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Global stabilization in nonlinear discrete systems with time-delay

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Abstract A class of scalar nonlinear difference equations with delay is considered. Sufficient conditions for the global asymptotic stability of a unique equilibrium are given. Applications in economics and other fields lead to consideration of associated optimal control problems. An optimal control problem of maximizing a consumption functional is stated. The existence of optimal solutions is established and their stability (the turnpike property) is proved.

Keywords Scalar difference equations with delay · Global asymptotic stability · Turnpike property in time-delay systems · Optimal control

Mathematics Subject Classification (2000) Primary: 39A22 · 39A30 · 39A60; Secondary: 37E5 · 49K21

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1 Introduction

Consider the following difference equation with delay

$$x_{n+1} = a\,x_n + f(x_{n-K}), \tag{1}$$

where $0 < a < 1$, function $f : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous on the positive semiaxis $\mathbb{R}^+ = \{x \in \mathbb{R} : x \geq 0\}$, and $K \geq 1$ is the integer delay.

The present paper deals with two aspects of dynamics of solutions of Eq. (1). The first one concerns the global asymptotic stability of its unique positive equilibrium. The analysis is based on considering Eq. (1) as a singular perturbation of a limiting 1D map and using the corresponding properties of the latter. This approach uses ideas and prior results obtained in [4,11] and further develops them for this particular case of Eq. (1). The corresponding results are contained in Sect. 2. The second aspect concerns an optimal control problem for Eq. (1) when the nonlinearity f is subject to control in the presence of a functional which has to be maximized. This type of problems has strong applied motivations, in particular, - in problems originating in economics. These details are contained in Sect. 3.

Difference equations have numerous applications in various fields including mathematical biology and economics; see e.g., [1,2,5,8–10,14,20] and further references therein. One of many important applications of Eq. (1) is in economics, which we briefly describe here. Let x_n be an economic output, such as a capital or a commodity being produced and measured at discrete time intervals (say daily, monthly, or annually). At any time n the output x_{n+1} on the next step $n+1$ consists of two components, a fraction of the current output, ax_n, and the output due to the delay factors in the system, $f(x_{n-K})$. The first one is a fixed ratio of the commodity produced during the immediately preceding n^{th} step, which is used in the production on the next step. The second component $f(x_{n-K})$ is the delayed one, which results from the production output K steps back. The latter may be caused by various delay factors specific to particular production or investment circumstances, such as times necessary for completing at least one cycle of production, storage and transportation times, or times it takes for investments to maturate, etc. This leads to difference models described by Eq. (1).

2 Preliminaries and global asymptotic stability

In this section we introduce basic definitions, recall standard notions, and state necessary results concerning the difference Eq. (1).

Difference Eq. (1) is assumed to have a unique positive equilibrium X^*, which is given by the equation $X^* = f(X^*)/(1-a)$. In addition the nonlinearity f satisfies the following assumption

$$f(x) > (1-a)x, \text{ if } 0 < x < X^* \quad \text{and} \quad f(x) < (1-a)x, \quad \text{if } x > X^*. \tag{2}$$

In order for Eq. (1) to have a solution x_n for $n > 0$ one needs to be given a set of initial values $x_{-K}, x_{-K+1}, \ldots, x_{-1}, x_0$; ($x_i \in \mathbb{R}^+ \ \forall i$). We shall call this set of initial data an initial string $\mathbf{x}_0 := \{x_{-K}, x_{-K+1}, \ldots, x_{-1}, x_0\}$. Given set $S \subseteq \mathbb{R}^+$ we say that $\mathbf{x}_0 \in S$ if $x_i \in S$ for all $i \in \{0, -1, \ldots, -K\}$.

For arbitrary initial string $\mathbf{x}_0$ the corresponding solution x_n is found for all $n \in \mathbb{Z}^+$ from Eq. (1) by consecutive iterations. The segment $\{x_1, x_2, \ldots, x_K, x_{K+1}\}$ of the solution is called the first string $\mathbf{x}_1$. Likewise, the segment $\{x_{K+2}, x_{K+3}, \ldots, x_{2K+1}, x_{2K+2}\}$ is called the second string $\mathbf{x}_2$, etc. [4]. Clearly, $x_n \geq 0$ for all $n \geq 1$ if the initial data for $\mathbf{x}_0$ are all in $\mathbb{R}^+$.

Difference Eq. (1) is equivalent to the following one

$$\mu \Delta x_n = -x_{n+1} + F(x_{n-K}), \tag{3}$$

where $\mu = a/(1-a)$, $F(x) = \frac{1}{1-a} f(x)$, and $\Delta x_n := x_{n+1} - x_n$ is the standard difference. This way Eq. (3) can be treated as a singular perturbation of the limiting difference equation as $\mu \to +0\,(a \to 0+)$:

$$x_{n+1} = F(x_{n-K}). \tag{4}$$

Here we recall some basic definitions and notions related to interval maps which are necessary for the exposition in this section. For additional details and other basics of 1D maps we refer the reader to monographs [3,6,23].

Fixed point X^* of map F is called *attracting* if there exists its open (with respect to $\mathbb{R}^+$) neighborhood $\mathcal{U}$ such that $F(\mathcal{U}) \subseteq \mathcal{U}$ and $\lim_{n\to\infty} F^n(x) = X^*$ for every $x \in \mathcal{U}$. Here F^n stands for the nth iteration of F, $F^n := F \circ F^{n-1} = F(F^{n-1})$. The largest interval $\mathcal{U} \subseteq \mathbb{R}^+$ with the above property is called the *domain of immediate attraction* of the fixed point X^*. An interval $I \subseteq \mathbb{R}^+$ is said to be invariant under F if $F(x) \in I$ for all $x \in I$.

In this subsection we shall employ an approach introduced and used in papers [4,11].

Proposition 2.1 (Invariance) *Let $I = [a, b]$ be an invariant interval of the map F, $F(I) \subseteq I$. For every initial string $\mathbf{x}_0 \in I$ the corresponding solution x_n of Eq.* (1) *satisfies $x_n \in I \;\; \forall n \in \mathbb{N}$.*

Proof This property can be deduced from relevant statements in papers [4,11]. It is also easily seen from the fact that the difference operator Δx_n is "directed" inside the interval I on its boundary $\{a, b\}$. Indeed, assume that the initial string satisfies $\mathbf{x}_0 \in I$, and let $N > 1$ be the first time when the corresponding solution x_n leaves the invariant interval I. To be definite, assume $x_N > b$ and $x_i \in I \;\; \forall i < N$. Then $\Delta x_{N-1} = x_N - x_{N-1} > 0$. On the other hand, Eq. (3) shows that $\Delta x_{N-1} = (1/\mu)[x_N - x_{N-1}] = (1/\mu)[-x_N + F(x_{N-K-1})] \le 0$, a contradiction. The case $x_N < a$ is similar and left to the reader. □

Proposition 2.2 *Let L be a closed bounded interval such that $F(L) \subseteq L$. Assume also that none of the endpoints of the interval $F(L)$ is a fixed point. Then for every initial string $\mathbf{x}_0 \in L$ there exists time $N = N(F, \mathbf{x}_0, \mu)$ such that the corresponding solution of Eq.* (3) *satisfies $x_n \in F(L)$ for all $n \ge N$.*

Proof Let $L := [\alpha, \beta]$ be a closed bounded interval with $F(L) := [\gamma, \delta] \subset L$. Let an initial string $\mathbf{x}_0 = \{x_{-K}, \ldots, x_{-1}, x_0\} \in L$ be given. Then, due to the Invariance Property (Proposition 2.1), one sees that the corresponding solution $x_n(\mathbf{x}_0)$ satisfies $x_n \in L \; \forall n \in \mathbb{N}$.

Suppose that $x_0 \in F(L)$. Then, exactly like in the proof of Proposition 2.1 one can show that $x_n \in F(L)$ for all $n \ge 0$. Indeed, suppose not, and let M be the first time when the solution leaves the interval $F(L)$. To be definite, assume that $x_M \in [\alpha, \gamma)$ and $x_n \in [\gamma, \delta]$ for all $n < M$. Then $\Delta x_{M-1} = x_M - x_{M-1} < 0$. On the other hand, since $F(x_{M-K}) \in F(L) = [\gamma, \delta]$, Eq. (3) shows that $\Delta x_{M-1} = (1/\mu)[-x_M + F(x_{M-K})] > 0$, a contradiction. The other possibility $x_M \in (\delta, \beta]$ leads to a contradiction in a similar way.

Suppose next that $x_0 \notin F(L)$. We claim that there exists time $M > 0$ such that $x_M \in F(L)$. Then, in view of the previous argument, the corresponding solution x_n will satisfy $x_n \in F(L)$ for all $n \ge M$. Suppose not. To be definite let $x_0 \in [\alpha, \gamma)$ and $x_n \notin [\gamma, \delta]$ for all $n > 0$. We claim that $x_n < \gamma$ for all $n > 0$. Indeed, if on the contrary $x_1 = x_1(\mathbf{x}_0) > \delta$, consider a modified initial string $\tilde{\boldsymbol{x}}_{\mathbf{0}} := \{x_{-K}, \ldots, x_{-1}, x_0, \gamma\}$. Since $\gamma \in F(L)$ one has, by the above

reasoning, that $x_n(\tilde{\mathbf{x}}_0) \in F(L)$ for all $n > 0$, so that $x_1(\tilde{\mathbf{x}}_0) \leq \delta$. From Eq. (1) it is easily seen that $x_1(\mathbf{x}_0) < x_1(\tilde{\mathbf{x}}_0)$, a contradiction with $x_1(\mathbf{x}_0) > \delta$. Since $x_1 \in F(L)$ one applies the induction argument to conclude that $\alpha \leq x_n < \gamma$ for all $n > 0$. Then Eq. (3) shows that $\Delta x_n = (1/\mu)[-x_{n+1} + F(x_{n-K})] \geq 0, \forall n \in \mathbb{N}$. Thus, the solution x_n is increasing and is bounded from above. Let $\lim_{x_n} = \bar{x} \leq \gamma$. Then, since $\lim_{n\to\infty} \Delta x_n = 0$, Eq. (3) implies that $0 = -\bar{x} + F(\bar{x})$. That is, $\bar{x} = \gamma$ is a fixed point of F, a contradiction. □

Proposition 2.3 *Let L be a closed bounded interval such that $F(L) \subseteq L$. If one of the endpoints of the interval $F(L)$ is a fixed point $x = X^*$ then for every initial string $\mathbf{x}_0 \in L$ and the corresponding solution $x_n = x_n(\mathbf{x}_0)$ of Eq. (3) either the conclusion of Proposition 2.2 holds or* $\lim_{n\to\infty} x_n = X^*$.

Proof Indeed, this is proved exactly the same way as Proposition 2.2, except that the possibility $\lim_{n\to\infty} x_n = \bar{x} = \gamma$ is allowed (or $\lim_{n\to\infty} x_n = \bar{x} = \delta$). □

Proposition 2.4 (Global Asymptotic Stability) *Suppose that interval map F has an attracting fixed point X^* with the interval J being the domain of immediate attraction:*

$$F(X^*) = X^*, \quad \lim_{n\to\infty} F^n(x) = X^* \ \ \forall x \in J \ni X^*.$$

Then for every initial string $\mathbf{x}_0 \in J$ the corresponding solution $x_n = x_n(\mathbf{x}_0)$ has the property $\lim_{n\to\infty} x_n = X^*$.

Proof Let an initial string $\mathbf{x}_0 = \{x_{-K}, \ldots, x_{-1}, x_0\}$ be given such that $\mathbf{x}_0 \in J$. Then one can find a closed bounded interval $I_0 \subset J$ such that $\mathbf{x}_0 \in I_0$ and $F(I_0) \subseteq I_0$. Since J is the domain of immediate attraction of X^* one also has

$$I_0 \supseteq F(I_0) \supseteq F^2(I_0) \supseteq \cdots \supseteq F^i(I_0) \supseteq \cdots \quad \text{and} \quad \cap_{i\geq 0} F^i(I_0) = X^*. \tag{5}$$

By using the induction argument, for every $i \in \mathbb{N}$ one sets $F^i(I_0) := L$ and applies either Proposition 2.2 or Proposition 2.3 to show that $\lim_{n\to\infty} x_n = X^*$. □

Corollary 2.5 *Suppose that map F is increasing in $\mathbb{R}^+$, has a unique positive fixed point X^*, and satisfies the condition*

$$F(x) > x, \quad \text{if } 0 < x < X^* \quad \text{and} \quad F(x) < x, \quad \text{if } x > X^*. \tag{6}$$

Then the equilibrium $x_n \equiv X^*$ of Eq. (3) is globally asymptotically stable for every delay $K \geq 1$:

$$\lim_{n\to\infty} x_n(\mathbf{x}_0) = X^* \ \ \textit{for arbitrary initial string} \ \ \mathbf{x}_0 \in \mathbb{R}^+.$$

Proof Let the initial string $\mathbf{x}_0 = \{x_{-K}, \ldots, x_{-1}, x_0\}$ be given. Denote $M := \max\{x_0, x_{-1}, \ldots, x_{-K}\}$ and $m := \min\{x_0, x_{-1}, \ldots, x_{-K}\}$. If $m < X^* < M$ set $K := [m, M]$. Then, in view of the hypothesis (6) and the monotonicity of F, one can easily see that

$$K \subset F(K) \subset F^2(K) \subset \cdots \subset F^n(K) \subset \cdots \quad \text{and} \quad \cap_{n\geq 0} F^n(K) = X^*.$$

Therefore, the reasoning as in the proof of Proposition 2.4 applies. If $m \leq M \leq X^*$ or $X^* \leq m \leq M$ one sets $K := [m, X^*]$ or $K := [X^*, M]$, respectively, and concludes the same chain of inclusions and the applicability of the same reasoning from Proposition 2.4. □

3 Convergence of solutions in optimal control problems

Consider the following difference equation

$$x_{n+1} = a\, x_n + u_n\, f(x_{n-K}), \quad n = 0, 1, 2, \ldots \tag{7}$$

where the parameter a and the function f are the same as for Eq. (1), and u_n is a control sequence with values in [0, 1] that is motivated by practical applications.

Many models in various areas lead to differential delay equations of the form (7). We refer the reader to a partial list of applications given in papers [1,7,22]. Equation (7) also serves as a general model of several economical processes, which in particular includes the modified Ramsey model with delay that will be considered in this paper. The continuous version of this model is considered in [18] (see also [12]).

Let $\mathbf{x_0} = \{x_{-K}, \ldots, x_{-1}, x_0\} \subseteq \mathbb{R}^+$ be any fixed initial string. For arbitrary control $\mathbf{u} = (u_n)_{n=0}^{\infty}$ system (7) defines a unique solution $\mathbf{x} = \mathbf{x}(\mathbf{x}_0, \mathbf{u}) = (x_n)_{n=0}^{\infty}$, which is found by consecutive iterations for all $n > 0$.

Introduce the following consumption functional $C(\mathbf{x}_0, \mathbf{u})$:

$$C(\mathbf{x}_0, \mathbf{u}) = \liminf_{n\to\infty} (1 - u_n)\, f(x_{n-K}). \tag{8}$$

This type of functional defined by "lim inf" is considered in several papers including [15,21]. It turns out that this functional is more preferable in the study of asymptotic stability of optimal solutions; we refer to [16,19] and references therein for more information about the advantages of this functional compared with integral type of functionals.

The optimal control problem that we consider in this section can be formulated as follows. Given initial string $\mathbf{x_0}$ find control $\mathbf{u}$ such that the consumption functional C achieves its maximum over the solutions to system (7):

$$\text{Maximize}: \ C(\mathbf{x}_0, \mathbf{u}), \quad \text{subject to (7)}. \tag{9}$$

The economic interpretation of the optimal control problem (9) can be as follows. At every time n the delay component $f(x_{n-K})$ of the commodity produced can be controlled in such a way that u_n fraction of it, $u_n\, f(x_{n-K})$, is put back into the production cycle. This leads to an equation of the type (1), which assumes now the form of Eq. (7). The remaining part of the commodity, $(1 - u_n)\, f(x_{n-K})$, is consumed. One would like to find a control $u_n, n \geq 0$, that maximizes the minimal level of consumption for sufficiently large time periods. Mathematically this leads to functional (8).

As in the previous section, we assume that f is a continuous function satisfying relation (2) with the fixed point $X^* = f(X^*)/(1 - a)$. This in particular means that

$$f(x) - (1 - a)x > 0, \quad \forall x \in (0, X^*).$$

Define the positive number c^* and the set $\mathcal{T}$ by

$$c^* = \max\{f(x) - (1 - a)x : \ x \in [0, X^*]\}; \tag{10}$$

$$\mathcal{T} = \{x \in [0, X^*] : \ f(x) - (1 - a)x = c^*\}. \tag{11}$$

The value c^* will be interpreted as maximum steady consumption that could be achieved in problem (9). Accordingly, each point $x \in \mathcal{T}$ is as a steady state (equilibrium) guaranteing this consumption c^*. Clearly, $\mathcal{T}$ is a closed set and it may contain more than one point.

A special case is when the set $\mathcal{T}$ consists of only one point, say x^*. In this case

$$c^* = f(x^*) - (1 - a)x^*.$$

Moreover, $f(x^*) > 0$ and if one sets $u^* = (1-a)x^*/f(x^*)$ then $u^* \in [0, 1)$ and

$$x^* = ax^* + u^* f(x^*).$$

Therefore, x^* is an equilibrium point of Eq. (7) corresponding to the stable control $u_n = u^*$.

Proposition 3.1 *Suppose that $f : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and (2) holds. Then solutions $\mathbf{x}$ to (7) are bounded; that is, there is a number $M < \infty$ such that*

$$\limsup_{n\to\infty} x_n \le M, \quad \forall \mathbf{x}.$$

In addition, if f is increasing then

$$\limsup_{n\to\infty} x_n < X^*, \quad \forall \mathbf{x}.$$

Proof Consider solution $\mathbf{x} = \mathbf{x}(\mathbf{x}_0, \mathbf{u}) = (x_n)_{n=0}^{\infty}$ to (7) corresponding to the initial string $\mathbf{x}_0 = \{x_{-K}, \ldots, x_{-1}, x_0\}$ and control $\mathbf{u} = (u_n)_{n=0}^{\infty}$.

(i) First we show that x_n is bounded.

Denote

$$L := \max\{f(x) : \ x \in [0, X^*]\} < +\infty,$$

and let

$$M := \max\left\{x_{-K}, \ldots, x_{-1}, x_0, \frac{1}{1-a}L\right\}.$$

Assume that x_n is not bounded. Then there is an index m such that $x_{m+1} > M$ and $x_n \le M$ for all $n < m+1$; in particular, $x_{m+1} > x_m$. From (7) we have

$$0 < x_{m+1} - x_m = -(1-a)x_m + u_m f(x_{m-K}) \le -(1-a)x_m + f(x_{m-K}). \tag{12}$$

If $x_{m-K} > X^*$ then from assumption (2) it follows that $f(x_{m-K}) < (1-a)x_{m-K}$. Since $x_{m-K} < x_{m+1}$ we obtain

$$f(x_{m-K}) < (1-a)x_{m+1}.$$

This inequality also holds in the case when $x_{m-K} \in [0, X^*]$. Indeed, in this case

$$f(x_{m-K}) \le L \le (1-a)M < (1-a)x_{m+1}.$$

Thus, from (12) we have

$$0 < x_{m+1} - x_m < -(1-a)x_m + (1-a)x_{m+1} = (1-a)(x_{m+1} - x_m),$$

a contradiction since $0 < a < 1$.

Therefore, x_n is bounded. Set $p := \limsup_{n\to\infty} x_n < \infty$.

(ii) Next we show that $p \le X^*$ when f is increasing.

Since $u_n \le 1$, from (7) we have

$$x_{n+1} \le ax_n + f(x_{n-K}), \quad n = 0, 1, 2, \ldots. \tag{13}$$

By the definition of p there is a subsequence $k_m \to \infty$ satisfying the following conditions:

$$x_{k_m+1} \to p; \ x_{k_m} \to x' \le p; \ \text{and } x_{k_m-K} \to x'' \le p. \tag{14}$$

Since f is increasing, the inequality $f(x'') \leq f(p)$ holds. Then substituting n with k_m in (13) and taking the limit we obtain

$$p \leq ax' + f(x'') \leq ap + f(p)$$

or $(1-a)p \leq f(p)$. This yields $p \leq X^*$ due to assumption (2). Proposition is proved. □

Proposition 3.2 *Suppose that $f : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and (2) holds. Then for every solution $\mathbf{x} = \mathbf{x}(\mathbf{x}_0, \mathbf{u})$ to (7) the inequality*

$$C(\mathbf{x}_0, \mathbf{u}) \leq c^*$$

holds; here c^* is defined by (10).

Proof Consider a solution $\mathbf{x} = \mathbf{x}(\mathbf{x}_0, \mathbf{u}) = (x_n)_{n=0}^{\infty}$ to (7). Recall that

$$C(\mathbf{x}_0, \mathbf{u}) = \liminf_{n\to\infty} c_n;$$

where

$$c_n = (1-u_n) f(x_{n-K}). \tag{15}$$

From (7) and (15) it follows

$$c_n = f(x_{n-K}) - x_{n+1} + ax_n. \tag{16}$$

Denote $p := \limsup_{n\to\infty} x_n < \infty$, and let $k_m \to \infty$ be a sequence satisfying (14) considered in the proof of Proposition 3.1. From (16) we have

$$\lim_{m\to\infty} c_{k_m} = f(x'') - p + ax' \leq f(x'') - (1-a)p \leq f(x'') - (1-a)x''.$$

By the definition of c^* we have $f(x'') - (1-a)x'' \leq c^*$. Therefore,

$$C(\mathbf{x}_0, \mathbf{u}) = \liminf_{n\to\infty} c_n \leq \lim_{m\to\infty} c_{k_m} \leq c^*.$$

Proposition is proved. □

Proposition 3.3 *Suppose that $f : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and (2) holds. Let $\mathbf{x} = \mathbf{x}(\mathbf{x}_0, \mathbf{u}) = (x_n)_{n=0}^{\infty}$ be a solution to (7) such that $C(\mathbf{x}_0, \mathbf{u}) = c^*$. Then*

$$x_{n+1} \leq ax_n + (1-a)x_{n-K} + \xi_n, \quad \forall n; \tag{17}$$

where $\xi_n > 0$ and $\xi_n \to 0$ as $n \to \infty$.

Moreover

$$\limsup_{n\to\infty} x_n \in \mathcal{T}. \tag{18}$$

Here $\mathcal{T}$ is the set of optimal equilibrium points defined by (11).

Proof Since $\liminf_{n\to\infty} c_n = c^*$, there is a sequence of positive numbers $\xi_n \to 0$ such that $c_n \geq c^* - \xi_n$. Then from (16)

$$f(x_{n-K}) - x_{n+1} + ax_n \geq c^* - \xi_n, \quad \forall n.$$

On the other hand by definition of c^* and assumption (2) it follows that $f(x) - (1-a)x \leq c^*$ for all x, and in particular

$$f(x_{n-K}) - (1-a)x_{n-K} \leq c^*.$$

Therefore, from the last two inequalities we have

$$(1-a)x_{n-K} - x_{n+1} + ax_n \geq -\xi_n, \quad \forall n;$$

which leads to (17).

Now we shall prove (18). Denote $p := \limsup_{n\to\infty} x_n$ and consider the sequence $k_m \to \infty$ satisfying (14) in the proof of Proposition 3.1:

$$x_{k_m+1} \to p; \; x_{k_m} \to x' \leq p; \text{ and } x_{k_m-K} \to x'' \leq p.$$

We have (see also (16))

$$c^* = \liminf_{n\to\infty} c_n \leq \lim_{m\to\infty} c_{k_m} = f(x'') - p + ax' \leq f(x'') - (1-a)p. \tag{19}$$

As $a < 1$ and $x'' \leq p$ the last inequality yields

$$c^* \leq f(x'') - (1-a)x''.$$

This means that $x'' \in \mathcal{T}$, in view of (11) and assumption (2). Therefore

$$f(x'') - (1-a)x'' = c^*.$$

Now, if $x'' < p$ then going back to (19) we obtain a contradiction in the form

$$c^* \leq f(x'') - (1-a)p < f(x'') - (1-a)x'' = c^*.$$

Therefore, $p = x'' \in \mathcal{T}$, that is, (18) is true. Proposition is proved. □

Proposition 3.3 describes the structure of optimal solutions $\mathbf{x}$ satisfying $C(\mathbf{x}_0, \mathbf{u}) = c^*$. In what follows we consider the case when the set of optimal equilibrium points $\mathcal{T}$ has an empty interior. In this case the convergence of optimal solutions to some steady state will be proved. In the literature, such a property of optimal solutions is called the turnpike property [13,19,24]. We note that it is extremely difficult to prove the stability in the case when there is more than one optimal equilibrium point (even in the absence of time-delay), especially for integral type of functionals (see for example [17]).

Proposition 3.4 *Suppose that $f : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous, hypothesis (2) holds, and the set of optimal equilibrium points $\mathcal{T}$ has an empty interior* $\text{int}\mathcal{T} = \emptyset$. *If $\mathbf{x} = \mathbf{x}(\mathbf{x}_0, \mathbf{u}) = (x_n)_{n=1}^{\infty}$ is a solution to (7) such that $C(\mathbf{x}_0, \mathbf{u}) = c^*$, then it converges to some optimal equilibrium point; that is,*

$$\lim_{n\to\infty} x_n = x \in \mathcal{T}. \tag{20}$$

Proof Denote

$$q := \liminf_{n\to\infty} x_n \quad \text{and} \quad p := \limsup_{n\to\infty} x_n.$$

From Proposition 3.3 it follows that $p \in \mathcal{T}$.

If $q = p$ then (20) is true. Consider the case $q < p$.

Take any positive number $\eta \in (0, \overline{\eta}]$, where $\overline{\eta}$ is defined as follows:

$$\overline{\eta} := \min\left\{p - \overline{q}, \; \frac{p-q}{2}\right\} > 0.$$

Here $\overline{q} := a^K q + (1 - a^K)p$ and K is the delay in system (7).

We will use relation (17) in Proposition 3.3. Since $\xi_n \to 0$ there is a number N_1 such that

$$\sum_{j=0}^{K-1} \xi_{n+j} < \frac{1}{4}\eta, \quad \forall n \geq N_1. \tag{21}$$

By definition of q and p, given any small positive number $\varepsilon < \frac{1}{4}\eta$ there is $n_1 > N_1$ such that the following two inequalities hold:

$$x_{n_1} \leq q + \varepsilon \quad \text{and} \quad x_n \leq p + \varepsilon, \quad \forall n \geq n_1 - K.$$

From (17) we have

$$x_{n_1+1} \leq a x_{n_1} + (1-a)x_{n_1-K} + \xi_{n_1} \leq a(q+\varepsilon) + (1-a)(p+\varepsilon) + \xi_{n_1}.$$

Consequently,

$$\begin{aligned} x_{n_1+2} &\leq a x_{n_1+1} + (1-a)x_{n_1-K+1} + \xi_{n_1+1} \\ &\leq a[a(q+\varepsilon) + (1-a)(p+\varepsilon) + \xi_{n_1}] + (1-a)(p+\varepsilon) + \xi_{n_1+1} \\ &= a^2(q+\varepsilon) + (1-a^2)(p+\varepsilon) + a\xi_{n_1} + \xi_{n_1+1} \\ &\leq a^2 q + (1-a^2)p + \varepsilon + \sum_{j=0}^{1} \xi_{n_1+j}. \end{aligned}$$

Therefore, for any $i = 1, \ldots, K-1$ we have

$$x_{n_1+i} \leq a^i q + (1-a^i)p + \varepsilon + \sum_{j=0}^{i-1} \xi_{n_1+j}.$$

Clearly, $a^i q + (1-a^i)p \leq a^K q + (1-a^K)p = \overline{q}$ for all $i = 1, \ldots, K-1$. Moreover, since $\eta \leq p - \overline{q}$ and $\varepsilon < \frac{1}{4}\eta$, taking into account (21) we obtain

$$x_{n_1+i} \leq a^K q + (1-a^K)p + \frac{1}{4}\eta + \frac{1}{4}\eta \leq p - \frac{1}{2}\eta.$$

On the other hand, $\eta \leq (p-q)/2$ and therefore

$$x_{n_1} \leq q + \varepsilon \leq p - 2\eta + \frac{1}{4}\eta < p - \frac{1}{2}\eta.$$

Denote $\widetilde{p} := p - \frac{1}{2}\eta < p$. We have shown that, starting from initial point x_{n_1}, the next K consecutive terms x_{n_1+i}, $i = 0, 1, \ldots, K-1$, satisfy the inequality $x_{n_1+i} \leq \widetilde{p}$.

Now consider point p. Since it is a limit point of x_n, there are infinitely many terms $x_{\widetilde{n}}$, with $\widetilde{n} > n_1 + K - 1$, satisfying inequality $x_{\widetilde{n}} > \widetilde{p}$. Let $k_1 + 1 > n_1 + K - 1$ be the first such term; that is, the following relations hold:

$$x_{k_1+1} > \widetilde{p}, \quad x_{k_1} \leq \widetilde{p}, \quad x_{k_1-K} \leq \widetilde{p}. \tag{22}$$

On the other hand, q is also a limit point of the sequence x_n, then there is a number $n_2 > k_1$ such that

$$x_{n_2} \leq q + \varepsilon.$$

Thus, we can repeat the above procedure by starting from index n_2 instead of n_1. As a result, we can find $k_2 + 1 > n_2 + K - 1$ such that the relations in (22) are satisfied for the index k_2.

Continuing this procedure we can construct a sequence $k_m \to \infty$ satisfying (22); that is,

$$x_{k_m+1} > \widetilde{p}, \quad x_{k_m} \le \widetilde{p}, \quad x_{k_m-K} \le \widetilde{p}, \quad \forall m \in \mathbb{Z}^+. \tag{23}$$

Moreover, we can choose a subsequences of k_m such that all the sequences in (23) are convergent. For the sake of simplicity, let

$$x_{k_m+1} \to \widetilde{x} \ge \widetilde{p}, \quad x_{k_m} \to x' \le \widetilde{p}, \quad x_{k_m-K} \to x'' \le \widetilde{p}. \tag{24}$$

Therefore,

$$\begin{aligned} c^* &= \liminf_{n\to\infty} c_n \le \lim_{m\to\infty} c_{k_m} \\ &= \lim_{m\to\infty} [f(x_{k_m-K}) - x_{k_m+1} + a x_{k_m}] = f(x'') - \widetilde{x} + a x'; \end{aligned}$$

or

$$c^* \le f(x'') - \widetilde{p} + a\widetilde{p}. \tag{25}$$

Since $c^* \ge f(x'') - (1-a)x''$ from (25) it follows that $x'' \ge \widetilde{p}$ or $x'' = \widetilde{p}$, due to (24). Taking this into account in (25) we obtain

$$c^* \le f(\widetilde{p}) - (1-a)\widetilde{p}$$

which means that $\widetilde{p} \in \mathcal{T}$.

Thus we have shown that

$$p - \frac{1}{2}\eta \in \mathcal{T}, \quad \forall \eta \in (0, \overline{\eta}].$$

This contradicts the assumption that $\mathcal{T}$ has an empty interior.

Proposition is proved. □

Proposition 3.5 *Suppose that $f : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and (2) holds. Then given any initial string $\mathbf{x}_0$ and any optimal equilibrium $\widetilde{x} \in \mathcal{T}$, there is a control $\mathbf{u}$ such that the corresponding solution $\mathbf{x} = \mathbf{x}(\mathbf{x}_0, \mathbf{u}) = (x_n)_{n=1}^{\infty}$ to (7) converges to that equilibrium; that is,*

$$\lim_{n\to\infty} x_n = \widetilde{x} \in \mathcal{T}.$$

Proof In the case when $\mathcal{T} = \{x^*\}$ and f is increasing the proof follows from Corollary 2.5. Here we prove this result for a more general case by choosing an appropriate control sequence u_n.

Take any $\widetilde{x} \in \mathcal{T}$. From (2) and (11) it follows that $\widetilde{x} < X^*$.

If $\widetilde{x} = 0$ then it is not difficult to observe that given any initial string $\mathbf{x}_0$ the solution $\mathbf{x} = \mathbf{x}(\mathbf{x}_0, \mathbf{u})$ to (7) corresponding to the control $u_n = 0, \forall n$, converges to zero; that is, the proposition is true.

Consider the case $\widetilde{x} > 0$. In this case from assumption (2) it follows that

$$f(x) > (1-a)x > 0, \quad \forall x \in (0, \widetilde{x}]. \tag{26}$$

Given any initial string $\mathbf{x}_0$ consider control $\widetilde{u}_n = 0$, $n = 0, 1, 2, \ldots$. We have

$$x_n = a^{n-1} x_0 \to 0.$$

Since $\widetilde{x} > 0$ there is a number n_1 such that

$$x_{n_1-j} \in (0, \widetilde{x}], \quad \forall j = 0, 1, \ldots, K.$$

Clearly,

$$\delta := \min\{x_{n_1-j} : j = 0, 1, \ldots, K\} > 0. \tag{27}$$

For $n \geq n_1$ we define the sequence $\widetilde{u}_n$ as follows:

$$\widetilde{u}_n = \min\left\{1, \frac{\widetilde{x} - ax_n}{f(x_{n-K})}\right\}, \quad n \geq n_1.$$

Clearly, as long as $\widetilde{x} \geq ax_n$ the values of $\widetilde{u}_n$ stay in the interval [0, 1]; that is, $\widetilde{u}_n$ can be used as a control parameter. Consider a sequence defined by

$$x_{n+1} = ax_n + \widetilde{u}_n f(x_{n-K}), \quad n \geq n_1. \tag{28}$$

First we show that this sequence is a solution to (7); that is, $\widetilde{u}_n \in [0, 1]$ for all $n \geq n_1$.

Consider the term $x_{n_1+1} = ax_{n_1} + \widetilde{u}_{n_1} f(x_{n_1-K})$. There are two possible cases.

- If $\widetilde{x} - ax_{n_1} \leq f(x_{n_1-K})$; that is, $\widetilde{u}_{n_1} = \frac{\widetilde{x} - ax_{n_1}}{f(x_{n_1-K})}$, then
$$x_{n_1+1} = ax_{n_1} + \widetilde{u}_{n_1} f(x_{n_1-K}) = \widetilde{x}.$$
- If $\widetilde{x} - ax_{n_1} > f(x_{n_1-K})$; that is, $\widetilde{u}_{n_1} = 1$, then
$$x_{n_1+1} = ax_{n_1} + \widetilde{u}_{n_1} f(x_{n_1-K}) = ax_{n_1} + f(x_{n_1-K}) < \widetilde{x}.$$

Moreover, from (26) and (27) one has

$$x_{n_1+1} = ax_{n_1} + f(x_{n_1-K}) > ax_{n_1} + (1-a)x_{n_1-K} \geq \delta.$$

Therefore, in both cases $x_{n_1+1} \in (\delta, \widetilde{x}]$. This in particular means that $\widetilde{x} - ax_{n_1+1} > 0$ or $\widetilde{u}_{n_1+1} \in (0, 1]$.

Continuing this process we obtain sequences $\widetilde{u}_n$ and x_n such that

$$\widetilde{u}_n \in (0, 1] \text{ and } x_n \in (\delta, \widetilde{x}], \quad \forall n \geq n_1.$$

Moreover,

$$\widetilde{u}_n = 1 \quad \text{if} \quad x_{n+1} < \widetilde{x}. \tag{29}$$

Thus, (28) defines a solution to (7). Now we show that this solution converges to $\widetilde{x}$: $x_n \to \widetilde{x}$.

Denote $q := \liminf x_n$ and, on the contrary assume, that $q < \widetilde{x}$. From $x_n \in (\delta, \widetilde{x}]$, $\forall n \geq n_1$, we know that $q \geq \delta > 0$. Consider a subsequence $k_m \to \infty$ such that

$$x_{k_m+1} \to q, \quad x_{k_m} \to x' \in [q, \widetilde{x}], \quad x_{k_m-K} \to x'' \in [q, \widetilde{x}].$$

Since for sufficiently large k_m the inequality $x_{k_m+1} < \widetilde{x}$ holds, from (29) we have $\widetilde{u}_{k_m} = 1$. Then $x_{k_m+1} = ax_{k_m} + f(x_{k_m-K})$, and by taking the limit

$$q = ax' + f(x'').$$

As $x'' \in (0, \widetilde{x}]$ from (26) it follows $f(x'') > (1-a)x''$. Then

$$q > ax' + (1-a)x''.$$

Since $x' \geq q$ and $x'' \geq q$, the last inequality yields $q > q$ that is a contradiction.

Proposition is proved. □

From this Proposition, we obtain that given any initial string, there are solutions to (7) converging to an optimal equilibrium $x^* \in \mathcal{T}$. Moreover, for any solution $\mathbf{x}^* = \mathbf{x}(\mathbf{x}_0, \mathbf{u}^*)$ that converges to x^*, it is not difficult to see that the corresponding control sequence $\mathbf{u}^*$ should also converge and

$$\lim_{n\to\infty} u_n^* = u^* = \frac{(1-a)x^*}{f(x^*)}.$$

In this case we have

$$C(\mathbf{x}_0, \mathbf{u}^*) = \liminf_{n\to\infty}(1-u_n^*)f(x_{n-K}^*) = \left(1 - \frac{(1-a)x^*}{f(x^*)}\right) f(x^*) = c^*.$$

Thus, according to Proposition 3.2, the solution $\mathbf{x}^*$ is optimal.

Therefore, we have established the following result.

Theorem 3.6 *Suppose that $f : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and condition (2) holds. Then*

- *functional (8) is bounded above over the solutions to (7); that is, the inequality*

$$C(\mathbf{x}_0, \mathbf{u}) \le c^*$$

  holds for all $\mathbf{x}_0$ and $\mathbf{u}$;
- *given any initial string $\mathbf{x}_0$, there exists an optimal control $\mathbf{u}_{\mathbf{x}_0}$ to problem (9) such that functional (8) achieves its maximum possible value; that is,*

$$C(\mathbf{x}_0, \mathbf{u}_{\mathbf{x}_0}) = c^*;$$

- *in addition, if the set of optimal equilibrium points defined by (11) has an empty interior, $\mathrm{int}\mathcal{T} = \emptyset$, then all optimal solutions converge to some optimal equilibrium; in particular, if the optimal equilibrium is unique; that is, $\mathcal{T} = \{x^*\}$, then*

$$\lim_{n\to\infty} x_n = x^*$$

  for all optimal solutions $\mathbf{x}$.

3.1 Example

Consider problem (9) where function f is the standard convex nonlinearity $f(x) = A\,x^\alpha$, with $A > 0$ and $\alpha \in (0, 1)$. The corresponding difference Eq. (7) assumes the form

$$x_{n+1} = a\,x_n + u_n\,A[x_{n-K}]^\alpha.$$

For any $u \in (0, 1]$, this equation has a unique positive equilibrium x^u given by

$$x^u = \left(\frac{A}{1-a}\right)^{\frac{1}{1-\alpha}} \cdot u^{\frac{1}{1-\alpha}}.$$

Proposition 3.5 guarantees that there is a control $\mathbf{u}$ such that the corresponding solution x_n converges to the equilibrium x^u. In addition, Corollary 2.5 states that one of such controls is the stable/constant control $u_n \equiv u$. The functional (8) in this case has the form

$$C(\mathbf{x}_0, \mathbf{u}) = \liminf_{n\to\infty}(1-u_n)f(x_{n-K}) = B\,(1-u)u^{\frac{\alpha}{1-\alpha}},$$

where $B = A^{1/(1-\alpha)}/(1-a)^{\alpha/(1-\alpha)}$. The unique maximum value of the expression $(1-u)u^{\frac{\alpha}{1-\alpha}}$ is achieved when $u = \alpha$.

Then, from Theorem 3.6 it follows that given any initial string the control defined by $\mathbf{u} = (u_n)_{n=1}^{\infty}$, with $u_n = \alpha, \forall n$, is an optimal control. Moreover, any optimal solution x_n satisfies

$$\lim_{n\to\infty} x_n = \left(\frac{A}{1-a}\right)^{\frac{1}{1-\alpha}} \cdot \alpha^{\frac{1}{1-\alpha}}.$$

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