3 > 0 \text{ ev.} \quad (3.6.2)$$

Proof of (3.6.1): The assumptions and Corollary 3.4,a) imply that x_2 tends to zero. The function defined by $a(t) := g_1(x_2(t))$ satisfies $a > 0 \text{ ev.}$, and $\dot{a} < 0 \text{ ev.}$ (since $g'_1(0) > 0$). It follows from $x_1 > 0$, from the first equation, and from Proposition 3.1,(iii) that $\dot{x}_1 < 0 \text{ ev.}$

Proof of (3.6.2): $\dot{x}_1 < 0 \text{ ev.}$, $x_1 > 0$ and Corollary 3.4,a) imply that x_1 goes to zero. The function defined by $a(t) := g_3(x_1(t-1))$ satisfies $a(t) \rightarrow 0$, $a < 0 \text{ ev.}$, and $\dot{a} > 0 \text{ ev.}$ Now the assumption $x_3 < 0 \text{ ev.}$, the third equation, and the second part of Proposition 3.1,(iii) show that $\dot{x}_3 > 0 \text{ ev.}$

Proof of the claim:

Ad (i): This implication follows directly from $x_1 > 0 \text{ ev.}$, from the equations of system (S), and from the feedback conditions on the functions g_i .

Ad (ii): $x_2 < 0 \text{ ev.}$ and $x_1 > 0 \text{ ev.}$, together with the first equation and Proposition 3.1,(ii) imply $\dot{x}_1 < 0 \text{ ev.}$ Now the assumption $x_3 < 0 \text{ ev.}$ and (3.6.2) show $\dot{x}_3 > 0 \text{ ev.}$ Further, as above, Corollary 3.4,a) implies convergence of all components to zero. The function defined by $a(t) := g_2(x_3(t))$ tends to zero, and $a < 0 \text{ ev.}$, $\dot{a} > 0 \text{ ev.}$ Using Proposition 3.1,(iii) and the assumption $x_2 < 0 \text{ ev.}$, one obtains $\dot{x}_2 > 0 \text{ ev.}$

Ad (iii): $x_3 > 0 \text{ ev.}$ and the third equation, together with $x_1 > 0$ on $[0, \infty)$, show that $\dot{x}_3 < 0 \text{ ev.}$, so x_3 converges. Corollary 3.4,a) implies convergence to zero. Setting $a(t) := g_2(x_3(t))$, we have $a(t) \rightarrow 0$, and $a > 0 \text{ ev.}$, $\dot{a} < 0 \text{ ev.}$ Now $x_2 > 0 \text{ ev.}$ and Proposition 3.1, (iii) show $\dot{x}_2 < 0 \text{ ev.}$ Finally, the property $\dot{x}_1 < 0 \text{ ev.}$ follows from (3.6.1).

Ad (iv): $x_3 < 0 \text{ ev.}$, $x_2 > 0 \text{ ev.}$ and the second equation imply $\dot{x}_2 < 0 \text{ ev.}$ From (3.6.1), one obtains $\dot{x}_1 < 0 \text{ ev.}$ Now (3.6.2) shows $\dot{x}_3 > 0 \text{ ev.}$

The claim is proved. It follows now from Corollary 3.4,a) that all components converge to zero, and the claim shows that $|x_i|$ is eventually strictly decaying. \square

Now we prove a lower estimate on the decay of solutions of a particular type.

Lemma 3.7. *Let $X = (x_1, x_2, x_3)$ be a nonzero solution of system (S) with the following properties*

- a) $x_1 > 0 \text{ ev.}$
- b) $x_2 > 0$ and $x_3 > 0 \text{ ev.}$, or $x_2 > 0$ and $x_3 < 0 \text{ ev.}$, or $x_2 < 0$ and $x_3 < 0 \text{ ev.}$

Then $X_t \rightarrow 0$ ($t \rightarrow \infty$), and there exists $c > 0$ such that

$$\forall t \geq 0 : \|X_{t+1}\| \geq c\|X_t\|.$$

Proof. 1. Convergence to zero follows from Proposition 3.6. There exists $t_0 > 0$ such that the inequality $x_1 > 0$, and one of the properties listed in b), hold on $[t_0, \infty)$. Further, in view of Proposition 3.6, we can assume that each $|x_i|$ decays strictly (to zero) on $[t_0, \infty)$.

From Remark 3.3, all segments X_t ($t \geq 0$) are nonzero. Thus, the function $[0, \max\{t_0, 2\}] \ni t \mapsto \|X_{t+1}\|/\|X_t\|$ is well-defined, continuous, and attains a positive minimum \tilde{c} . Hence $\|X_{t+1}\| \geq \tilde{c}\|X_t\|$ for $t \in [0, \max\{t_0, 2\}]$.

2. Convergence to zero implies that there exists $R > 0$ such that $\forall t \geq 0 : \|X_t\| \leq R$. There exists $L > 0$ such that for $i = 1, 2, 3$ and $x \in [-R, R]$ one has

$$|g_i(x)| \leq L|x|. \quad (3.7.1)$$

3. We set $\tau_1 := 0, \tau_2 := 0, \tau_3 := 1$. Then, for $i = 1, 2, 3$ and $t \geq 0$ one has

$$\begin{aligned} \left(e^{\mu_i(\cdot)} |x_i(\cdot)| \right)'(t) &= \text{sign}(x_i) [e^{\mu_i(\cdot)} x_i(\cdot)]'(t) \\ &= \text{sign}(x_i) e^{\mu_i t} [\mu_i x_i(t) + \dot{x}_i(t)] \\ &= \text{sign}(x_i) e^{\mu_i t} g_i(x_{i+1}(t - \tau_i)), \end{aligned} \quad (3.7.2)$$

where the index $i + 1$ is to be taken mod 3. Hence,

$$\text{sign} \left(e^{\mu_i(\cdot)} |x_i(\cdot)| \right)' = \text{sign}(x_i) \text{sign}[g_i(x_{i+1}(\cdot - \tau_i))] \text{ on } [1, \infty).$$

We obtain the following implication, which holds for $i = 1, 2, 3$, and where $i + 1$ is to be read mod 3:

$$\begin{aligned} \text{sign}(x_i(\cdot)) &= \text{sign}[g_i(x_{i+1}(\cdot - \tau_i))] \text{ on } [1, \infty) \text{ and } t \geq 1, \theta \geq 0 \\ \implies |x_i(t + \theta)| &\geq e^{-\mu_i \theta} |x_i(t)|. \end{aligned} \quad (3.7.3)$$

4. We consider $T \geq \max\{t_0, 2\}$ now, and we compare $\|X_{T+1}\|$ to $\|X_T\|$. The definition of $\|\cdot\|$ and the monotonicity of x_1 imply that

$$\|X_T\| \in \{|x_2(T)|, |x_3(T)|, \max_{t \in [T-1, T]} |x_1(t)|\} = \{|x_2(T)|, |x_3(T)|, x_1(T-1)\}.$$

We distinguish different cases according to which of the last three numbers equals $\|X_T\|$, and in each case we have to consider different subcases according to which of the possibilities from assumption b) takes place. We briefly write ' $x_i < 0$ ' or ' $x_i > 0$ ' instead of ' $x_i < 0$ (> 0) on $[t_0, \infty)$ '.

Case A: $\|X_T\| = |x_3(T)|$.

Subcase A1: $x_3 < 0$. Then, using that $g_3(x_1(\cdot - 1)) < 0$ on $[1, \infty)$ together with (3.7.3), one sees that

$$\|X_{T+1}\| \geq |x_3(T+1)| \geq e^{-\mu_3} |x_3(T)| = e^{-\mu_3} \|X_T\|.$$

Subcase A2: $x_3 > 0$. Then, according to assumption b), $x_2 > 0$. Now $T \geq 2$ and (3.7.3) with $i := 1$ imply that for $t \in [T-1, T]$ one has

$$x_1(T) \geq e^{-\mu_1(T-t)} x_1(t) \geq e^{-\mu_1} x_1(t). \quad (3.7.4)$$

We set $\gamma := \frac{e^{-\mu_3}}{2Le^{\mu_1}} > 0$. If $x_1(T) \geq \gamma \|X_T\|$ then

$$\|X_{T+1}\| \geq \max_{t \in [T, T+1]} x_1(t) = x_1(T) \geq \gamma \|X_T\|.$$

If $x_1(T) < \gamma \|X_T\|$ then, from (3.7.4), $x_1(t) \leq e^{\mu_1} \gamma \|X_T\|$ for $t \in [T-1, T]$. It follows from (3.7.2) that for $t \in [T, T+1]$ one has

$$\begin{aligned} \left| \left(e^{\mu_3(\cdot)} x_3(\cdot) \right)'(t) \right| &\leq e^{\mu_3 t} |g_3(x_1(t-1))| \leq e^{\mu_3 t} L |x_1(t-1)| \\ &\leq e^{\mu_3 t} L e^{\mu_1} \gamma \|X_T\| = e^{\mu_3 t} L e^{\mu_1} \gamma x_3(T) \\ &\leq e^{\mu_3(T+1)} L e^{\mu_1} \gamma x_3(T). \end{aligned}$$

Hence $e^{\mu_3(T+1)}x_3(T+1) \geq e^{\mu_3 T}x_3(T) - e^{\mu_3(T+1)}Le^{\mu_1}\gamma x_3(T)$, or $x_3(T+1) \geq (e^{-\mu_3} - Le^{\mu_1}\gamma)x_3(T) = (e^{-\mu_3}/2)x_3(T)$. It follows that

$$\|X_{T+1}\| \geq |x_3(T+1)| \geq \frac{e^{-\mu_3}}{2}x_3(T) = \frac{e^{-\mu_3}}{2}\|X_T\|.$$

Case B: $\|X_T\| = |x_2(T)|$.

Subcase B1: $x_2 > 0$, $x_3 > 0$. Then $g_2(x_3(\cdot)) > 0$, and (3.7.2) shows that $e^{\mu_2(\cdot)}x_2(\cdot)$ is increasing. Hence $e^{\mu_2(T+1)}x_2(T+1) \geq e^{\mu_2 T}x_2(T)$, and

$$\|X_{T+1}\| \geq |x_2(T+1)| \geq e^{-\mu_2}x_2(T) = e^{-\mu_2}\|X_T\|.$$

Subcase B2: $x_2 < 0$, $x_3 < 0$. Then, again from (3.7.2), we obtain that $e^{\mu_2(\cdot)}|x_2(\cdot)|$ is increasing. As in subcase B1, it follows that

$$\|X_{T+1}\| \geq e^{-\mu_2}\|X_T\|.$$

Subcase B3: $x_2 > 0$, $x_3 < 0$. Then the property $g_3(x_1(\cdot - 1)) < 0$ on $[1, \infty)$ and (3.7.2) together imply that $|x_3(T)| \leq e^{\mu_3}|x_3(T+1)|$. Set $\gamma := e^{-\mu_2}/2L$ now. If $|x_3(T)| \geq \gamma|x_2(T)|$ then

$$\|X_{T+1}\| \geq |x_3(T+1)| \geq e^{-\mu_3}|x_3(T)| \geq e^{-\mu_3}\gamma|x_2(T)| = e^{-\mu_3}\gamma\|X_T\|.$$

If $|x_3(T)| < \gamma|x_2(T)|$ then monotonicity of x_3 implies $|x_3(t)| \leq \gamma|x_2(T)|$ for $t \in [T, T+1]$. Using (3.7.2) we get that for $t \in [T, T+1]$

$$\left| \left(e^{\mu_2(\cdot)}x_2(\cdot) \right)'(t) \right| = e^{\mu_2 t}|g_2(x_3(t))| \leq e^{\mu_2(T+1)}L|x_3(t)| \leq e^{\mu_2(T+1)}L\gamma|x_2(T)|.$$

It follows that $e^{\mu_2(T+1)}x_2(T+1) \geq e^{\mu_2 T}x_2(T) - e^{\mu_2(T+1)}L\gamma x_2(T)$, so $x_2(T+1) \geq (e^{-\mu_2} - L\gamma)x_2(T) = (e^{-\mu_2}/2)x_2(T)$, and finally

$$\|X_{T+1}\| \geq |x_2(T+1)| \geq (e^{-\mu_2}/2)x_2(T) = (e^{-\mu_2}/2)\|X_T\|.$$

Case C: $\|X_T\| = |x_1(T-1)|$.

Subcase C1: $x_2 > 0$. Then $g_1(x_2(\cdot)) > 0$, and (3.7.2) shows that $e^{\mu_1(\cdot)}x_1(\cdot)$ is increasing. Hence $\|X_{T+1}\| \geq x_1(T) \geq e^{-\mu_1}x_1(T-1) = e^{-\mu_1}\|X_T\|$.

Subcase C2: $x_2 < 0$. Then $x_3 < 0$. Set $\gamma := e^{-\mu_1}/2L$. Assume first that $|x_2(T-1)| \geq \gamma x_1(T-1)$. We have $g_2(x_3(\cdot)) < 0$, and (3.7.2) shows that $e^{\mu_2(\cdot)}|x_2(\cdot)|$ is increasing. Hence $|x_2(T+1)| \geq e^{-2\mu_2}|x_2(T-1)|$, and

$$\|X_{T+1}\| \geq |x_2(T+1)| \geq e^{-2\mu_2}\gamma x_1(T-1) = e^{-2\mu_2}\gamma\|X_T\|.$$

Assume now that $|x_2(T-1)| < \gamma x_1(T-1)$. The monotonicity of x_2 implies $|x_2(t)| \leq \gamma x_1(T-1)$ for $t \in [T-1, T]$, and from (3.7.2) one gets for $t \in [T-1, T]$

$$\left| \left(e^{\mu_1(\cdot)}x_1(\cdot) \right)'(t) \right| = e^{\mu_1 t}|g_1(x_2(t))| \leq e^{\mu_1 T}L\gamma|x_1(T-1)|.$$

Hence $|e^{\mu_1 T}x_1(T) - e^{\mu_1(T-1)}x_1(T-1)| \leq e^{\mu_1 T}L\gamma|x_1(T-1)|$, and $x_1(T) \geq (e^{-\mu_1} - L\gamma)x_1(T-1) = (e^{-\mu_1}/2)x_1(T-1)$. Consequently,

$$\|X_{T+1}\| \geq x_1(T) \geq (e^{-\mu_1}/2)x_1(T-1) = (e^{-\mu_1}/2)\|X_T\|.$$

5. In all cases A, B, and C, we have obtained a constant \hat{c} such that $\|X_{T+1}\| \geq \hat{c}\|X_T\|$. Defining c as the minimum of the set containing these constants together with \tilde{c} , the assertion of the lemma follows. \square

Lemma 3.8. *Let $X = (x_1, x_2, x_3)$ be a nonzero solution of system (S) with initial value $X_0 = (\varphi, x_2^0, x_3^0)$, and such that $\varphi \geq 0, x_2^0 \geq 0, x_3^0 \geq 0$.*

a) There exists $\delta > 0$ such that $x_1 > 0$ and $x_2 > 0$ on $(0, \delta]$.

b) If $x_1 > 0$ on $(0, \infty)$ then X satisfies the conditions of Lemma 3.7.

Proof. Ad a): *Case A:* $x_1(0) > 0$ and $x_2(0) > 0$. Then the assertion is trivial.

Case B: $x_1(0) = 0$, $x_2(0) > 0$. Then the assertion follows from $\dot{x}_1(0) > 0$.

Case C: $x_1(0) = 0 = x_2(0)$. Then the definition of \mathfrak{K} shows that $\varphi = 0$, and the fact that X is nonzero implies $x_3(0) > 0$. hence $\dot{x}_2(0) > 0$, and hence $x_2 > 0$ on some interval $(0, \delta]$. Formula (3.2) shows that $x_1 > 0$ on $(0, \delta]$.

Ad b): Assume now $x_1 > 0$ on $(0, \infty)$. In view of a), we can assume that $x_1(0) > 0$ and $x_2(0) > 0$ (consider X_δ instead of X_0). From $\varphi \geq 0$ and from $x_1 > 0$ on $[0, \infty)$, we infer

$$g_3(x_1(t-1)) \leq 0 \text{ for } t \in [0, 1] \text{ and } g_3(x_1(t-1)) < 0 \text{ for } t \in [1, \infty). \quad (3.8.1)$$

Case A: $x_3 > 0$ on $[0, \infty)$. Then $x_2(\delta) > 0$ and the second equation of system (S) together with Proposition 3.1, (i) show that $x_2 > 0$ on $[\delta, \infty)$.

Case B: x_3 has a first zero $z_1^3 \geq 0$. Formula (3.2), applied to the third equation with $t_0 := z_1^3$, and (3.8.1) show that one has

$$x_3 \leq 0 \text{ on } [z_1^3, \infty), \text{ and } x_3 < 0 \text{ ev.} \quad (3.8.2)$$

Note that $x_3 \geq 0$ on $[0, z_1^3]$. If $x_2 > 0$ on $[\delta, \infty)$, we are done. Otherwise, x_2 has a first zero z_1^2 in (δ, ∞) . In case $\delta < z_1^3$, the construction of δ and Proposition 3.1, (i) show that $x_2 > 0$ on $(0, z_1^3]$, and hence

$$z_1^2 > z_1^3.$$

In case $z_1^3 \leq \delta$, the last inequality holds since $z_1^2 > \delta$. It follows now from (3.8.2) and from formula (3.2) that

$$x_2 \leq 0 \text{ on } [z_1^2, \infty), \text{ and } x_2 < 0 \text{ ev.} \quad (3.8.3)$$

Note that in all cases one of the possibilities listed in Lemma 3.7, b) takes place.

The proof is complete. \square

Next, we introduce a cone in \mathcal{C} . (The oscillation properties of solutions will define a map of this cone into itself which has a nontrivial fixed point.) Set

$$\mathfrak{K} := \{\psi = (\varphi, x_2^0, x_3^0) \in \mathcal{C} \mid \varphi(-1) = 0, s \mapsto \exp(\mu_1 s)\varphi(s) \text{ is increasing on } [-1, 0], \\ x_2^0 \geq 0, x_3^0 \geq 0\}.$$

It is clear that \mathfrak{K} is closed in \mathcal{C} and convex, that $\mathbb{R}_0^+ \mathfrak{K} \subset \mathfrak{K}$, and

$\mathfrak{K} \cap (-\mathfrak{K}) = \{0\}$. Hence \mathfrak{K} is a cone. Note that $(\varphi, x_2^0, x_3^0) \in \mathfrak{K}$ implies $\varphi \geq 0$.

The next proposition shows that, under the semiflow on \mathcal{C} generated by system (S), initial segments in $\mathfrak{K} \setminus \{0\}$ evolve to segments in \mathfrak{K} with $x_1(0) > 0$.

Proposition 3.9. *Let $X = (x_1, x_2, x_3)$ be a nonzero solution of (S) with $X_0 = (\varphi, x_2(0), x_3(0)) \in \mathfrak{K}$. Define $\xi := \max \{s \in [-1, 0] \mid \varphi(s) = 0\} + 1$. Then the following properties hold.*

- (i) $x_1 > 0$ on $(\xi - 1, \xi]$.
- (ii) $x_2 \geq 0$ and $x_3 \geq 0$ on $[0, \xi]$.
- (iii) $X_\xi \in \mathfrak{K}$.

Proof. For $t \in [0, \xi]$ we have $x_1(t-1) = 0$, and hence $\dot{x}_3(t) = -\mu_3 x_3(t)$, so $x_3(t) = e^{-\mu_3 t} x_3(0)$. In particular, $x_3 \geq 0$ on $[0, \xi]$. From the second equation and from Proposition 3.1, (i), we obtain that $x_2 \geq 0$ on $[0, \xi]$. Property (ii) is proved.

Proof of property (i): If $\xi = 0$ then the definition of \mathfrak{K} shows $x_1 > 0$ on $(-1, 0]$.

If $\xi = 1$ then $x_1 = 0$ on $[-1, 0]$, and hence $x_2(0) > 0$ or $x_3(0) > 0$. In both cases, the second and third equation of (S) imply $x_2 > 0$ on $(0, \xi]$. The first equation and formula (3.2) then show that $x_1 > 0$ on $(0, \xi]$.

Assume now $\xi \in (0, 1)$. Then $x_1 > 0$ on $(\xi - 1, 0]$ follows from the definition of \mathfrak{K} . Now $x_1(0) > 0, x_2 \geq 0$ on $[0, \xi]$ and Proposition 3.1, (i) yield $x_1 > 0$ on $[0, \xi]$. Property (i) is proved.

Proof of (iii): Set now $\psi := x_{1,\xi}$. For $s \in [-1, -\xi]$ we have

$$\exp(\mu_1 s)\psi(s) = \exp(\mu_1 s)x_1(\xi + s) = \exp(-\mu_1 \xi) \exp[\mu_1(\xi + s)]x_1(\xi + s),$$

and the last expression is an increasing function of s on $[-1, -\xi]$, due to the definition of \mathfrak{K} . Further, the function $s \mapsto \exp(\mu_1 s)\psi(s)$ is differentiable on $[-\xi, 0]$ (at $-\xi$, we mean the derivative from the right), and for $t \in [-\xi, 0]$ one has

$$\begin{aligned} \frac{d}{ds} \Big|_{s=t} [s \mapsto \exp(\mu_1 s)\psi(s)] &= \exp(\mu_1 t)[\dot{x}_1(\xi + t) + \mu_1 x_1(\xi + t)] \\ &= \exp(\mu_1 t)g_1(x_2(\xi + t)) \geq 0. \end{aligned}$$

It follows that $s \mapsto \exp(\mu_1 s)\psi(s)$ is increasing on $[-1, 0]$. Together with $x_2(\xi) \geq 0$ and $x_3(\xi) \geq 0$, we obtain $X_\xi \in \mathfrak{K}$. \square

We can now describe the behavior of solutions starting in \mathfrak{K} , and construct a return map. For $X_0 = (\varphi, x_2(0), x_3(0)) \in \mathfrak{K}$, we set $\|X_0\|_1 := \varphi(0) + x_2(0) + x_3(0)$. (This expression does not define a norm on all of \mathcal{C} .) Recall the definition of the functions γ_j in Remark 3.5, and the constants $K = -k_1 k_2 k_3$ and K_c (see Lemma 2.3) associated to system (S).

Lemma 3.10. *Let $X = (x_1, x_2, x_3)$ be a nonzero solution of (S) with initial value $X_0 = (\varphi, x_2(0), x_3(0)) \in \mathfrak{K}$. If $K > K_c$, then x_1 has a first zero $z_1^1 = z_1^1(X_0)$ in $(0, \infty)$, and the following properties hold.*

- a) $-X_{z_1^1+1} \in \mathfrak{K}$, and $\dot{x}_1(z_1^1) < 0$.
- b) The map

$$Q_S : \mathfrak{K} \rightarrow -\mathfrak{K}, \quad X_0 \mapsto \begin{cases} X_{z_1^1(X_0)+1} & \text{if } X_0 \neq 0 \\ 0 & \text{if } X_0 = 0 \end{cases}$$

is continuous. Further, there exists a constant $\kappa_0 > 0$ such that for $X_0 \in \mathfrak{K}$ with $\|X_0\|_1 \leq 1$ one has

$$\max \left\{ \|X_t\| \mid t \in [0, z_1^1(X_0) + 1] \right\} \leq \kappa_0 \|X_0\|_1.$$

- c) If $|g_3|$ is bounded by a constant $\bar{\gamma}_3$ then, setting

$$c_3 := \bar{\gamma}_3/\mu_3, \quad c_2 := \gamma_2(c_3)/\mu_2, \quad c_1 := \gamma_1(c_2)/\mu_1$$

and $\mathfrak{B} := \left\{ (\psi, x_2^0, x_3^0) \in \mathfrak{K} \mid |\psi| \leq c_1, |x_2^0| \leq c_2, |x_3^0| \leq c_3 \right\}$, one has $Q_S(\mathfrak{K}) \subset -\mathfrak{B}$, and Q_S is compact.

Proof. 1. It follows from $K > K_c$ that all characteristic values of the linearized system (L) (see Lemma 2.3 a),(i)) have nonzero imaginary parts. Now Theorem 1 from [4] implies that all nonzero solutions of (L) oscillate in the sense of the definition from [4]. This means that either the solution is equal to zero after finite time, or one of the coordinates has arbitrarily large zeroes. From Remark 3.3, we know that the first possibility is excluded. Now Corollary 3.4, b) and c) show that nonzero solutions of (L) oscillate in the sense of Definition 3.2.

2. We know from Lemma 3.8,a) that $x_1 > 0$ on some interval $(0, \delta]$. Assume now that $x_1 > 0$ on $(0, \infty)$. Using Lemma 3.8 and Lemma 3.7, we see that $X(t) \rightarrow 0$ as $t \rightarrow \infty$, and there exists a constant $c > 0$ such that

$$\forall t \geq 0 : \|X_{t+1}\| \geq c\|X_t\|.$$

There exist constants $R, L_1 > 0$ such that $\|X_t\| \leq R$ for $t \geq 0$, and such that $|g_1(x)| \leq L_1|x|$ if $|x| \leq R$. Further, from Proposition 3.6, there exists $t_0 \geq 1$ such that $|x_2|$ decays monotonously on $[t_0 - 1, \infty)$. For $n \in \mathbb{N}$, $n \geq t_0$, we define $\varphi_n \in C$ and $x_{2,n}, x_{3,n} \in \mathbb{R}$ by

$$X_n = (\varphi_n, x_{2,n}, x_{3,n}).$$

Then $\varphi_n \in C^1$. From the first equation of system (S), from monotonicity of $|x_2|$, and from the above lower estimate, we infer

$$\begin{aligned} |\varphi'_n| &\leq \mu_1|\varphi_n| + |g_1(x_2(\cdot))|_{[n-1, n]}|^{C^0} \\ &\leq \mu_1\|X_n\| + L_1|x_2|_{[n-1, n]}|^{C^0} = \mu_1\|X_n\| + L_1|x_2(n-1)| \\ &\leq \mu_1\|X_n\| + L_1\|X_{n-1}\| \leq (\mu_1 + L_1c^{-1})\|X_n\|. \end{aligned}$$

Setting $\tilde{\varphi}_n := \varphi_n/\|X_n\|$, we have

$$|\tilde{\varphi}_n| \leq 1 \text{ and } |\tilde{\varphi}'_n| \leq \mu_1 + L_1c^{-1} \quad (n \in \mathbb{N}, n \geq t_0).$$

Set now $\psi_n := X_n/\|X_n\| = (\tilde{\varphi}_n, x_{2,n}/\|X_n\|, x_{3,n}/\|X_n\|)$ ($n \in \mathbb{N}$). It follows from the estimates for $|\varphi_n|$ and $|\varphi'_n|$, from the Arzélà-Ascoli Theorem, and from the Bolzano-Weierstraß Theorem that there exists a subsequence (ψ_{n_k}) of (ψ_n) and $\psi_* \in \mathcal{C}$ with $\psi_{n_k} \rightarrow \psi_*$ as $k \rightarrow \infty$ (with respect to $\|\cdot\|$). Now $\|\psi_{n_k}\| = 1$ implies $\|\psi_*\| = 1$. In particular, in view of step 1, the solution z^* of system (L) with initial value ψ_* oscillates, so there exists $T > 0$ such that each component of z^* takes values of both signs on $[0, T]$.

3. The functions $X(n_k + \cdot)|_{[-1, \infty)}$ ($k \in \mathbb{N}$) satisfy the linear nonautonomous system

$$(k) \begin{cases} \dot{\xi}_1(t) + \mu_1\xi_1(t) = a_{1,k}(t)\xi_2(t) \\ \dot{\xi}_2(t) + \mu_2\xi_2(t) = a_{2,k}(t)\xi_3(t) \\ \dot{\xi}_3(t) + \mu_3\xi_3(t) = a_{3,k}(t)\xi_1(t-1), \end{cases}$$

where $a_{1,k}(t) := \int_0^1 g'_1(sx_2(n_k+t)) ds$, $a_{2,k}(t) := \int_0^1 g'_2(sx_3(n_k+t)) ds$, and $a_{3,k}(t) := \int_0^1 g'_3(sx_1(n_k+t-1)) ds$.

Convergence of X to 0 implies

$$a_{1,k}(t) \rightarrow g'_1(0), \quad a_{2,k}(t) \rightarrow g'_2(0), \quad a_{3,k}(t) \rightarrow g'_3(0)$$

for $k \rightarrow \infty$, uniformly on $[0, \infty)$.

Linearity of system (k) implies that the solution $X^{*,k}$ of (k) with initial value $\psi_{n,k}$ at time 0 is given by

$$X^{*,k}(t) = \|X_{n_k}\|^{-1} \cdot X(n_k + t) \quad (k \in \mathbb{N}, t \geq 0).$$

The convergence of the coefficient functions $a_{j,k}$ ($j = 1, 2, 3$) together with $\psi_{n,k} \rightarrow \psi_*$ implies that $X^{*,k}|_{[0, T]} \rightarrow z^*|_{[0, T]}$ in the sup-norm, as $k \rightarrow \infty$. Thus, there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$, every component of $X^{*,k}$ takes values of both signs on $[0, T]$, since this is true for z^* . From the relation between $X^{*,k}$

and $X(n_k + \cdot)$, we now see that all components of X take values of both signs on $[n_k, n_k + T]$ for $k \geq k_0$, in contradiction to $x_1 > 0$ on $(0, \infty)$.

Hence x_1 has a first zero z_1^1 in $(0, \infty)$.

4. Proof of a): With ξ defined as in Proposition 3.9, we have $\xi - 1 \leq 0$ and $x_1 > 0$ on $(\xi - 1, \xi]$, in particular, $z_1^1 > \xi$. Further, $X_\xi \in \mathfrak{K}$, so $x_2(\xi) \geq 0$ and $x_3(\xi) \geq 0$.

Claim 1: There exists a first zero $z_1^3 \geq \xi$ such that $x_3(z_1^3) = 0$ and $x_3 > 0$ on $[\xi, z_1^3)$, if $z_1^3 > \xi$.

If not, then $x_3 > 0$ on $[\xi, \infty)$. Proposition 3.1(i) and the second equation of system (S) show that $x_2 > 0$ on (ξ, ∞) , which in turn implies, by virtue of the first equation of system (S), that $x_1 > 0$ on $[\xi, \infty)$. This contradicts the existence of z_1^1 .

Claim 2: $x_2 > 0$ on $(\xi, z_1^3]$ if $z_1^3 > \xi$, and $x_1 > 0$ on $(\xi - 1, z_1^3]$. In particular,

$$z_1^1 > z_1^3.$$

We know already that $x_1 > 0$ on $(\xi - 1, \xi]$. The first two inequalities follow from $x_2(\xi) \geq 0$, from Claim 1, and from $x_1(\xi) > 0$, using Proposition 3.1(i), and the second and first equation of (S). The last inequality is a consequence of the second.

Claim 3: $x_3 < 0$ on $(z_1^3, z_1^3 + 1]$. This follows from the third equation of (S), together with Claim 2 and formula (3.2).

Claim 4: There exists a first zero z_1^2 of x_2 in $[z_1^3, \infty)$. If not then Claim 2 implies $x_2 > 0$ on $[z_1^3, \infty)$. Then the first equation of (S) and Claim 2 show that $x_1 > 0$ on $(\xi - 1, \infty)$, a contradiction to the existence of z_1^1 .

Claim 5: $x_1 > 0$ on $[z_1^3, z_1^2]$, so $z_1^1 > z_1^2$, and $x_3 < 0$ on $(z_1^3, z_1^1 + 1]$. The inequality for x_1 follows from Claim 2, from the first equation of (S), and from Proposition 3.1(i). It is then clear that $z_1^1 > z_1^2$. The inequality for x_3 follows from Claim 3 and from $x_1 > 0$ on $(\xi - 1, z_1^1)$, using the third equation of (S) and, again, Proposition 3.1(i).

Claim 6: $x_2 < 0$ on $(z_1^2, z_1^1 + 1]$. From Claim 4, we know that $z_1^2 \geq z_1^3$. Claim 6 follows from the second equation of (S) and from formula (3.2), using Claim 5.

Set now $\psi := x_1(z_1^1 + 1 + \cdot)|_{[-1, 0]}$. Obviously $\psi(-1) = 0$, and for $t \in [-1, 0]$ one has

$$\begin{aligned} \frac{d}{ds} \Big|_{s=t} [s \mapsto \exp(\mu_1 s) \psi(s)] &= \exp(\mu_1 t) [\dot{\psi}(t) + \mu_1 \psi(t)] \\ &= \exp(\mu_1 t) [\dot{x}_1(z_1^1 + 1 + t) + \mu_1 x_1(z_1^1 + 1 + t)] \\ &= \exp(\mu_1 t) g_1(x_2(z_1^1 + 1 + t)) < 0, \end{aligned}$$

since $x_2 < 0$ on $(z_1^2, z_1^1 + 1]$ and $z_1^2 < z_1^1$. Further, from Claims 5 and 6, we know $x_2(z_1^1 + 1) < 0$ and $x_3(z_1^1 + 1) < 0$. Thus, we have $-X_{z_1^1+1} = -(\psi, x_2(z_1^1 + 1), x_3(z_1^1 + 1)) \in \mathfrak{K}$. It follows from Claim 6 that $\dot{x}_1(z_1^1) = \dot{\psi}(-1) < 0$. (Assertion a) is proved.)

Proof of b) and c): The fact that z_1^1 is a simple zero of x_1 implies that the map $\mathfrak{K} \setminus \{0\} \ni X_0 \mapsto z_1^1(X_0) \in \mathbb{R}$ is continuous. It follows now from continuity of the semiflow generated by system (S) that Q_S is continuous at points in $\mathfrak{K} \setminus \{0\}$. We now provide estimates which show the continuity at zero, and also the remaining assertions. Note that the functions defined by

$$\tilde{\gamma}_j(r) := \max \left\{ |g'_j(x)| \mid |x| \leq r \right\} \text{ for } r \geq 0, j = 1, 2, 3$$

are increasing, and, with γ_j from Remark 3.5, one has $\gamma_j(x) \leq \tilde{\gamma}_j(|x|)|x|$ for $x \in \mathbb{R}$, $j = 1, 2, 3$.

On $[0, z_1^3]$, we have $x_1 \geq 0$ (since $z_1^1 > z_1^3$), and hence $\dot{x}_3 \leq 0$, so $|x_3| \leq x_3(0)$ on $[0, z_1^3]$. From formula (3.2), one now gets for $t \in [0, z_1^3]$

$$\begin{aligned} |x_2(t)| &\leq x_2(0) + \int_0^t e^{-\mu_2(t-s)} \gamma_2(x_3(0)) ds \\ &\leq x_2(0) + \gamma_2(x_3(0))/\mu_2 \\ &\leq x_2(0) + \tilde{\gamma}_2(1)x_3(0)/\mu_2 \leq (1 + \tilde{\gamma}_2(1)/\mu_2)[x_2(0) + x_3(0)] \\ &\leq \kappa_2 \|X_0\|_1, \end{aligned} \quad (3.10.1)$$

with $\kappa_2 := (1 + \gamma_2(1)/\mu_2)$. Similarly, one obtains from (3.10.1) (using monotonicity of $\tilde{\gamma}_1$ and $\|X_0\|_1 \leq 1$) that on $[0, z_1^3]$

$$\begin{aligned} |x_1| &\leq x_1(0) + \gamma_1[\kappa_2 \|X_0\|_1]/\mu_1 \\ &\leq x_1(0) + \tilde{\gamma}_1(\kappa_2)\kappa_2 \|X_0\|_1/\mu_1 \\ &\leq (1 + \tilde{\gamma}_1(\kappa_2)\kappa_2/\mu_1) \|X_0\|_1 = \kappa_1 \|X_0\|_1, \end{aligned} \quad (3.10.2)$$

with $\kappa_1 := 1 + \tilde{\gamma}_1(\kappa_2)\kappa_2/\mu_1$. On $[z_1^3, z_1^2]$, the properties $x_3 \leq 0$ (see Claim 5) and $x_2 \geq 0$ imply $\dot{x}_2 \leq 0$, so the estimate in (3.10.1) even holds on $[0, z_1^2]$. Similarly, $x_2 \leq 0$ on $[z_1^2, z_1^1 + 1]$ (see Claim 6) implies that (3.10.2) holds on $[0, z_1^1]$. Now we conclude from Proposition 3.1.(v) that on $[z_1^3, z_1^1 + 1]$ one has

$$|x_3| \leq \gamma_3[\kappa_1 \|X_0\|_1]/\mu_3 \leq \tilde{\gamma}_3(\kappa_1)\kappa_1 \|X_0\|_1/\mu_3 = \kappa_3 \|X_0\|_1, \quad (3.10.3)$$

with $\kappa_3 := \tilde{\gamma}_3(\kappa_1)\kappa_1/\mu_3$.

Using Proposition 3.1.(v) two more times, we obtain

$$|x_2| \leq \frac{\gamma_2(\kappa_3 \|X_0\|_1)}{\mu_2} \leq \frac{\tilde{\gamma}_2(\kappa_3)}{\mu_2} \kappa_3 \|X_0\|_1 \text{ on } [z_1^2, z_1^1 + 1], \quad (3.10.4)$$

and, on $[z_1^1, z_1^1 + 1]$,

$$|x_1| \leq \frac{1}{\mu_1} \gamma_1\left[\frac{\tilde{\gamma}_2(\kappa_3)}{\mu_2} \kappa_3 \|X_0\|_1\right] \leq \frac{1}{\mu_1} \tilde{\gamma}_1\left[\frac{\tilde{\gamma}_2(\kappa_3)}{\mu_2} \kappa_3\right] \frac{\tilde{\gamma}_2(\kappa_3)}{\mu_2} \kappa_3 \|X_0\|_1. \quad (3.10.5)$$

Obviously $\|X_0\|_1 \rightarrow 0$ as $\|X_0\| \rightarrow 0$, so it follows from (3.10.3)-(3.10.5) that Q_S is also continuous at zero, and hence continuous. The second part of Assertion b) also follows from (3.10.1)-(3.10.5), regarding the facts that $|x_3| \leq |x_3(0)|$ on $[0, z_1^3]$, and that estimates (3.10.1) and (3.10.2) are valid on $[0, z_1^2]$ and $[0, z_1^1]$, respectively. Assertion b) is proved.

Assume now that $|g_3| \leq \bar{\gamma}_3$, and define c_j as in Assertion c). Then, instead of (3.10.3), we have

$$|x_3| \leq \bar{\gamma}_3/\mu_3 = c_3 \text{ on } [z_1^3, z_1^1 + 1],$$

and in analogy to (3.10.4) and (3.10.5) we now have $|x_2| \leq \gamma_2(c_3)/\mu_2 = c_2$ on $[z_1^2, z_1^1 + 1]$ and $|x_1| \leq \gamma_1(c_2)/\mu_1 = c_1$ on $[z_1^1, z_1^1 + 1]$. We see from these estimates that Q_S maps into the bounded set $-\mathfrak{B}$. Further, for $t \in [z_1^1, z_1^1 + 1]$, the first equation of system (S) shows that

$$|\dot{x}_1(t)| \leq \mu_1 c_1 + \gamma_1(c_2). \quad (3.10.6)$$

Compactness of Q_S now follows from boundedness $-\mathfrak{B}$, from (3.10.6) and the theorem of Arzelà-Ascoli, together with the Bolzano-Weierstraß Theorem. \square

Corollary 3.11. Assume that $K > K_c$.

- a) For $X_0 \in \mathfrak{K} \setminus \{0\}$, the first component of the corresponding solution (x_1, x_2, x_3) has a second zero $z_2^1 = z_2^1(X_0)$ in $(0, \infty)$, and $X_{z_2^1(X_0)+1} \in \mathfrak{K}$.

b) *The map*

$$P_S : \mathfrak{K} \rightarrow \mathfrak{K}, X_0 \mapsto \begin{cases} X_{z_2^1(X_0)+1} & \text{if } X_0 \neq 0 \\ 0 & \text{if } X_0 = 0 \end{cases}$$

is continuous.

- c) *There exist constants $\kappa > 0$ and $\delta_1 \in (0, 1]$ such that for $X_0 \in \mathfrak{K}$ with $\|X_0\|_1 \leq \delta_1$ one has $\max \left\{ \|X_t\| \mid t \in [0, z_1^2(X_0) + 1] \right\} \leq \kappa \|X_0\|_1$.*
- d) *If $|g_3|$ is bounded by a constant $\bar{\gamma}_3$ then, with \mathfrak{B} as in Lemma 3.10, the map P_S takes values in \mathfrak{B} and is compact.*

Proof. Recall the system (S^-) defined in Remark 3.5. From Lemma 3.10, we obtain a map Q_{S^-} for this system. Note that the constants c_j and the set \mathfrak{B} are the same as for system (S). Let now $X_0 \in \mathfrak{K} \setminus \{0\}$, with corresponding solution $X = (x_1, x_2, x_3)$ of system (S), and define Y by $Y_t = -X_{z_1^1(X_0)+1+t}$ for $t \geq 0$. Then, using Lemma 3.10, a), we see that

$$Y_0 = -X_{z_1^1(X_0)+1} = -Q_S(X_0) \in \mathfrak{K}.$$

Further (compare Remark 3.5), $Y = (y_1, y_2, y_3)$ is a solution of (S^-) . Application of Lemma 3.10 to the system (S^-) and to Y yields a first zero $z_1^1(Y_0)$ of y_1 , and $Q_{S^-}(Y_0) = Y_{z_1^1(Y_0)+1} \in -\mathfrak{K}$. The number $z_2^1(X_0) := z_1^1(X_0) + 1 + z_1^1(Y_0)$ is the second zero of x_1 in $(0, \infty)$, and we have

$$P_S(X_0) = X_{z_2^1(X_0)+1} = -Y_{z_1^1(Y_0)+1} = -Q_{S^-}(Y_0) = -Q_{S^-}(-Q_S(X_0)).$$

Assertion a) is proved. Clearly, the last identity also holds for $X_0 = 0$, so that we have

$$P_S = -Q_{S^-} \circ (-Q_S).$$

Assertions b) and d) now follow from Lemma 3.10.

For Assertion c), note that $\|Y_0\|_1 \leq 3\|Y_0\|$ for $Y_0 \in \mathfrak{K}$. Let κ_0 be as in Lemma 3.10, b). Now, if $X_0 \in \mathfrak{K}$ and $\|X_0\|_1 \leq 1/(3\kappa_0 + 1)$ then Lemma 3.10, b) gives (for the corresponding solution X)

$$\max \left\{ \|X_t\| \mid t \in [0, z_1^1(X_0) + 1] \right\} \leq \kappa_0 \|X_0\|_1.$$

In particular, $\|Q_S(X_0)\| \leq \kappa_0 \|X_0\|_1$, and hence

$$\|Q_S(X_0)\|_1 \leq 3\|Q_S(X_0)\| \leq 3\kappa_0 \|X_0\|_1 \leq 1.$$

Application of the same part of Lemma 3.10 to Q_{S^-} gives a constant κ_0^- such that

$$\max \left\{ \|X_t\| \mid t \in [z_1^1(X_0) + 1, z_2^1(X_0) + 1] \right\} \leq \kappa_0^- \|Q_S(X_0)\|_1 \leq 3\kappa_0^- \kappa_0 \|X_0\|_1.$$

Hence we have $\|P_S(X_0)\| \leq \kappa_0^- \|Q_S(X_0)\|_1 \leq 3\kappa_0^- \kappa_0 \|X_0\|_1$. The assertion follows with $\delta_1 := 1/(3\kappa_0 + 1) \in (0, 1]$ and $\kappa := \max\{\kappa_0, 3\kappa_0^- \kappa_0\}$. \square

Clearly, a nonzero fixed point of an iterate of P_S is an initial value of a non-constant periodic solution to system (S). We obtain such a fixed point in the next section.

4. Periodic solutions from the Schauder fixed point theorem. In this last section we construct a convex, compact set, which does not contain zero, and which is invariant under an iterate of the map P_S .

Recall the numbers $k_j = g'_j(0)$, $j = 1, 2, 3$, and define the nonlinear parts h_j of the functions g_j by

$$g_j(x) = k_j x + h_j(x) \quad (x \in \mathbb{R}), \quad j = 1, 2, 3.$$

Recall also the Laplace transform defined by

$$(\mathcal{L}x)(\lambda) := \int_0^\infty e^{-\lambda t} x(t) dt,$$

for $x : [0, \infty) \rightarrow \mathbb{C}$ and $\lambda \in \mathbb{C}$ such that the integral converges (as improper Riemann integral). The Laplace transform is extended to \mathbb{C}^n -valued functions by componentwise application. For a matrix $A \in \mathbb{C}^{n \times n}$ and a \mathbb{C}^n -valued function x such that $\mathcal{L}x$ exists, one has

$$\mathcal{L}(Ax)(\lambda) = A(\mathcal{L}x)(\lambda).$$

The following additional properties hold (if all occurring derivatives and integrals exist, and x is defined at least on $[-1, \infty)$ for the second property):

$$(\mathcal{L}\dot{x})(\lambda) = -x(0) + \lambda(\mathcal{L}x)(\lambda) \quad (\mathcal{L}1)$$

$$[\mathcal{L}x(\cdot - 1)](\lambda) = e^{-\lambda} \left[\int_{-1}^0 e^{-\lambda t} x(t) dt + (\mathcal{L}x)(\lambda) \right]. \quad (\mathcal{L}2)$$

Proposition 4.1. *Let $X = (x_1, x_2, x_3)$ be a solution of system (S), and let $\lambda \in \Sigma_K$ be a solution of the characteristic equation with $\operatorname{Re}(\lambda) > 0$. Further, assume that the integrals $\int_0^\infty x_j(t)e^{-\lambda t} dt$ and $\int_0^\infty \dot{x}_j(t)e^{-\lambda t} dt$ converge (as improper Riemann integrals) for $j = 1, 2, 3$. Then*

$$\begin{aligned} & -k_1 k_2 x_3(0) - k_1(\lambda + \mu_3)x_2(0) - (\lambda + \mu_3)(\lambda + \mu_2)x_1(0) \\ & + K \int_{-1}^0 e^{-\lambda(t+1)} x_1(t) dt \\ & = (\lambda + \mu_3)(\lambda + \mu_2)\mathcal{L}(h_1 \circ x_2)(\lambda) + k_1(\lambda + \mu_3)\mathcal{L}(h_2 \circ x_3)(\lambda) \\ & + k_1 k_2 \mathcal{L}(h_3 \circ x_1(\cdot - 1))(\lambda). \end{aligned} \quad (4.1.1)$$

Proof. System (S) can be written in the form

$$\dot{x}(t) = Ax(t) + Bx(t-1) + h(t),$$

where

$$A = \begin{pmatrix} -\mu_1 & k_1 & 0 \\ 0 & -\mu_2 & k_2 \\ 0 & 0 & -\mu_3 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ k_3 & 0 & 0 \end{pmatrix},$$

and $h(t)$ is an abbreviation for $\begin{pmatrix} h_1(x_2(t)) \\ h_2(x_3(t)) \\ h_3(x_1(t-1)) \end{pmatrix}$. Applying the Laplace transform

and using (L1) and (L2), we get

$$\begin{aligned}
& -x(0) + \lambda(\mathcal{L}x)(\lambda) \\
& = A(\mathcal{L}x)(\lambda) + B(\mathcal{L}x(\cdot - 1))(\lambda) + (\mathcal{L}h)(\lambda) \\
& = A(\mathcal{L}x)(\lambda) + Be^{-\lambda} \left[\int_{-1}^0 e^{-\lambda t} x(t) dt + (\mathcal{L}x)(\lambda) \right] + (\mathcal{L}h)(\lambda),
\end{aligned}$$

so with the 3×3 unit matrix I we have

$$(\lambda I - A - e^{-\lambda} B)(\mathcal{L}x)(\lambda) = x(0) + Be^{-\lambda} \int_{-1}^0 e^{-\lambda t} x(t) dt + (\mathcal{L}h)(\lambda).$$

We write ξ for $(\mathcal{L}x)(\lambda)$ and η for the right hand side of the above equation in \mathbb{R}^3 . Inserting the explicit form of A and B , we obtain

$$\begin{pmatrix} \lambda + \mu_1 & -k_1 & 0 \\ 0 & \lambda + \mu_2 & -k_2 \\ -k_3 e^{-\lambda} & 0 & \lambda + \mu_3 \end{pmatrix} \xi = \eta.$$

Recall that $(\lambda + \mu_1)(\lambda + \mu_2)(\lambda + \mu_3) - k_1 k_2 k_3 e^{-\lambda} = 0$. The following three steps bring the above matrix into upper triangular form, with the last row equal to zero:

- 1) Multiplication of the last row with $k_1 k_2$.
- 2) Adding the first row, multiplied by $(\lambda + \mu_2)(\lambda + \mu_3)$, to the third row.
- 3) Adding the second row, multiplied by $k_1(\lambda + \mu_3)$, to the third row.

Applying the same manipulations to $\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix}$, the resulting equation in the third component is

$$0 = k_1 k_2 \eta_3 + (\lambda + \mu_2)(\lambda + \mu_3) \eta_1 + k_1(\lambda + \mu_3) \eta_2.$$

Recalling the definitions of η and of h , and using the explicit form of B , we conclude that

$$\begin{aligned}
0 &= k_1 k_2 [x_3(0) + k_3 e^{-\lambda} \int_{-1}^0 e^{-\lambda t} x_1(t) dt + \mathcal{L}(h_3 \circ x_1(\cdot - 1))(\lambda)] \\
&+ (\lambda + \mu_2)(\lambda + \mu_3) [x_1(0) + \mathcal{L}(h_1 \circ x_2)(\lambda)] \\
&+ k_1(\lambda + \mu_3) [x_2(0) + \mathcal{L}(h_2 \circ x_3)(\lambda)].
\end{aligned}$$

Collecting all the terms which are linear in x on one side of the equation and recalling that $k_1 k_2 k_3 = -K$, the asserted formula is obtained. \square

Motivated by Proposition 4.1, we study the left-hand side $LHS(X_0)$ and the right-hand side $RHS(X_0)$ of equation (4.1.1) for segments $X_0 \in \mathfrak{K}$, with corresponding solution $X = (x_1, x_2, x_3)$ of system (S).

Recall the numbers K_c and K_0^* introduced in Section 2 (Proposition 2.2. and Lemma 2.3). We assume now that $K > K_0^*$, so that the linear system (L) has an eigenvalue $\lambda = \rho + i\omega$ with $\rho > 0$ and $\omega \in (0, \pi)$. There exists a constant $M_2 > 0$ such that for $i = 1, 2, 3$ and $|u| \leq 1$ one has

$$|h_i(u)| \leq M_2 |u|^2.$$

Further, define the positive constants

$$\begin{aligned} M_1 &:= \{ |(\lambda + \mu_3)(\lambda + \mu_2)| + |k_1(\lambda + \mu_3)| + k_1 k_2 \} / |\lambda|, \\ M^* &:= M_1 M_2, \\ m_1 &:= \min\{k_1, \mu_3 + \mu_2 + 2\rho\}\omega, \\ \tilde{m}_2 &:= (k_1 k_2)^{-1} [k_1 |(\lambda + \mu_3)| + |(\lambda + \mu_3)(\lambda + \mu_2)| + K e^{\mu_1}]^{-1}, \\ m_2 &:= \frac{1}{2} (1 + \tilde{m}_2)^{-1}, \\ m^* &:= \min\{m_1 m_2 / 2, k_1 k_2 / 4\}. \end{aligned}$$

(Note that $m_2 \in (0, 1)$.)

Proposition 4.2. *Let $X = (x_1, x_2, x_3)$ be a solution of system (S) with $X_0 \in \mathfrak{K}$. For $\delta \in (0, 1]$, one has the implication*

$$\forall t \geq 0 : \|X_t\| \leq \delta \implies |RHS(X_0)| \leq M^* \delta^2.$$

Proof. It follows from the definition of \mathcal{L} that if $\operatorname{Re}(\lambda) > 0$ and u is a bounded function then $|(\mathcal{L}u)(\lambda)| \leq \|u\|_\infty / |\lambda|$. Using the definition of M_1, M_2 and the property $\|X_t\| \leq \delta \leq 1$ ($t \geq 0$), we conclude that

$$\begin{aligned} |RHS(X_0)| &\leq [|(\lambda + \mu_3)(\lambda + \mu_2)| + |k_1(\lambda + \mu_3)| + k_1 k_2] \cdot \\ &\quad \cdot \max\{ \|h_1 \circ x_2\|_\infty, \|h_2 \circ x_3\|_\infty, \|h_3 \circ x_1(\cdot - 1)\|_\infty \} / |\lambda| \\ &\leq M_1 M_2 \delta^2 = M^* \delta^2. \end{aligned}$$

□

Proposition 4.3. *Assume $K > \max\{K_c, K_0^*\}$, and, as above, let $\lambda = \rho + i\omega$ be the eigenvalue of the linear system with $\rho > 0, \omega \in (0, \pi)$. With m^* as above, one has*

$$\forall X_0 \in \mathfrak{K} : |LHS(X_0)| \geq m^* \|X_0\|_1.$$

Proof. First case: $x_1(0) + x_2(0) \geq m_2 x_3(0)$. Then

$$\begin{aligned} |LHS(X_0)| &\geq |\operatorname{Im}(LHS(X_0))| \\ &= | -k_1 \omega x_2(0) - (\mu_3 + \mu_2 + 2\rho) \omega x_1(0) + \\ &\quad + \operatorname{Im}(K \int_{-1}^0 e^{-\lambda(t+1)} x_1(t) dt) | \\ &= | -k_1 \omega x_2(0) - (\mu_3 + \mu_2 + 2\rho) \omega x_1(0) - \\ &\quad - K \int_{-1}^0 e^{-\rho(t+1)} \sin(\omega(t+1)) x_1(t) dt |. \end{aligned}$$

Note that, since $\omega \in (0, \pi)$ and $x_1 \geq 0$ on $[-1, 0]$, the integral in the last expression is nonnegative. Using the definition of m_1 , we conclude

$$|LHS(X_0)| \geq |k_1 \omega x_2(0) + (\mu_3 + \mu_2 + 2\rho) \omega x_1(0)| \geq m_1 |x_2(0) + x_1(0)|.$$

With the assumption of the present case (together with the property $m_2 \leq 1$), one gets

$$\begin{aligned} |LHS(X_0)| &\geq (m_1/2)(x_2(0) + x_1(0) + m_2 x_3(0)) \\ &\geq (m_1 m_2 / 2) [x_2(0) + x_1(0) + x_3(0)]. \end{aligned} \tag{4.3.1}$$

Second case: $x_1(0) + x_2(0) < m_2 x_3(0)$. The fact that $e^{\mu_1} x_1(\cdot)$ is increasing on $[-1, 0]$ implies that $x_1(t) \leq e^{\mu_1} x_1(0)$ for $t \in [-1, 0]$, and we have

$$\left| K \int_{-1}^0 e^{-\lambda(t+1)} x_1(t) dt \right| \leq K e^{\mu_1} x_1(0).$$

Hence, with the definition of \tilde{m}_2 and m_2 and the assumption of the second case, it follows that

$$\begin{aligned} & |LHS(X_0)| \\ & \geq k_1 k_2 x_3(0) - k_1 |(\lambda + \mu_3) x_2(0) - (\lambda + \mu_3)(\lambda + \mu_2) x_1(0)| \\ & \quad - K e^{\mu_1} x_1(0) \\ & \geq k_1 k_2 \{ x_3(0) - (k_1 k_2)^{-1} [k_1 |(\lambda + \mu_3)| + |(\lambda + \mu_3)(\lambda + \mu_2)| + \\ & \quad + K e^{\mu_1}] (x_1(0) + x_2(0)) \} \\ & = k_1 k_2 [x_3(0) - \tilde{m}_2 (x_1(0) + x_2(0))] \\ & \geq k_1 k_2 [x_3(0) - (1 + \tilde{m}_2) (x_1(0) + x_2(0))] \\ & = k_1 k_2 [x_3(0) - (m_2^{-1}/2) (x_1(0) + x_2(0))] \geq k_1 k_2 x_3(0)/2 \\ & \geq (k_1 k_2/4) [x_3(0) + x_1(0) + x_2(0)]. \end{aligned}$$

The assertion follows now from the definition of m^* , from the last estimate and from (4.3.1). \square

Corollary 4.4. *Assume $K > \max\{K_c, K_0^*\}$. There exists $\delta_2 > 0$ with the following property:*

$$\forall X_0 \in \mathfrak{K} \setminus \{0\}, \|X_0\|_1 \leq \delta_2 \exists j(X_0) \in \mathbb{N} : \|P_S^{j(X_0)}(X_0)\|_1 > 2\|X_0\|_1.$$

Proof. Recall $\delta_1 \in (0, 1]$ and κ from Corollary 3.11,c), and the numbers m^* and M^* from Propositions 4.2 and 4.3. Choose $\delta_2 > 0$ such that $\max\{2\delta_2, 2\kappa\delta_2\} \leq \delta_1$, and

$$\forall \delta \in (0, \delta_2] : M^* 4\kappa^2 \delta^2 < m^* \delta. \quad (4.4.1)$$

Assume now that $X_0 \in \mathfrak{K} \setminus \{0\}$ satisfies $\|X_0\|_1 \leq \delta_2$ and $\|P_S^j(X_0)\|_1 \leq 2\|X_0\|_1$ for all $j \in \mathbb{N}_0$. Set $\delta := \|X_0\|_1$, so $\delta \leq \delta_2$. We have

$$\|P_S^j(X_0)\|_1 \leq 2\delta \leq 2\delta_2 \leq \delta_1 \text{ for all } j \in \mathbb{N}_0,$$

and using Corollary 3.11,c) we obtain for the corresponding solution X

$$\forall t \geq 0 : \|X_t\| \leq \kappa \cdot 2\delta.$$

Now $\kappa 2\delta \leq \kappa 2\delta_2 \leq \delta_1 \leq 1$, and Proposition 4.2 yields

$$|RHS(X_0)| \leq M^* 4\kappa^2 \delta^2.$$

On the other hand, the assumption $K > \max\{K_c, K_0^*\}$ and Proposition 4.3 show that

$$|LHS(X_0)| \geq m^* \|X_0\|_1 = m^* \delta,$$

in contradiction to (4.4.1). Hence there must exist a $j \in \mathbb{N}$ with

$$\|P_S^j(X_0)\|_1 > 2\|X_0\|_1. \quad \square$$

We can now prove the remaining part of the main theorem. The construction of an invariant set for an iterate of P_S is inspired by the proof of Lemma 1 from [2], p. 576.

Proof of Theorem 1.1, Part b): Under the assumptions of the theorem we have the compact map P_S with image contained in \mathfrak{B} as in Corollary 3.11. It follows from compactness of $\overline{P_S(\mathfrak{K})}$ and the Mazur Theorem that

$$\mathfrak{B}^* := \overline{\text{co}}(P_S(\mathfrak{K}))$$

(the closure of the convex hull of $P_S(\mathfrak{K})$) is compact (and convex). (See, e.g., [3], p. 416, and observe that $\overline{\text{co}}(A) = \overline{\text{co}}(\bar{A})$ for a subset A of a Banach space.) We have $P_S(\mathfrak{B}^*) \subset P_S(\mathfrak{K}) \subset \mathfrak{B}^*$, and $\mathfrak{B}^* \subset \mathfrak{K}$. With δ_2 from Corollary 4.4, choose $\delta_3 \in (0, \delta_2]$ such that \mathfrak{B}^* contains points X_0 with $\|X_0\|_1 > \delta_3$. From Corollary 4.4 and from the fact that P_S maps only zero to zero, one sees that no nonzero point X_0 in \mathfrak{B}^* satisfies $\|P_S^j(X_0)\|_1 \leq \delta_3$ for all $j \in \mathbb{N}$. Thus, for all X_0 in the compact set $\mathfrak{B}_{\delta_3}^* := \{Y_0 \in \mathfrak{B}^* \mid \|Y_0\|_1 \geq \delta_3\}$, there exists $\tilde{j}(X_0) \in \mathbb{N}$ with $\|P_S^{\tilde{j}(X_0)}(X_0)\|_1 > \delta_3$. Compactness of this set and continuity of P_S imply that there exists $j_1 \in \mathbb{N}$ such that

$$\forall X_0 \in \mathfrak{B}_{\delta_3}^* \exists j(X_0) \in \{1, \dots, j_1\} : \|P_S^{j(X_0)}(X_0)\|_1 > \delta_3.$$

Setting $\mathfrak{B}_1 := \bigcup_{j=0}^{j_1} P_S^j(\mathfrak{B}_{\delta_3}^*)$, we have $\mathfrak{B}_1 \subset \mathfrak{B}_{\delta_3}^* \cup P_S(\mathfrak{K}) \subset \mathfrak{B}^*$.

Claim: $P_S(\mathfrak{B}_1) \subset \mathfrak{B}_1$.

Proof. Assume $X_0 \in \mathfrak{B}_1$. If $X_0 \in \bigcup_{j=0}^{j_1-1} P_S^j(\mathfrak{B}_{\delta_3}^*)$ then

$$P_S(X_0) \in \bigcup_{j=1}^{j_1} P_S^j(\mathfrak{B}_{\delta_3}^*) \subset \mathfrak{B}_1.$$

Otherwise, one has $X_0 \in P_S^{j_1}(\mathfrak{B}_{\delta_3}^*)$, so there exists $Y_0 \in \mathfrak{B}_{\delta_3}^*$ with

$$X_0 = P_S^{j_1}(Y_0) = P_S^{j_1-j(Y_0)}(P_S^{j(Y_0)}(Y_0)) \in P_S^{j_1-j(Y_0)}(\mathfrak{B}_{\delta_3}^*).$$

Hence, $j(Y_0) \in \{1, \dots, j_1\}$ implies

$$P_S(X_0) \in P_S^{j_1-j(Y_0)+1}(\mathfrak{B}_{\delta_3}^*) \subset \bigcup_{j=1}^{j_1} P_S^j(\mathfrak{B}_{\delta_3}^*) \subset \mathfrak{B}_1.$$

The claim is proved.

The set \mathfrak{B}_1 is compact and does not contain zero, hence there exists $\delta_4 \in (0, \delta_3]$ such that $\forall X_0 \in \mathfrak{B}_1 : \|X_0\|_1 > \delta_4$. It is obvious from the definition of $\|\cdot\|_1$ that the set $\{X_0 \in \mathfrak{K} \mid \|X_0\|_1 \geq \delta_4\}$ is convex. Convexity and compactness of \mathfrak{B}^* , together with $\mathfrak{B}^* \subset \mathfrak{K}$, imply that the set $\mathfrak{B}_{\delta_4}^* := \{X_0 \in \mathfrak{B}^* \mid \|X_0\|_1 \geq \delta_4\}$ is also compact and convex. It follows from $\mathfrak{B}_1 \subset \mathfrak{B}^*$ and the choice of δ_4 that $\mathfrak{B}_1 \subset \mathfrak{B}_{\delta_4}^*$.

Analogous to the number j_1 from above, there exists $j_2 \in \mathbb{N}$ with the property

$$\forall X_0 \in \mathfrak{B}_{\delta_4}^* \exists k(X_0) \in \{1, \dots, j_2\} : \|P_S^{k(X_0)}(X_0)\|_1 > \delta_3.$$

Recalling that $P_S(\mathfrak{B}^*) \subset \mathfrak{B}^*$, we see that for $X_0 \in \mathfrak{B}_{\delta_4}^*$ one has $P_S^{k(X_0)}(X_0) \in \mathfrak{B}_{\delta_3}^* \subset \mathfrak{B}_1$. Now the invariance of \mathfrak{B}_1 implies that also $P_S^j(X_0) \in \mathfrak{B}_1$ for all $j \geq k(X_0)$, so we conclude that

$$P_S^{j_2}(\mathfrak{B}_{\delta_4}^*) \subset \mathfrak{B}_1 \subset \mathfrak{B}_{\delta_4}^*.$$

Schauder's fixed point theorem now shows the existence of a fixed point of $P_S^{j_2}$ in $\mathfrak{B}_{\delta_4}^*$, which is not zero, and hence yields a nonconstant periodic solution of system (S). \square

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