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Periodic solutions of a singular differential delay equation with the Farey-type nonlinearity

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Abstract

We address the problem of existence of periodic solutions for the differential delay equation

$$\varepsilon\dot{x}(t) + x(t) = f(x(t-1)), \quad 0 < \varepsilon \ll 1,$$

with the Farey nonlinearity $f(x)$ of the form

$$f(x) = \begin{cases} mx + A & \text{if } x \leqslant 0, \\ mx - B & \text{if } x > 0, \end{cases}$$

where $|m| < 1$, $A > 0$, $B > 0$. We show that when the map $x \mapsto f(x)$ has an attracting 2-cycle then the delay differential equation has a periodic solution, which is close to the square wave corresponding to the limit (as $\varepsilon \to 0^+$) difference equation $x(t) = f(x(t-1))$.
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1. Introduction

Consider the following scalar differential delay equation:

$$\varepsilon\dot{x}(t) + x(t) = f(x(t-1)), \tag{1}$$

with the singular parameter $0 < \varepsilon$ and the Farey-type nonlinearity $f(x)$. The Farey nonlinearity f is defined by

$$f(x) = \begin{cases} mx + A & \text{if } x \leqslant 0, \\ mx - B & \text{if } x > 0, \end{cases} \tag{2}$$

where $0 < m < 1$, $A > 0$ and $B > 0$. Eq. (1) and its modifications appear in a variety of important applications as the exact reduction of nonlinear boundary value problems for the one-dimensional wave equations (see, e.g., [7,9]). In the particular case of the Farey-type nonlinearity it is a mathematical model of a Chua circuit with delay [8].

The limiting case of Eq. (1) as $\varepsilon \to 0+$ results in the pure difference equation

$$x(t) = f(x(t-1)) \tag{3}$$

with the continuous argument $t \in \mathbb{R}^+ := \{t : t \geqslant 0\}$.

Asymptotic behavior of solutions of Eq. (3) is determined by the one-dimensional map

$$f : x \mapsto f(x), \quad x \in \mathbb{R}. \tag{4}$$

The dynamics of the map (4) with f given by (2) is simple: it has only one cycle that is globally attracting [2,3,8]. There is a continuous dependence on the parameter $\varepsilon > 0$ for Eq. (1) as $\varepsilon \to 0+$, which results in certain closeness properties between solutions of Eqs. (1) and (3) on finite time intervals [2,3,5]. In view of the closeness, it is natural to expect that Eq. (1) has a periodic solution that is close, for all sufficiently small $\varepsilon > 0$, to the attracting cycle of the map f.

When f is continuous, satisfies a negative feedback condition, and has a stable 2-cycle, the existence of a stable periodic solution of (1) whose limiting profile as $\varepsilon \to 0$ is a square wave of period two was investigated by several authors. We refer the reader to the work by Mallet-Paret and Nussbaum [6] and Chow et al. [1]. For some open problems and more references, see the report of Hale [4].

However, for nonlinearity (2) the above results do not apply. In the present paper, we prove the existence of periodic solutions close to the attracting 2-cycle of the map f following a different approach. Moreover, we think that our techniques will permit us to solve the problem when f has an attracting cycle of period greater than two. We will address this task in a forthcoming paper.

2. Preliminaries

In this section we introduce basic definitions and preliminary results necessary for the proof of our main results in subsequent sections.

Under the solution of Eq. (1) we mean a continuous function defined on $[-1, \infty)$, which is continuously differentiable and satisfies the equation for all $t > 0$ except possibly at a countable number of points.

Solutions of Eq. (1) are found by direct integration. Let an initial function $\varphi(t) \in C([-1, 0], \mathbb{R}) := C$ be, such that its graph has a finite number of points of intersection with the discontinuity set of the

nonlinearity $f(x)$, i.e. the set $\{t \in [-1, 0] : \varphi(t) = 0\}$ is finite. Then $f(\varphi(t-1))$, $t \in [0, 1]$, is piece-wise continuous with a finite number of discontinuity points. Therefore, the piece-wise smooth solution of Eq. (1), denoted by $x_\varphi^\varepsilon(t)$ in this paper, exists on the interval $[0, 1]$. If it satisfies the same assumptions on the interval $[0, 1]$ as φ does on the interval $[-1, 0]$, then the solution can be derived by integration on the next unit time interval $[1, 2]$, and so on. We do not specifically address here the question of existence of solutions of Eq. (1) for all $t \geqslant 0$. However, our considerations that follow, the initial functions, and other assumptions are such that the corresponding solutions exist on the entire positive semiaxis.

Given $\varphi \in C$, the corresponding solution $x = x_\varphi^\varepsilon(t)$ can be viewed at a particular time t as a point in $\mathbb{R}$ or as an element of the phase space C given by $x_t := x_\varphi^\varepsilon(t+s)$, $s \in [-1, 0]$.

Solutions of Eq. (1) possess some of the same basic properties as solutions of the corresponding equation with a continuous $f(x)$. One of them is the *invariance property*.

Assume that map (4) has an invariant interval I, i.e. $f(x) \in I \ \forall x \in I$. Consider the subset C_I of the phase space C for Eq. (1) defined by

$$C_I := \{\varphi \in C : \varphi(t) \in I \ \forall t \in [-1, 0]\}.$$

Then the set C_I is invariant under the semiflow defined by Eq. (1), i.e.,

Proposition 1 (*Invariance*). *For every $\varphi \in C_I$ and arbitrary $\varepsilon > 0$ the corresponding solution $x_\varphi^\varepsilon(t)$ of Eq.* (1) *satisfies $x_\varphi^\varepsilon(t) \in I \ \forall t \geqslant 0$.*

The proof of this simple fact can be found in [5, Theorem 2.1].

In addition to the invariance property, the *global attractivity* property holds for Eq. (1) with the Farey nonlinearity f. A general form of this property is given in [5]; we will state here its simpler version appropriate for the Farey map (2).

Assume that none of the endpoints of a closed invariant interval I is a fixed point, and that I is a global attractor: $f(I) \subseteq I$ and $f^n(x) \in I$ for all $x \in \mathbb{R}$ and some finite $n = n(x) \in \mathbb{N}$. Then the following property holds for solutions of Eq. (1).

Proposition 2 (*Global attractivity*). *For every $\varphi \in C$ and arbitrary $\varepsilon > 0$ there exists a finite time $T = T(\varphi, \varepsilon) \geqslant 0$ such that $x_\varphi^\varepsilon(t) \in I \ \forall t \geqslant T$.*

Proof of this statement can be found in [5, Lemma 2.1] for the case of continuous $f(x)$, which easily extends to the case of Farey nonlinearity (2) with a few minor changes.

Note that the Farey map (2) has the invariant interval $I_0 = [-B, A]$, which is also a global attractor. Therefore, in view of Propositions 1 and 2, the subset C_{I_0} of the phase space C is invariant and a global attractor under the semiflow defined by Eq. (1). In the sequel we will restrict our considerations to the set C_{I_0} only.

For every initial function $\psi(t) \in C([-1, 0), \mathbb{R})$ the corresponding solution $x = x_\psi(t)$ of Eq. (3) exists for all $t \geqslant 0$ and is found by iterations. The invariance property obviously holds for the solutions of Eq. (3): if $\psi(t) \in C([-1, 0), \mathbb{R})$ is such that $\psi(t) \in I \ \forall t \in [-1, 0)$, then the corresponding solution $x_\psi(t)$ satisfies $x_\psi(t) \in I \ \forall t \geqslant 0$.

Note that for a continuous $f(x)$ the solution $x_\psi(t)$ will be continuous for all $t \geqslant -1$ if the consistency condition, $\lim_{t \to 0+} \psi(t) = f(\psi(-1))$, is satisfied (see [5,9] for more details). When this consistency

condition is not satisfied, the solution $x_\psi(t)$ is discontinuous at the integer points $t = i$, $i \in \mathbb{N}$, with discontinuities of first kind: $\lim_{t\to i-0} x_\psi(t) \neq x_\psi(i+0)$.

The situation is somewhat different for the difference equation (3) with the Farey nonlinearity (2): if the initial function $\psi(t) \in C([-1,0),\mathbb{R})$ intersects with $x = 0$ at some point $s \in (-1,0)$, the corresponding solution $x_\psi(t)$ becomes discontinuous at $t = s+1$. Therefore, in order for the solution $x_\psi(t)$ not to have discontinuities at points other than the integer points $t = i$, $i \in \mathbb{N}$, one has to additionally assume that the set of values of the initial function ψ does not have points in common with the set $\mathscr{D} := \{x \in \mathbb{R} : f^n(x) = 0 \text{ for some } n \in \mathbb{N}\}$ of all preimages of the discontinuity point $x = 0$ of the nonlinearity $f(x)$. In particular, this is always true if $\psi(t) \in C([-1,0), I_0)$ and $\psi(t) \neq 0$ for all $t \in [-1,0)$.

Our next step is to state a closeness result as $\varepsilon \to 0+$ between the solutions of Eqs. (1) and (3) on finite time intervals.

Let $\varphi \in C_{I_0}$ be such that the corresponding solution $x^\varepsilon_\varphi(t)$ of Eq. (1) exists for all $t \geqslant 0$. Consider initial functions $\psi(t) \in C([-1,0), I_0)$ such that $\psi(t) \neq 0$ for all $t \in [-1,0)$. The solution $x^\varepsilon_\varphi(t)$, $\varphi \in C$ of Eq. (1) is continuous for all $t \geqslant -1$ while the solution $x_\psi(t)$ of Eq. (3) is typically discontinuous at every integer point $t = i$, $i \in \mathbb{N}$. We would like to compare the solutions $x^\varepsilon_\varphi(t)$ and $x_\psi(t)$ on finite time intervals $[0,T]$ for arbitrary but fixed $T > 0$ in the uniform metric. Obviously, such a comparison may be possible everywhere except at the integer points $t = i$, $i \in \mathbb{N}$.

Given $T > 0$ and $\sigma > 0$ consider the set

$$J^\sigma_T := [0,T] \bigg\backslash \bigcup_{i=0}^{[T]} U_\sigma(i), \quad \text{where } U_\sigma(i) = (i-\sigma, i+\sigma).$$

Let $\|\phi\|_S = \sup\{|\phi(t)|,\ t \in S\}$, where $\phi(t)$ is a real-valued function defined on the set $S \subseteq \mathbb{R}$.

Now we are in a position to state a continuous dependence on ε result between the solutions of Eqs. (1) and (3).

Theorem 3. *Let the initial functions $\varphi \in C_{I_0}$ and $\psi(t) \in C([-1,0), I_0)$ be as defined above. For every $T > 0$, $\sigma > 0$ and arbitrary $\eta > 0$ there exist $\varepsilon' > 0$ and $\delta > 0$ such that $\|x^\varepsilon_\varphi - x_\psi\|_{J^\sigma_T} < \eta$ for all $0 < \varepsilon \leqslant \varepsilon'$ provided $\|\varphi - \psi\|_{[-1,0)} < \delta$.*

The proof of this theorem can be derived as a corollary from more general closeness results in [5, p. 187].

3. Existence and asymptotics of periodic solutions

In this section we deal with the case when the Farey map (2) has a globally attracting cycle of period 2. We first establish a necessary and sufficient condition for the existence of such a 2-cycle. Then, we prove the existence of a slowly oscillating periodic solution of Eq. (1) with period $T = 2 + O(\varepsilon)$ for all sufficiently small $\varepsilon > 0$.

Lemma 4. *The one-dimensional map* (4) *with Farey nonlinearity* (2) *has a cycle of period* 2 *if and only if $mA < B$ and $mB < A$.*

Proof. (a) *Necessity*: Assume that points $x_1 < 0 < x_2$ form a cycle of period 2 of the map f. Then $f^2(x_2) = (A - mB)/(1 - m^2) = x_2$. Since $x_2 > 0$ we conclude that $A - mB > 0$, i.e. $mB < A$. Analogously one shows that $mA < B$.

(b) *Sufficiency*: Assume that $mA < B$ and $mB < A$. Let $x_1 = (mA - B)/(1 - m^2) < 0$ and $x_2 = (A - mB)/(1 - m^2) > 0$. It is immediate to check that $f(x_1) = x_2$ and $f(x_2) = x_1$, so therefore (4) has a cycle of period 2. □

Next we state the main result of this subsection regarding the existence and asymptotic shape as $\varepsilon \to 0+$ of periodic solutions of Eq. (1) corresponding to the globally attracting cycle of period 2 of the map f.

Theorem 5. *Assume that the map f has a globally attracting cycle of period 2 formed by the points $x_1 < 0 < x_2$.*

There exists an $\varepsilon_0 > 0$ such that for every $0 < \varepsilon \leqslant \varepsilon_0$ Eq. (1) has a slowly oscillating periodic solution $p(t)$ with period $\omega = 2 + O(\varepsilon)$ and the following properties:

(i) *$p(t)$ has zeros at 0, z_1, and z_2 such that $p(t) > 0$ in $(0, z_1)$ and $p(t) < 0$ in (z_1, z_2), where $z_1 > 1$, $z_2 - z_1 > 1$, and $\omega = z_2$;*

(ii) *there exist positive numbers l_1, l_2, L_1, L_2 independent of ε such that $l_1\varepsilon \leqslant z_1 - 1 \leqslant L_1\varepsilon$, $l_2\varepsilon \leqslant z_2 - z_1 - 1 \leqslant L_2\varepsilon$;*

(iii) *for every $\mu > 0$ there exist $\varepsilon' > 0$ and $\delta > 0$ such that*

$$|p(t) - x_2| \leqslant \mu \quad \forall t \in [\delta, z_1 - \delta] \quad \textit{and} \quad |p(t) - x_1| \leqslant \mu \quad \forall t \in [z_1 + \delta, z_2 - \delta],$$

for all $0 < \varepsilon \leqslant \varepsilon'$; moreover, $\delta \to 0$ as $\varepsilon \to 0+$.

The proof of this theorem will be done in several steps by using the two lemmas below and the continuous dependence result from the previous section, Theorem 3.

For $h > 0$, let us introduce the following subsets of initial functions:

$$C^+ := \{\varphi \in C_{I_0} : \varphi(s) > 0 \ \forall s \in [-1, 0]\},$$

$$C^- := \{\varphi \in C_{I_0} : \varphi(s) < 0 \ \forall s \in [-1, 0]\},$$

$$C_h^+ := \{\varphi \in C_{I_0} : \varphi(s) \geqslant h \ \forall s \in [-1, 0]\},$$

$$C_h^- := \{\varphi \in C_{I_0} : \varphi(s) \leqslant -h \ \forall s \in [-1, 0]\}.$$

Lemma 6. *There exist $\varepsilon_0^- > 0$ and $h_0^- > 0$ such that for all $0 < \varepsilon \leqslant \varepsilon_0^-$ and $0 < h \leqslant h_0^-$ the following holds*: *For every initial function $\varphi \in C_h^+$ there exists a time $t_- = t_-(\varepsilon, h) > 0$ such that $x_\varphi^\varepsilon(t) \leqslant -h \ \forall t \in [t_- - 1, t_-]$.*

Proof. Let $h \in (0, A)$ and $\varphi \in C_h^+$ be arbitrary but fixed.

Observe first that given an initial point (t_0, x_0) and an initial function $\phi \in C^+$, the corresponding solution $x(t) = x_\phi^\varepsilon(t)$ of Eq. (1) can be estimated from above and below as follows:

$$x_-^l(t) \leqslant x(t) \leqslant x_-^u(t), \quad t \in [t_0, t_0 + 1], \tag{5}$$

where

$$x_-^l(t) = -B + (x_0 + B)\exp\{-(t - t_0)/\varepsilon\}, \quad x_-^u(t) = -b + (x_0 + b)\exp\{-(t - t_0)/\varepsilon\}.$$

This is a simple implication of the two-sided estimate for the nonlinearity f

$$-B \leqslant f(x) \leqslant -b = mA - B < 0, \quad x \in (0, A]$$

and a comparison argument for the solution of the initial value problem

$$\varepsilon\dot{x}(t) + x(t) = f(x(t-1)), \quad x(t_0) = x_0, \ x(t) = \phi(t), \ \ t < t_0\,,$$

for the values of $t \in [t_0, t_0 + 1]$.

Note next that, given an initial function $\varphi \in C^+$, there exists an $\varepsilon_1 > 0$ such that for all $0 < \varepsilon \leqslant \varepsilon_1$ there exists a first time $z_1 = z_1(\varepsilon, \varphi)$ such that $x(z_1) = 0$, $x(t)$ is decreasing in $[0, z_1]$, and z_1 is a simple zero of the solution $x = x_\varphi^\varepsilon(t)$. Indeed, the strictly decreasing nature of $x(t)$ in $[0, z_1]$ follows from Eq. (1) and the fact that $f(x) < 0$ for $x \in (0, A)$:

$$\varepsilon\dot{x}(t) = -x(t) + f(x(t-1)) < 0. \tag{6}$$

The existence of $\varepsilon_1 > 0$ and of a uniform bound on z_1 follows from the upper estimate in (5). Indeed, in order to have $z_1 \leqslant 1$ $\forall\varphi \in C_{I_0}$ one can choose $t_0 = 0$, $x_0 = A$, and assume $x_-^u(1) = 0$ to be satisfied. This gives $\varepsilon_1 = 1/\ln[(A + b)/b]$.

The fact that z_1 is a simple zero of $x_\varphi^\varepsilon(t)$ follows from (6) since $x(z_1 - 1) > 0$.

The above argument about the existence of z_1 also implies that for arbitrary $\varphi \in C_{I_0}$ and every $h \in (0, A)$ there exists a time moment t_1 such that $0 \leqslant t_1 < z_1$, $x_\varphi^\varepsilon(t_1) = h$, $x_\varphi^\varepsilon(t) > h$ in $[0, t_1)$ for all $0 < \varepsilon \leqslant \varepsilon_1$.

Next we will show that there exists $\varepsilon_2 > 0$ such that for every $\varphi \in C_{I_0}$, all $0 < \varepsilon \leqslant \varepsilon_2$, and any $h \in (0, \min\{b, A\})$ the solution $x_\varphi^\varepsilon(t)$ has a finite time moment $t_2 > z_1$ such that $t_2 < 1$, $x_\varphi^\varepsilon(t)$ is decreasing in $[t_1, t_2]$, $x_\varphi^\varepsilon(t_2) = -h$, and $x_\varphi^\varepsilon(t) < -h$ $\forall t \in (t_2, 1]$. Indeed, this follows from the inequality $x_\varphi^\varepsilon(t) \leqslant x_-^u(t)$ $\forall t \in [0, 1]$ and the assumption that $x_-^u(1) < -h$. The latter gives $\varepsilon_2 < 1/\ln[(b+h)/(b-h)]$. Moreover, since $x_\varphi^\varepsilon(t) \leqslant x_-^u(t)$ $\forall t \in [z_1, z_1 + 1]$ one finds that

$$x_\varphi^\varepsilon(z_1 + 1) := x_1 \leqslant -b + b\,\exp\{-1/\varepsilon\} := x_1^u < -h.$$

The decreasing nature of $x_\varphi^\varepsilon(t)$ in $[t_1, t_2]$ follows from (6).

Our next step is to derive an upper estimate for the transition time of the solution $x = x_\varphi^\varepsilon(t)$ between the levels $x = 0$ and $-h$, i.e. for $t_2 - z_1$. Obviously, the largest transition time is achieved for the upper bound $x_-^u(t)$, whose estimation results in $t_2 - z_1 \leqslant \varepsilon\,\ln[b/(b - h)]$.

In the next step, we will show that there exists $h_1 > 0$ and $\varepsilon_3 > 0$ such that for all $0 < h \leqslant h_1$ and $0 < \varepsilon \leqslant \varepsilon_3$ the solution $x_\varphi^\varepsilon(t)$ satisfies

$$x_\varphi^\varepsilon(t) \leqslant -h, \quad \forall t \in [z_1 + 1, t_2 + 1]. \tag{7}$$

To prove this we first notice that, similar to the fastest and slowest rates of decay of solutions of Eq. (1) with the initial functions from C^+ described by the inequalities (5), there is also a two-sided estimate for the rates of growth when the initial functions are in C^-. Namely, given an initial point (t_0, x_0) and an

initial function $\psi \in C^-$, the corresponding solution $x(t) = x_\psi^\varepsilon(t)$ of Eq. (1) can be estimated from above and below as follows:

$$x_+^l(t) \leqslant x(t) \leqslant x_+^u(t), \quad t \in [t_0, t_0 + 1], \tag{8}$$

where

$$x_+^l(t) = a + (x_0 - a)\exp\{-(t - t_0)/\varepsilon\}, \quad x_+^u(t) = A + (x_0 - A)\exp\{-(t - t_0)/\varepsilon\},$$

with $a = A - mB > 0$. Therefore, on the interval $[z_1 + 1, t_2 + 1]$ one has the estimate $x_\varphi^\varepsilon(t) \leqslant x_+^u(t)$. Inequality (7) will be satisfied if one assumes that $x_+^u(t_2 + 1) \leqslant -h$. The last inequality is equivalent to

$$A + (x_1^u - A)\frac{b - h}{b} \leqslant -h \quad \text{where } x_1^u = -b + b\exp\{-1/\varepsilon\}.$$

The latter can be reduced to

$$H(\varepsilon, h) := Ab + (b\ \exp\{-1/\varepsilon\} - b - A)(b - h) + bh \leqslant 0.$$

Since $H(\varepsilon, h)$ is continuous and $\lim_{\varepsilon \to 0+, h \to 0+} H(\varepsilon, h) = -b^2 < 0$, the existence of h_1 and ε_3 follows.

To complete the proof of Lemma 6 one sets $t_- = t_2 + 1$, $\varepsilon_0^- = \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$, and $h_0^- = \min\{A, b, h_1\}$. □

Lemma 7. *There exist $\varepsilon_0^+ > 0$ and $h_0^+ > 0$ such that for all $0 < \varepsilon \leqslant \varepsilon_0^+$ and $0 < h \leqslant h_0^+$ the following holds*:

For every initial function $\psi \in C_h^-$ there exists a time $t_+ = t_+(\varepsilon, h) > 0$ such that $x_\psi^\varepsilon(t) \geqslant h$ $\forall t \in [t_+ - 1, t_+]$.

Proof. The proof of this lemma is very similar to the proof of Lemma 6. We leave the details to the reader. □

Next, with $h = \min\{h_0^-, h_0^+\}$, we will construct a map $\mathscr{F}$ of the set C_h^+ into itself whose fixed points give rise to periodic solutions of Eq. (1).

By using Lemma 6 we define the following map $\mathscr{F}_1$ from C_h^+ into C_h^-:

$$(\mathscr{F}_1\varphi)(t) = x_\varphi^\varepsilon(t_- + t), \quad t \in [-1, 0].$$

Similarly, Lemma 7 allows us to define the map $\mathscr{F}_2$ from C_h^- into C_h^+ by

$$(\mathscr{F}_2\psi)(t) = x_\psi^\varepsilon(t_+ + t), \quad t \in [-1, 0].$$

Therefore, the composite map $\mathscr{F} = \mathscr{F}_2 \circ \mathscr{F}_1$ is well defined and maps C_h^+ into itself. By the Schauder fixed point theorem, it has a fixed point $\varphi_0 \in C_h^+$. With the initial function φ_0, the corresponding solution $x_{\varphi_0}^\varepsilon(t)$ is obviously periodic.

The estimates on the magnitude of z_1 as claimed in part (ii) of the theorem are derived from inequalities (5). The smallest value of z_1 is obtained if one uses $x_-^l(t)$, $t_0 = 0$, $x_0 = h$, and the largest one is derived from $x_-^u(t)$, $t_0 = 0$, $x_0 = A$. An easy calculation shows that

$$l_1 = \ln\frac{B + h}{B}, \quad L_1 = \ln\frac{b + A}{b}.$$

The values of

$$l_2 = \ln \frac{A+h}{A}, \quad L_2 = \ln \frac{a+B}{a}$$

are obtained in a similar way. This proves part (ii) of the theorem. Moreover,

$$|\omega - 2| = |z_2 - 2| = |z_2 - z_1 - 1 + z_1 - 1| \leqslant (L_1 + L_2)\varepsilon,$$

and therefore $\omega = 2 + O(\varepsilon)$.

The asymptotic shape of the periodic solutions, as described in part (iii) of Theorem 5, follows from the continuous dependence result, Theorem 3.

Indeed, the periodic solutions result from initial functions $\varphi_0 \in C_h^+$ satisfying $A \geqslant \varphi_0(t) \geqslant h$ $\forall t \in [-1, 0]$. As the 2-cycle $\{x_1, x_2\}$ is globally attracting, for arbitrary $\mu > 0$ there exists a positive integer $N = N(\mu)$ such that $f^{2N}(x) \in (x_2 - \mu/2, x_2 + \mu/2)$ for all $x \in [h, A]$. In Theorem 3, take $T = 2N$ and $\eta = \mu/2$. Consider also the corresponding solution $x_{\varphi_0}(t)$ of the difference Eq. (3) with the same initial function φ_0. Then, for arbitrarily small $\sigma > 0$, on the interval $[2N - 1 + \sigma, 2N - \sigma]$, one has

$$\|x^{\varepsilon}_{\varphi_0} - x_2\| \leqslant \|x^{\varepsilon}_{\varphi_0} - x_{\varphi_0}\| + \|x_{\varphi_0} - x_2\| = \|x^{\varepsilon}_{\varphi_0} - x_{\varphi_0}\| + \|f^{2N}(\varphi_0) - x_2\| \leqslant \mu,$$

which proves the first inequality in (iii) with $\delta = \sigma + O(\varepsilon)$. The second one is proved exactly the same way, by using the initial function $\mathscr{F}_1(\varphi_0)$, where $\mathscr{F}_1$ is the earlier defined map. This completes the proof of Theorem 5.

Remarks. 1. Theorem 5 holds true also for the case $-1 < m \leqslant 0$ with some minor changes in the proof. In this case the map f has a globally attracting cycle $\{x_1, x_2\}$ of period 2 with the interval $I_0 := [x_1, x_2]$ being invariant and globally attracting. Other major steps of the proof remain the same, and we leave details to the reader.

2. Note that neither uniqueness nor stability of the periodic solutions is claimed in Theorem 5. In fact, in the case of piece-wise constant f with a globally attracting cycle of period 2, Eq. (1) can possess multiple periodic solutions (some unstable) and chaos [5].

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