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On the existence of non-monotone non-oscillating wavefronts

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ABSTRACT

We present a monostable delayed reaction–diffusion equation with the unimodal birth function which admits only non-monotone wavefronts. Moreover, these fronts are either eventually monotone (in particular, such is the minimal wave) or slowly oscillating. Hence, for the Mackey–Glass type diffusive equations, we answer affirmatively the question about the existence of non-monotone non-oscillating wavefronts. As it was recently established by Hasik et al. and Ducrot et al., the same question has a negative answer for the KPP-Fisher equation with a single delay.

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1. Introduction and main results

This work deals with the traveling waves for the diffusive Mackey–Glass type equation

$$u_t(t,\mathbf{x}) = \Delta u(t,\mathbf{x}) - u(t,\mathbf{x}) + g\big(u(t-h,\mathbf{x})\big), \quad u(t,\mathbf{x}) \geqslant 0,\ \mathbf{x} \in \mathbb{R}^m. \tag{1}$$

The population model (1) was extensively studied (including its non-local version) during the past decade, e.g. see [10,11,16–18,24] and references therein. Notice that the non-negativity condition $u(t,\mathbf{x}) \geqslant 0$ of (1) is due to the biological interpretation of u as the size of an adult population. In this paper we are mostly concerned with classical positive solutions to (1) of the special form $u(t,\mathbf{x}) = \phi(ct + \mathbf{n}\cdot\mathbf{x})$, $c > 0$, $|\mathbf{n}| = 1$, where ϕ additionally satisfies the boundary conditions $\phi(-\infty) = 0$, $\phi(+\infty) = \kappa$. Such solutions of Eq. (1) are called traveling fronts or simply wavefronts. The function $\phi : \mathbb{R} \to \mathbb{R}_+$ is said to be the profile of the wavefront $u(t,\mathbf{x}) = \phi(ct + \mathbf{n}\cdot\mathbf{x})$. It is easy to see that each profile ϕ is a positive heteroclinic solution of the delay differential equation

$$x''(t) - cx'(t) - x(t) + g\big(x(t-ch)\big) = 0, \quad t \in \mathbb{R}. \tag{2}$$

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The nonlinear term g in (1) and (2) plays the role of a *birth function* and therefore it is non-negative. Motivated by various concrete applications, throughout the paper we assume that g satisfies the following unimodality condition (see also Fig. 2):

(UM) $g : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and has only one positive local extremum point $x = \theta$ (global maximum). Furthermore, g has two equilibria $g(0) = 0$, $g(\kappa) = \kappa$ with $g'(0) > 1$, $g'(\kappa) < 1$ and additionally satisfies $g(x) > x$ for $x \in (0, \kappa)$ and $g(x) < x$ for $x > \kappa$.

Therefore, in view of the terminology used in the traveling waves theory, the diffusive Mackey–Glass type equation (1) is of monostable type [11]. In the particular case when g is monotone on the interval $[0, \kappa]$ there is quite satisfactory description of all wavefront solutions for Eq. (1) given by the following result.

Proposition 1. *(See [15,22].) Suppose that g satisfies* **(UM)** *and is strictly monotone on $[0, \kappa]$, see Fig. 2 (left). Then there is $c_* > 0$ (called the minimal speed of propagation) such that Eq. (1) has a unique (up to translation) wavefront $u(t, \mathbf{x}) = \phi(ct + \mathbf{n} \cdot \mathbf{x})$ for each $c \geqslant c_*$ and every $h \geqslant 0$. In addition, the profile ϕ is a strictly increasing function. If $c < c_*$ then Eq. (1) does not have any wavefront.*

We note that the stability of monotone fronts of (1) was successfully analyzed in [16–18].

In the case when $\theta \in (0, \kappa)$ (so that g is not monotone on $[0, \kappa]$ anymore), much less information on the traveling fronts to Eq. (1) is available. In particular, as far as we know, for a general function g satisfying the hypothesis **(UM)**, none of the four aspects (the existence of the minimal speed c_*, the uniqueness, the monotonicity properties, the wavefront stability) mentioned in Proposition 1 has received a satisfactory characterization. In this paper, we shed some new light on the description of possible geometric shapes of the wavefront profiles ϕ. Due to the biological interpretation of solutions to (1), the geometric properties of leading (invading) parts of wavefront profiles characterize the 'smoothness' of the expansion (invasion) processes. This fact shows the practical importance of our studies. A first picture of the wavefront monotonicity properties was obtained in [24] under the following additional condition:

(FC) The restriction $g : [g^2(\theta), g(\theta)] \to \mathbb{R}_+$ has the positive feedback with respect to the equilibrium κ: $(g(x) - \kappa)(x - \kappa) < 0$, $x \neq \kappa$. Here we use the notation $g^2(\theta)$ for $g(g(\theta))$.

More precisely, the following result holds:

Proposition 2. *(See [24].) Consider the case when* **(UM)** *holds and $g'(\kappa) < 0$. Let $u(t, \mathbf{x}) = \phi(ct + \mathbf{n} \cdot \mathbf{x})$ be a wavefront to Eq. (1). Then there exists $\tau_1 \in \mathbb{R} \cup \{+\infty\}$ such that $\phi'(s) > 0$ on $(-\infty, \tau_1)$. Furthermore, τ_1 is finite if and only if $\phi(\tau_1) > \kappa$. If, in addition, the birth function g satisfies* **(FC)***, then ϕ is eventually either monotone or slowly oscillating around κ. Finally, if τ_0 is the leftmost point where $\phi(\tau_0) = \theta$ then $\tau_1 - \tau_0 \geqslant ch$.*

It should be noted that the existence of oscillating traveling fronts in the delayed reaction–diffusion equations is by now a well-known fact confirmed both numerically and analytically. The subclass of slowly oscillating profiles is defined below:

Definition 3. Set $\mathbb{K} = [-ch, 0] \cup \{1\}$. For any $v \in C(\mathbb{K}) \setminus \{0\}$ we define the number of sign changes by

$$\mathrm{sc}(v) = \sup\bigl\{k \geqslant 1 : \text{there are } t_0 < \cdots < t_k \text{ such that } v(t_{i-1})v(t_i) < 0 \text{ for } i \geqslant 1\bigr\}.$$

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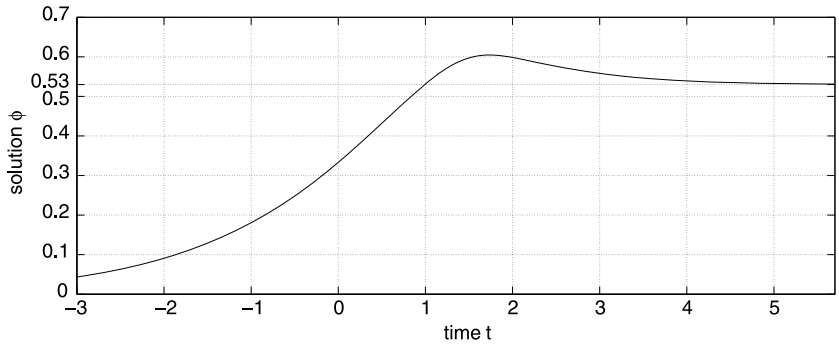

Fig. 1. Profile of a minimal, non-monotone and non-oscillating wavefront to Eq. (1).

We set $\mathrm{sc}(v) = 0$ if $v(s) \geqslant 0$ or $v(s) \leqslant 0$ for $s \in \mathbb{K}$. If $\varphi(t)$, $t \geqslant a - ch$, is a solution of (2), we set $(\bar{\varphi}_t)(s) = \varphi(t+s) - \kappa$ if $s \in [-ch, 0]$, and $(\bar{\varphi}_t)(1) = \varphi'(t)$. We say that $\varphi(t)$ is slowly oscillating about κ if $\varphi(t) - \kappa$ is oscillatory and for each $t \geqslant a$, we have either $\mathrm{sc}(\bar{\varphi}_t) = 1$ or $\mathrm{sc}(\bar{\varphi}_t) = 2$.

The studies carried out in [24] have left unanswered the question about the existence of non-monotone but eventually monotone traveling fronts for the model (1) (in particular, for the well-known diffusive Nicholson's blowflies equation with $g(x) = px\exp(-x)$). The new facts that have appeared after publication of [24] do not give an unconditional support to this conjecture. From one side, numerical simulations of wavefronts for more general non-local equations (e.g. the non-local KPP-Fisher equation [5]) indicate, in certain cases, the presence of non-monotone but eventually monotone traveling fronts. See also [2,3,6,8,13,19,21]. On the other hand, the recent works [7,14] establish analytically that the KPP-Fisher equation with a finite discrete delay can have wavefronts only with profiles which are either monotone or slowly oscillating around κ. It is noteworthy that the above mentioned results of [7,13] were predicted in [20].

In the present work we give a rigorous analytical justification of the existence of the proper eventually monotone wavefronts to the model (1), see Fig. 1. As a result, we answer affirmatively the question raised in [24]. And further, our main result below contains even more information.

Theorem 4. *There exists a piece-wise linear unimodal function g (see Fig. 2) satisfying* **(UM)**, **(FC)** *and positive numbers $h, c_* < c^*$ such that the following hold:*

(a) for each $c \geqslant c_*$ Eq. (1) has a unique wavefront $u(t, \mathbf{x}) = \phi(ct + \mathbf{x}\cdot\mathbf{n})$, $|\mathbf{n}| = 1$, and it does not have any wavefront propagating with the speed $c < c_*$;
(b) for each $c \in [c_*, c^*]$, the profile ϕ is non-monotone but eventually monotone (see Fig. 1, where the minimal front is depicted);
(c) for each $c > c^*$, the wavefront profile ϕ slowly oscillates around κ.

The proof of this theorem combines several ideas from [9,10,24]. It is given in the next section.

2. Proof of Theorem 4

A direct analysis of (2) shows that each local maximum $M_j = \phi(t_j)$ of the front profile $\phi(t)$ should satisfy the inequality

$$M_j = \phi''(t_j) - c\phi'(t_j) + g\big(\phi(t_j - ch)\big) \leqslant g\big(\phi(t_j - ch)\big) \leqslant g(\theta). \tag{3}$$

Therefore it suffices to consider g defined on the interval $[0, g(\theta)]$ only. In the simplest 'unimodal' case, the graph of g consists of two linear segments. This nonlinearity was already analyzed in [24]. Since, in such a case, g satisfies the following sub-tangency condition at κ:

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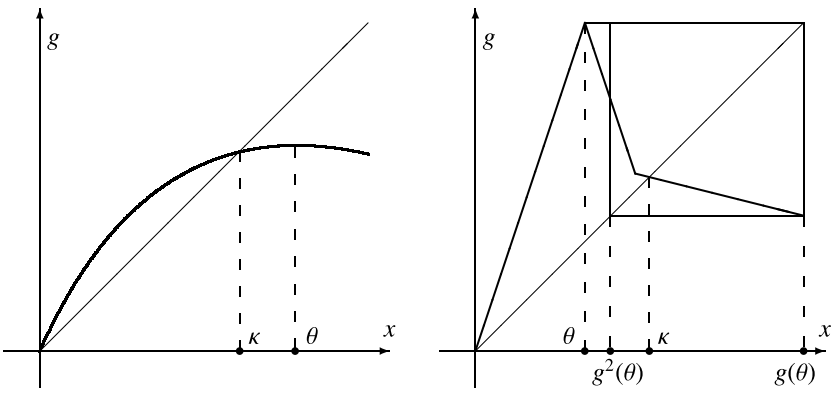

Fig. 2. Graph of the unimodal birth function g: the cases $\theta > \kappa$ (on the left) and $\theta < \kappa$ (on the right).

$$g(x) \leqslant \kappa + g'(\kappa)(x - \kappa), \quad x \in [0, \kappa], \tag{4}$$

each eventually monotone wavefront is in fact a monotone front, see [10] for more detail. Therefore, if we want to construct a piece-wise linear birth function g suitable for Theorem 4, its graph must contain at least three linear segments and does not satisfy the inequality (4), see Fig. 2 (right).

Analytically, $g(x)$ is defined by the formula

$$g(x) := \begin{cases} k_1 x, & 0 \leqslant x \leqslant \theta, \\ k_2 x + q_2, & \theta \leqslant x \leqslant \theta_1, \\ k_3 x + q_3, & \theta_1 < x \leqslant g(\theta). \end{cases} \tag{5}$$

Here real numbers q_j are chosen to assure the continuity of g. In what follows, we will seek the appropriate parameter values k_j, θ_j and (h, c) to obtain the desired shape of the profile. Note that one of the main restrictions on (h, c) was already found in [10], where it was proved that an eventually monotone wavefront in the Mackey–Glass type equation can appear only for (h, c) belonging to the connected closed domain $\mathcal{D}_{\mathfrak{L}}$ defined below:

Definition 5. $(h, c) \in \mathcal{D}_{\mathfrak{L}}$ if and only if each of the equations $\chi_0(z) := z^2 - cz - 1 + g'(0)e^{-zch} = 0$, $\chi_\kappa(z) := z^2 - cz - 1 + g'(\kappa)e^{-zch} = 0$, has exactly two real roots (counting the multiplicity) of the same sign: the positive roots $0 < \mu_2 \leqslant \mu_1$ for the first equation, and the negative roots $\lambda_2 \leqslant \lambda_1 < 0$ for the second one.

The following result (established in [10, Lemma 1.1] and [24, Lemma 21]) partially describes the structure of the set $\mathcal{D}_{\mathfrak{L}}$ and other properties of eigenvalues λ_j:

Lemma 6. *Suppose that $g'(\kappa) < 0$. Then there exists $c^* = c^*(h) \in (0, +\infty]$ such that the characteristic function $\chi_\kappa(z)$ has three real zeros $\lambda_1 \leqslant \lambda_2 < 0 < \lambda_3$ if and only if $c \leqslant c^*$. If c^* is finite and $c = c^*$, then χ_κ has a double zero $\lambda_1 = \lambda_2 < 0$, while χ_κ does not have any negative zero when $c > c^*$. Moreover, if $\lambda_j \in \mathbb{C}$ is a complex zero of χ_κ for $c \in (0, c^*]$ then $\Re\lambda_j < \lambda_2$ and $|\Im\lambda_j| > 2\pi/(ch)$.*

By [23, Theorem 4.5], for each $(h, c) \in \mathcal{D}_{\mathfrak{L}}$, Eq. (2) has at least one semi-wavefront solution (i.e. positive bounded solution $\phi(t)$ such that $\phi(-\infty) = 0$). Consider the finite numbers

$$0 \leqslant m := \liminf_{t \to +\infty} \phi(t) \leqslant \limsup_{t \to +\infty} \phi(t) =: M.$$

As it was established in [23, Lemma 4.3], it holds necessarily that $m > 0$ (in some cases, the latter inequality is also considered in the definition of a semi-wavefront, cf. [2,5]). To make the semi-wavefront ϕ converge

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to κ at $+\infty$ (equivalently, to prove that $m = M = \kappa$), we will impose an additional condition on g, h, c described in the next proposition. This condition is given in terms of g and a new piece-wise linear unimodal function $\sigma : [g^2(\theta), g(\theta)] \to [g^2(\theta), g(\theta)]$ defined by $\sigma(x) = \zeta^{-1}((1-\xi)g(x))$, where

$$\xi = \xi(h,c) = \frac{z_2 - z_1}{z_2 e^{-chz_1} - z_1 e^{-chz_2}} \in \left[e^{-h}, 1\right], \qquad \zeta(x) = x - \xi\psi(x),$$

$\psi : [g^2(\theta), g(\theta)] \to [\theta, g(\theta)]$ is the inverse of g restricted to the interval $[\theta, g(\theta)]$, and $z_1 = z_1(c) < 0 < z_2 = z_2(c)$ are the roots of the equation $z^2 - cz - 1 = 0$. Observe that the relations

$$(1-\xi)g\big(\big[g^2(\theta), g(\theta)\big]\big) \subseteq (1-\xi)\big[g^2(\theta), g(\theta)\big] \subseteq \big[g^2(\theta) - \xi g(\theta), g(\theta) - \xi\theta\big] = \zeta\big(\big[g^2(\theta), g(\theta)\big]\big)$$

imply that the mapping $\sigma : [g^2(\theta), g(\theta)] \to [g^2(\theta), g(\theta)]$ is well-defined.

Proposition 7. *Assume* **(UM)** *and the following global stability condition:*

(GA) *κ is the globally attracting fixed point for at least one of the following two one-dimensional maps: $g, \sigma : [g^2(\theta), g(\theta)] \to [g^2(\theta), g(\theta)]$.*

Then every semi-wavefront solution of (2) converges to κ at infinity: $\phi(+\infty) = \kappa$.

Proof. A demonstration of this result constitutes the main part of the proof of Theorem 5.1 from [23]. For the reader's convenience, here we outline briefly the main ideas of this proof.

First, we observe that $M = \lim_{j\to+\infty} M_j$, $m = \lim_{j\to+\infty} m_j$ for some sequences $m_j = \phi(s_j)$, $M_j = \phi(t_j)$, $s_j, t_j \to +\infty$, of the local extremum values. Taking the limit in (3) as $j \to +\infty$, we find that, for each positive ϵ, it holds that $M \in g([m-\epsilon, M+\epsilon])$. Thus $M \in g([m, M])$. Analogously, we have that $m \in g([m, M])$. In consequence, $[m, M] \subseteq g([m, M])$. But then, due to the global attractivity properties of κ, it holds that $m = M = \kappa$ because of

$$[m, M] \subseteq g\big([m, M]\big) \subseteq g^2\big([m, M]\big) \subseteq \cdots \subseteq g^n\big([m, M]\big) \to \kappa, \quad n \to +\infty.$$

By the same reason, the relations $m, M \in \sigma([m, M])$ will imply $m = M = \kappa$. In order to prove that $M \in \sigma([m, M])$, we first note that (3) implies $\phi(t_j - ch) \leqslant \psi(M_j)$. Next, since $\phi'(t_j) = 0$, the variation of constants formula applied to the equation $\phi''(t) - c\phi'(t) - \phi(t) = -g(\phi(t - ch))$ yields

$$M_j = \phi(t_j) = \xi\left\{\phi(t_j - ch) + \frac{1}{z_2 - z_1}\int_{t_j - ch}^{t_j} \big(e^{z_1(t_j - ch - u)} - e^{z_2(t_j - ch - u)}\big) g\big(\phi(u - ch)\big)\right\}.$$

Therefore, for each fixed $\epsilon > 0$ and all sufficiently large j, we obtain that

$$\begin{aligned} M_j = \phi(t_j) &\leqslant \xi\left\{\psi(M_j) + \frac{1}{z_2 - z_1}\int_{t_j - ch}^{t_j} \big(e^{z_1(t_j - ch - u)} - e^{z_2(t_j - ch - u)}\big) \max_{s\in[m-\epsilon, M+\epsilon]} g(s)\right\} \\ &\leqslant \xi\psi(M_j) + (1-\xi) \max_{s\in[m-\epsilon, M+\epsilon]} g(s). \end{aligned}$$

Taking the limit in the last inequalities as $j \to +\infty$ (for a fixed $\epsilon > 0$) and then considering $\epsilon \to 0+$, we conclude that $\zeta(M) \leqslant (1-\xi)\max_{s\in[m,M]} g(s)$ and therefore $M \in \sigma([m, M])$. The inclusion $m \in \sigma([m, M])$ can be established in a similar way. $\square$

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Remark 8. Note that for the birth function g defined by (5) the hypotheses of Proposition 7 can be easily verified since the continuous graphs of both σ and g are piecewise linear. In order to verify the hypothesis **(GA)** in the case of unimodal C^3-smooth birth functions, the authors of [23] have systematically used the criterion of the negative Schwarz derivative.

The above discussion leads to our first auxiliary result:

Lemma 9. *Suppose that g is defined by (5) (see Fig. 2 (right)), that the hypotheses* **(UM)**, **(FC)** *and* **(GA)** *are satisfied and that $(h, c) \in \mathcal{D}_{\mathfrak{L}}$. Then there exists at least one traveling front $u(t, \mathbf{x}) = \phi(ct + \mathbf{x} \cdot \mathbf{n})$, $|\mathbf{n}| = 1$, to Eq. (1) and its profile ϕ must be eventually monotone.*

Proof. As we have already mentioned, the existence of at least one semi-wavefront ϕ for (1) is assured by [23, Theorem 4.5]. Due to Proposition 7, this semi-wavefront is actually a wavefront. Therefore we only have to prove the eventual monotonicity of ϕ. Suppose, on the contrary, that $\phi(t)$ is oscillating around κ. Since the feedback condition **(FC)** is satisfied, Proposition 2 shows that these decaying oscillations of $\phi(t)$ should be slow. In addition, we claim that the convergence of $\phi(t)$ to κ is not super-exponential. Indeed, by our construction, the difference $y(t) := \phi(t) - \kappa$ satisfies the linear homogeneous equation

$$y''(t) - cy'(t) - y(t) + k_3 y(t - ch) = 0, \tag{6}$$

for all sufficiently large positive t. Therefore, if $y(t)$ is a small (i.e. super-exponentially decaying) solution of (6), it should be identically zero for all large positive t (see Theorem 3.1 in [12, p. 76]). We can conclude that there exists a leftmost T such that $\phi(t) = \kappa$ for all $t \geqslant T$. But then, by using Eq. (2), we easily get a contradiction since $\phi(t) = \kappa$ for all $t \geqslant T - ch$.

Now, since $y(t)$ is not a small solution of (6), it can be approximated by a finite linear combination of the eigenfunctions

$$y(t) = a_1 e^{\lambda_1 t} + a_2 e^{\lambda_2 t} + a_j e^{\Re\lambda_j t} \sin(\Im\lambda_j t + a_4) + O\big(e^{(\Re\lambda_j - \delta)t}\big),$$

where $a_1, a_2, a_j \in R$ and $a_j \neq 0$, and $\delta > 0$ is sufficiently small, e.g. see [4, Theorem 6.7]. From our assumption about the oscillatory behavior of ϕ, we deduce that actually $a_1 = a_2 = 0$. Recalling now that $\Im|\lambda_j| > 2\pi/(ch)$, we obtain that $\phi(t) = \kappa + a_j e^{\Re\lambda_j t} \sin(\Im\lambda_j t + a_4) + O(e^{(\Re\lambda_j - \delta)t})$ is rapidly oscillating about κ, a contradiction. □

Before announcing our next result, we recall that, by Proposition 2, the leading part of the wavefront is monotone between the equilibria. Since, in addition, $\phi_s(t) := \phi(t + s)$ solves (2) for each fixed s, there is no loss of generality in assuming that $\phi(0) = \theta \in (0, \kappa)$ and that $\phi'(t) > 0$ for all $t \leqslant 0$. As a consequence, $\phi(t)$ satisfies the linear homogeneous equation

$$y''(t) - cy'(t) - y(t) + k_1 y(t - ch) = 0, \tag{7}$$

for all $t \leqslant ch$. This fact allows us to find an almost complete representation of ϕ for $t \leqslant ch$ described by the following lemma.

Lemma 10. *In addition to all assumptions of Lemma 9, suppose that $\phi(0) = \theta$ and that $\mu_2 \leqslant \mu_1$ are as in Definition 5. Let the unimodal continuous function g defined by (5) have the shape as shown in Fig. 2 (right). Then, for all $t \leqslant ch$, it holds that*

$$\phi(t) = pe^{\mu_2 t} + (\theta - p)e^{\mu_1 t} \quad \text{if } \mu_2 < \mu_1, \qquad \phi(t) = -qte^{\mu_1 t} + \theta e^{\mu_1 t} \quad \text{if } \mu_2 = \mu_1, \tag{8}$$

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for some p, q satisfying the inequalities

$$\theta < p \leqslant \frac{\mu_1\theta}{\mu_1 - \mu_2 e^{-ch(\mu_1-\mu_2)}}, \qquad 0 < q \leqslant \frac{\mu_1\theta}{1+\mu_1 ch}. \tag{9}$$

Proof. Since $\phi(-\infty) = 0$ and $\phi(t)$ is not a small solution at $-\infty$ by Theorem 3.1 in [12, p. 76], we find that ϕ can be represented by a finite sum

$$\phi(t) = \sum_{\Re\mu_j>0} c_j e^{\mu_j t}, \quad t \leqslant ch, \text{ if } \mu_2 < \mu_1, \qquad \phi(t) = c_0 t e^{\mu_1 t} + \sum_{\Re\mu_j>0} c_j e^{\mu_j t}, \quad t \leqslant ch, \text{ if } \mu_2 = \mu_1,$$

where μ_j are roots of the characteristic equation $z^2 - cz - 1 + g'(0)e^{-zch} = 0$ with the positive real parts. It is a well-known fact that the set of all such roots is finite, cf. [12]. Moreover, the positive roots μ_2 and μ_1 are dominating in the sense that $\Re\mu_j < \mu_2 \leqslant \mu_1$ for each $j > 2$ (see, e.g. [23, Lemma 2.3]). Thus we can conclude that, in fact,

$$\phi(t) = c_2 e^{\mu_2 t} + c_1 e^{\mu_1 t}, \quad t \leqslant ch, \text{ if } \mu_2 < \mu_1, \qquad \phi(t) = c_0 t e^{\mu_1 t} + c_1 e^{\mu_1 t}, \quad t \leqslant ch, \text{ if } \mu_2 = \mu_1.$$

Indeed, otherwise $\phi(t)$ will oscillate at $-\infty$. Taking into account that $\phi(0) = \theta$, we obtain formulas (8).

Next, in order to prove the first inequality for p in (9), we observe that the coefficient $c_1 := \theta - p$ can be calculated explicitly (see, e.g. [9, Lemma 28]) with the help of the bilateral Laplace transform:

$$(\theta - p)e^{\mu_1 t} = -\mathrm{Res}_{z=\mu_1}\left[\frac{e^{zt}}{\chi_\kappa(z)} \int_{-\infty}^{+\infty} e^{-zs} f(s)ds\right],$$

with $f(s) := g'(0)\phi(s - ch) - g(\phi(s - ch)) \geqslant 0$, $s \in \mathbb{R}$, satisfying $f(+\infty) = (g'(0) - 1)\kappa > 0$ and $\chi_\kappa(z) = z^2 - cz - 1 + k_1 e^{-zch}$. As a result, since μ_1 is a simple zero of χ_κ and $\chi'(\mu_1) > 0$, we find that

$$\theta - p = -\frac{1}{\chi'_\kappa(\mu_1)} \int_{-\infty}^{+\infty} e^{-\mu_1 s} f(s)ds < 0.$$

Finally, Proposition 2 guarantees that $\phi'(t) > 0$ for all $t \in [0, ch)$, see also [24, Lemma 10]. In particular, $\phi'(ch) \geqslant 0$ which amounts to the second inequality for p in (9).

Using the obtained restrictions on p, we easily find that, if $\mu_2 < \mu_1$, then

$$\inf_p \max_{t\in[0,ch]} \phi(t,p) = \inf_p \phi(ch,p) = \frac{g(\theta)}{1+\mu_1\mu_2}, \tag{10}$$

where $\phi(t,p) := pe^{\mu_2 t} + (\theta - p)e^{\mu_1 t}$ and the infimum is taken over the admissible interval for p given by (9). See also the proof of Corollary 11 below.

Similarly, if $\mu_1 = \mu_2$, we obtain

$$(-qt + \theta)e^{\mu_1 t} = -\mathrm{Res}_{z=\mu_1}\left[\frac{e^{zt}}{\chi_\kappa(z)} \int_{-\infty}^{+\infty} e^{-zs} f(s)ds\right],$$

and therefore

$$q = \frac{2}{\chi''_\kappa(\mu_1)} \int_{ch}^{+\infty} e^{-\mu_1 s} f(s)ds > 0.$$

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Now, the inequality for q in (9) is equivalent to the above mentioned property $\phi'(ch) \geqslant 0$ satisfied by each wavefront. □

Corollary 11. *Let all assumptions of Lemma 10 be satisfied and $c > c_*$. Then*

$$\phi(ch, c) \geqslant \frac{g(\theta)}{1 + \mu_1(c)\mu_2(c)}. \tag{11}$$

Proof. To prove (11), it suffices to use the expression for ϕ in (8) and the inequality for p in (9):

$$\begin{aligned}\phi(ch, c) &= p\big(e^{\mu_2 ch} - e^{\mu_1 ch}\big) + \theta e^{\mu_1 ch} \\ &\geqslant \frac{\mu_1\theta}{\mu_1 - \mu_2 e^{-ch(\mu_1-\mu_2)}}\big(e^{\mu_2 ch} - e^{\mu_1 ch}\big) + \theta e^{\mu_1 ch} = \frac{g(\theta)}{1 + \mu_1(c)\mu_2(c)}. \qquad \square\end{aligned}$$

Corollary 12. *Assume, in addition to conditions of Lemma 10, that each admissible wavefront to Eq. (1) is unique (up to translation). Then*

$$\phi(c_*h, c_*) \geqslant \frac{g(\theta)}{1 + \mu_1^2(c_*)}, \qquad 0 < q \leqslant \frac{\theta - g(\theta)e^{-\mu_1(c_*)c_*h}/(1 + \mu_1^2(c_*))}{c_*h}. \tag{12}$$

Proof. Let $u(t, \mathbf{x}) = \phi(\mathbf{x}\cdot\mathbf{n} + ct, c)$, $|\mathbf{n}| = 1$, $\phi(0, c) = \theta$, be the wavefront propagating at the velocity $c > c_*$. It is easy to see that each profile $\phi(t, c)$ satisfies the integral equation

$$\phi(t, c) = \frac{1}{z_2 - z_1}\left(\int_{-\infty}^{t} e^{z_1(t-s)} g\big(\phi(s - ch, c)\big)ds + \int_{t}^{+\infty} e^{z_2(t-s)} g\big(\phi(s - ch, c)\big)ds\right), \tag{13}$$

where $z_1 < 0 < z_2$ are roots of the equation $z^2 - cz - 1 = 0$. As a consequence,

$$\phi'(t, c) = \frac{1}{z_2 - z_1}\left(z_1 \int_{-\infty}^{t} e^{z_1(t-s)} g\big(\phi(s - ch, c)\big)ds + z_2 \int_{t}^{+\infty} e^{z_2(t-s)} g\big(\phi(s - ch, c)\big)ds\right).$$

A straightforward estimation shows that $|\phi'(t, c)| \leqslant g(\theta)/\sqrt{c^2 + 4}$ and $|\phi(t, c)| \leqslant g(\theta)$. Choose now a strictly decreasing sequence $c_j \to c_*$. By the Arzelà–Ascoli theorem, the sequence $\phi(t, c_j)$ has a subsequence $\phi(t, c_{j_k})$ which converges, uniformly on compact subsets of $\mathbb{R}$, to the continuous non-negative bounded function $\phi_0(t)$. It is clear that ϕ_0 is non-decreasing on $(-\infty, ch]$ and that $\phi_0(0) = \theta$. By the Lebesgue dominated convergence theorem, ϕ_0 satisfies Eq. (13) with $c = c_*$. Therefore ϕ_0 is a non-negative profile of a traveling wave propagating with the velocity c_*. Since the limit value $\phi_0(-\infty) < \theta$ and $\phi_0(t)$ must satisfy Eq. (2), we get $\phi_0(-\infty) = 0$. This means that actually $\phi_0(t)$ is a wavefront. But then, due to the uniqueness assumption, we have that $\phi_0(t) = \phi(t, c_*)$ and that

$$(-qc_*h + \theta)e^{\mu_1 c_* h} = \phi(c_*h, c_*) = \phi_0(c_*h, c_*) = \lim_{j\to+\infty} \phi(c_j h, c_j) \geqslant \lim_{j\to+\infty} \frac{g(\theta)}{1 + \mu_1(c_j)\mu_2(c_j)} = \frac{g(\theta)}{1 + \mu_1^2(c_*)}.$$

Inequalities (12) follow easily from these relations. □

The above considerations yield the following conclusion:

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Theorem 13. *Let the unimodal continuous function g be defined by (5), where $k_2 < k_3 < 0 < 1 < k_1$. In addition, suppose that the hypotheses* **(FC)** *and* **(GA)** *are satisfied, $(h, c) \in \mathcal{D}_{\mathfrak{L}}$, and*

$$\gamma(c) := \frac{g(\theta)}{1 + \mu_1(c)\mu_2(c)} > \kappa. \tag{14}$$

Then Eq. (1) has a non-empty set of traveling fronts propagating with the speed c (which can be the minimal one). Next, each such wavefront is non-monotone on $\mathbb{R}$ but eventually monotone. Furthermore, if either $|k_2| \leqslant k_1$ or the characteristic equation $z^2 - cz - 1 + |k_2|e^{-zch} = 0$ has two real positive roots (counting multiplicity), then there exists a unique (up to translation) wavefront propagating with the velocity c.

Proof. By Lemma 9, there exists at least one traveling front $u(t, \mathbf{x}) = \phi(\mathbf{x} \cdot \mathbf{n} + ct)$, $|\mathbf{n}| = 1$, to Eq. (1) and its profile ϕ must be eventually monotone. On the other hand, Corollaries 11, 12 and inequality (14) assure that $\phi(ch) > \kappa$ and therefore the profile ϕ is non-monotone. Finally, since $|g(s_1) - g(s_2)| \leqslant \max\{k_1, |k_2|\}|s_1 - s_2|$, $s_i \geqslant 0$, the uniqueness (up to a translation) of the wavefront propagating with the given velocity c follows from [1, Theorems 7, 8]. □

Proof of Theorem 4. Set $k_1 = -k_2 = 3$, $k_3 = -0.25$, $\theta = 1/3$, $h = 2$, $\kappa = 0.53$. Then the minimal speed $c_* = 0.712\ldots$ and the critical speed $c^* = 0.751\ldots$ can be found from the characteristic equations $z^2 - cz - 1 + 3e^{-2cz} = 0$, $z^2 - cz - 1 - 0.25e^{-2cz} = 0$. Recall that, by definition, $\{2\} \times [c_*, c^*] = \mathcal{D}_{\mathfrak{L}} \cap \{2\} \times \mathbb{R}_+$. We also have that $\mu_1(c_*) = \mu_2(c_*) = 0.926\ldots$. It is a well-known fact (cf. [23, Theorem 1.1]) that Eq. (1) does not have any semi-wavefront propagating at the velocity $c < c_*$. A straightforward (but a little bit tedious) evaluation of $\gamma(c)$ shows that inequality (14) holds for each $c \in [c_*, c^*]$. For completeness, we present the proof of this fact:

Lemma 14. *Consider the above defined g and let $h = 2$. Then*

$$\gamma(c) > \gamma_1(c) := \frac{1 + 1.53c^2}{2.55 + 1.53c^2} > \kappa = 0.53, \quad c \in \left[c_*, c^*\right] = [0.712\ldots, 0.751\ldots].$$

Proof. Set $\rho_j(c) = c\mu_j(c)$, then $0 < \rho_1(c) \leqslant \rho_2(c)$ are the only two real roots of the equation $1 + z - c^{-2}z^2 = 3e^{-2z}$. A direct computation shows that $\rho_1(c_*) = \rho_2(c_*) = 0.656\ldots$ and $\rho_1(c^*) = 0.537\ldots$, $\rho_2(c^*) = 0.867\ldots$. Now, for each fixed $z \in \mathbb{R}$ the function $P(c) := 1 + z - c^{-2}z^2$ is strictly increasing on $(0, +\infty)$, and therefore $\rho_1(c)$ is strictly decreasing and $\rho_2(c)$ is strictly increasing on $[c_*, c^*]$. In particular, $\rho_1(c), \rho_2(c) \in [\rho_1(c^*), \rho_2(c^*)] \subset [0.537, 0.868]$ for all $c \in [c_*, c^*]$. Consider next the quadratic polynomial

$$Q(z) = 3 \cdot e^{-2\rho_1(c_*)}\big(1 - 2.04\big(z - \rho_1(c_*)\big) + 1.9\big(z - \rho_1(c_*)\big)^2\big),$$

which is a small deformation of the second order Taylor approximation of the function $y = 3e^{-2z}$ at $z = \rho_1(c_*)$. It can be easily verified that $Q(z) > 3e^{-2z}$ for all $z \in [0.521, \rho_1(c_*))$ and $Q(z) < 3e^{-2z}$ for all $z \in (\rho_1(c_*), 0.885]$. As a consequence, for each $c \in [c_*, c^*]$, the equation $1 + z - c^{-2}z^2 = Q(z)$ has exactly two real roots $\tilde{z}_1(c) > \rho_1(c)$, $\tilde{z}_2(c) > \rho_2(c)$. Therefore

$$\mu_1(c)\mu_2(c) = c^{-2}\rho_1(c)\rho_2(c) < c^{-2}\tilde{z}_1(c)\tilde{z}_2(c) = \frac{Q(0) - 1}{1 + 5.7c^2e^{-2\rho_1(c_*)}} = \frac{1.549\ldots}{1 + 5.7c^2e^{-2\rho_1(c_*)}}$$

and $\gamma(c) > 1/(1 + c^{-2}\tilde{z}_1(c)\tilde{z}_2(c)) > \gamma_1(c) := (1 + 1.53c^2)/(2.55 + 1.53c^2) > \kappa$, $c \in [c_*, c^*] = [0.712\ldots, 0.751\ldots]$. □

Next, the graph of g shown in Fig. 2 (right) was drawn by using the above mentioned data; it is clear from its shape that κ is the global attractor of g. Indeed, the second iteration $g^2 : [g^2(\theta), g(\theta)] \to [g^2(\theta), g(\theta)]$

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is a piece-wise linear map, whose slopes cannot exceed $|k_2k_3| = 0.75$ in the absolute value. Thus, all the assumptions of Theorem 13 are satisfied for all $c \in [c_*, c^*]$, which proves statements (a), (b) of Theorem 4. Finally, part (c) follows from [24, Theorem 3]. □

Remark 15. It is comforting to observe that the conclusions of Theorem 4 agree with the statement of [9, Remark 2] which says that, in the case of existence of non-monotone and non-oscillating wavefronts, the equation

$$z^2 - c_*z - 1 - g'_\kappa e^{-2zc_*} = 0, \quad \text{where } g'_\kappa := \inf_{x\in(0,\kappa)} \big(g(x) - g(\kappa)\big)/(x-\kappa) = -2.69\ldots,$$

cannot have negative real roots.

To illustrate our theoretical results, in Fig. 1 we are presenting a graph of the minimal wavefront ($q = 0.12304\ldots$). In its derivation we have used the estimate $q \leqslant 0.1314\ldots$ which follows from (9), (12). The graph exhibits only one local extremum. We believe that it is possible to find g defined by (5) such that the associated wavefront will have two critical points. It seems that the number of the critical points cannot exceed 2 (at least for piece-wise linear g defined by (5)).

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