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Stability and instability criteria for Kaplan–Yorke solutions

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Abstract. We derive sufficient conditions for the stability and instability of periodic solutions $p : \mathbb{R} \to \mathbb{R}$ of Kaplan–Yorke type to the equation $\dot{x}(t) = \alpha f(x(t), x(t-1))$, where f is even in the first and odd in the second argument. The criteria are based on the monotonicity of the coefficient in a transformed version of the variational equation. For the special case of cubic f, we show that this monotonicity property is satisfied if and only if the set $\Big\{ (p(t), p(t-1)) \mid t \in \mathbb{R} \Big\} \subset \mathbb{R}^2$ is contained in a region E defined by a quadratic form (bounded by an an ellipse or a hyperbola). The coefficients of this quadratic form are expressible in terms of the Taylor coefficients of f. Further, the parameter α in the equation and the amplitude z of the periodic solution are related by an elliptic integral. Using the relation between this integral and the arithmetic-geometric mean, we obtain upper and lower estimates on this relation, and on the inverse function. Combining these estimates with the inequality that defines the region E, we obtain stability criteria explicit in terms of the Taylor coefficients of f. These criteria go well beyond local stability analysis, as examples show.

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1. Introduction

This paper deals with the existence and stability of periodic solutions of scalar differential delay equations of the form

$$\dot{x}(t) = \alpha f(x(t), x(t-1)), \tag{1.1}$$

where $\alpha > 0$ and the nonlinearity f satisfies the symmetry conditions

$$f(-x, y) = f(x, y) = -f(x, -y) \quad \forall (x, y) \in \mathbb{R}^2. \tag{2.1}$$

(The value 1 of the delay in equation (1.1) can be achieved by time rescaling.) If periodic solutions are obtained from topological methods, such as fixed point theorems, stability properties are usually not accessible. For periodic solutions

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with small amplitude, stability properties can be studied via Hopf bifurcation analysis [4, 11, 10]; the results are then local only.

The restrictions (2.1) allow to relate periodic solutions for equation (1.1) to periodic solutions of an associated two-dimensional system of ordinary differential equations (ODEs). For the special case where f depends only on y, this idea is due to Kaplan and Yorke [13]. It was an important tool in later publications [2, 5, 6, 7, 8, 9, 12] concerned with stability of periodic solutions.

These special periodic solutions are of period 4 and have a number of symmetries; here they are called KY solutions for brief. The quantity $z := \max_{t\in\mathbb{R}}\{p(t)\}$ is called the amplitude of a KY solution $p : \mathbb{R} \to \mathbb{R}$.

The parameter dependent equation (1.1) typically possesses a branch of KY solutions which can be described by a smooth curve in the $(\alpha, z)-$plane. The stability of a periodic solution is determined by the location of its Floquet multipliers in the complex plane [4, 11]. For a KY solution p (with period 4) these numbers are the eigenvalues of the time-4 operator defined by solutions of the variational equation along p. The Floquet multipliers are zeros of a characteristic function introduced in [14]. This function can be expressed in terms of the solution of a two-dimensional linear non-autonomous periodic system of ODEs built from the variational equation. Further details on KY solutions and the characteristic function are given below in Sections 2 and 3.

Criteria for stability and instability of KY solutions with small amplitude (that is, close to the Hopf bifurcation) were obtained in [7]. For the special case that f has the product form $f(x, y) = G(x) \cdot H(y)$, such criteria were also obtained in [8] for solutions with not necessarily small amplitude.

In this paper we describe the behavior of the KY branch for equation (1.1) (not only close to the bifurcation point), and we give stability results which assume neither smallness of the amplitude nor product form of the function f. Our analysis is mainly restricted to the case of at most cubic f. In view of the symmetry restrictions and the condition $\partial_2 f(0, 0) < 0$ (which we shall need, e.g., in order to have negative feedback at least in a neighborhood of zero), this means that f has the form

$$f(x, y) = y(-1 + \beta x^2 + \gamma y^2), \text{ with parameters } \beta, \gamma \in \mathbb{R}.$$

(Note that there is a third parameter α in the differential equation, and that the product form would require $\gamma = 0$ or $\beta = 0$.) Sections 2 and 3 describe the relation of KY solutions to those of the associated ODE, and give general stability criteria in terms of sign conditions on an expression involving partial derivatives of f along the periodic solution. So far, the results are not restricted to cubic f, but it turns out that for higher order f one cannot expect to obtain a KY branch, unless one imposes restrictions (equalities) on the higher order Taylor coefficients. The reason is that in the cubic case the associated ODE in $\mathbb{R}^2$ has a first integral, and a neighborhood of $(0, 0)$ is filled with periodic orbits, which correspond to the periodic solutions of the delay equations on the KY branch. This is not so for higher order f, where in general even the existence of one single

periodic solution of the ODE is questionable. In this sense, the Kaplan–Yorke method for $\dot{x}(t) = f(x(t), x(t-1))$ is limited to cubic nonlinearities f.

For the cubic case we obtain a rather complete picture of the global behavior of the KY branch and its stability properties in sections 4 and 5. This includes criteria for existence and stability or instability of KY solutions which are completely explicit in terms of the parameters α, β, γ. As in [7], the study of an elliptic integral is involved. Our analysis depends very much on a technique which was not used in this context so far, namely, the relation of the elliptic integral to the arithmetic-geometric mean of Gauß and Lagrange. Section 6 shows that our stability criteria compare quite well with numerical observations.

2. KY solutions

Assume that $f : \mathbb{R}^2 \to \mathbb{R}$ is C^1 and has the symmetries $f(x,y) = f(-x,y) = -f(x,-y)$ for $(x,y) \in \mathbb{R}^2$.

Definition 2.1. *By a Kaplan–Yorke (KY) solution of*

$$\dot{x}(t) = \alpha f(x(t), x(t-1)), \tag{2.1}$$

we mean a nonzero solution $p : \mathbb{R} \to \mathbb{R}$ of (2.1) *with period 4 and such that $p(0) = 0, p(t) = -p(t-2) = -p(-t)$ for $t \in \mathbb{R}$, and $p > 0$ on $(0,2)$.*

The Kaplan–Yorke method [13] relates the periodic solutions of the ODE

$$\dot{x} = f(x,y), \quad \dot{y} = -f(y,x) \tag{2.2}$$

in $\mathbb{R}^2$ to KY solutions of (2.1). The relevant connection between the two classes of solutions was described in [7], Sections 2 and 3, but under the assumption that $yf(x,y) < 0$ holds if $y \neq 0$. It is convenient for us to prove a version which does not use this condition. We point out some consequences of (2.1) first. The symmetry assumptions on f imply that $F : \mathbb{R}^2 \to \mathbb{R}^2, F(x,y) := \begin{pmatrix} f(x,y) \\ -f(y,x) \end{pmatrix}$ has the following properties: Setting

$$R_1(x,y) := (y,x),\ R_2(x,y) := (x,-y),\ R_3(x,y) := (-x,y),\ R_4(x,y) := (-y,-x)$$

for $(x,y) \in \mathbb{R}^2$, the vector field F is reversible with respect to $R_j, j = 1,...,4$ (i.e., $F \circ R_j = -R_j \circ F$), and moreover $F[-(x,y)] = -F(x,y)$. It follows that if $(x,y) : \mathbb{R} \to \mathbb{R}^2$ is a (periodic) solution of (2.2) then $t \mapsto R_j \circ (x,y)(-t) (j = 1,...,4)$ and $t \mapsto -(x,y)(t)$ define also (periodic) solutions. One sees that $R_i \circ R_j \circ (x,y)$ is also a solution for $i,j \in \{1,...,4\}$. Further, since system (2.2) is autonomous, for every $\tau \in \mathbb{R}$ the time translate $(x(\cdot + \tau), y(\cdot + \tau))$ of a (periodic) solution (x,y) is also a (periodic) solution.

Lemma 2.2. *a) Let p be a KY solution of* (2.1) *for some $\alpha > 0$, and set $z := p(1)$, so $z > 0$. Then $x(t) := p(t/\alpha), y(t) := p(t/\alpha - 1) = x(t - \alpha)$ defines a solution of*

(2.2) *with period* 4α*, and with the properties*

$$\begin{gathered} x(0)=0,\ y(0)=-z<0,\ x>0 \text{ and on } (0,\alpha], \\ y<0 \text{ on } [0,\alpha),\ \text{and } x(\alpha)=z,\ y(\alpha)=0. \end{gathered} \tag{2.2.1}$$

b) Conversely, if $\alpha, z>0$ *and if* $(x,y):\mathbb{R}\to\mathbb{R}^2$ *is a solution of* (2.2) *with the properties listed in* (2.2.1)*, then* (x,y) *has period* 4α*, and* $x(t)=y(t+\alpha)$ *for* $t\in\mathbb{R}$*, and* $p(t):=x(\alpha t)$ *defines a KY solution of* (2.1)*.*

Proof. Ad a): Assume that p is a KY solution (for some $\alpha>0$). Then $p>0$ on the interval $(0,2)$, and $p(\cdot)=-p(\cdot-2)$ implies $p<0$ on $(-2,0)$. Set $x(t):=p(t/\alpha)$, $y(t):=p(t/\alpha-1)$ for $t\in\mathbb{R}$, and $z:=p(1)$. Then $x(0)=p(0)=0$, $y(0)=p(-1)=-p(1)=-z<0$. For $t\in(0,\alpha)$, $x(t)\in p((0,1))$, so $x(t)>0$, and $y(t)\in p((-1,0))$, so $y(t)<0$. Further, $x(\alpha)=p(1)=z$, $y(\alpha)=p(0)=0$. (The properties in (2.2.1) are proved.)

Since

$$\begin{aligned} \dot x(t) &= \frac{1}{\alpha}\dot p(t/\alpha)=\frac{1}{\alpha}\alpha f(p(t/\alpha),p(t/\alpha-1))=f(x(t),y(t)),\\ \dot y(t) &= \frac{1}{\alpha}\dot p(t/\alpha-1)=\frac{1}{\alpha}\alpha f(p(t/\alpha-1),p(t/\alpha-2))\\ &= f(y(t),-p(t/\alpha))=-f(y(t),x(t)), \end{aligned}$$

we see that (x,y) solves (2.2), and clearly (x,y) has period 4α.

Ad b): Assume that $(x,y):\mathbb{R}\to\mathbb{R}^2$ is a solution of system (2.2) with the properties from (2.2.1). Then $t\mapsto R_2\circ R_1\circ(x,y)(t+\alpha)=[y(t+\alpha),-x(t+\alpha)]$ is also a solution of (2.2) which satisfies the same initial condition as (x,y), since $x(0)=0=y(\alpha)$, $y(0)=-z=-x(\alpha)$. It follows that $x(t)=y(t+\alpha)$, $y(t)=-x(t+\alpha)$ $(t\in\mathbb{R})$, from which we obtain that $x(t-2\alpha)=-x(t)$ $(t\in\mathbb{R})$. Consequently, both x and y have period 4α.

Further, $t\mapsto R_3\circ(x,y)(-t)=[-x(-t),y(-t)]$ is a solution of (2.2) satisfying the same initial condition as (x,y), and we conclude that $x(t)=-x(-t)$ $(t\in\mathbb{R})$.

Set now $p(t):=x(\alpha t)$ for $t\in\mathbb{R}$. Then p is nonzero, has period 4, and $p(0)=x(0)=0$. Regarding the symmetries of p, we have

$$\begin{gathered} p(t-2)=x(\alpha t-2\alpha)=-x(\alpha t)=-p(t),\\ p(-t)=x(-\alpha t)=-x(\alpha t)=-p(t). \end{gathered}$$

From (2.2.1), we know $x>0$ on $(0,\alpha]$ and $y<0$ on $[0,\alpha)$. For $t\in(\alpha,2\alpha)$ we have $t-\alpha\in(0,\alpha)$ and $x(t)=-y(t-\alpha)>0$. Together, one obtains $x>0$ on $(0,2\alpha)$. It follows that for $t\in(0,2)$ we have $p(t)=x(\alpha t)>0$. Finally, for all $t\in\mathbb{R}$,

$$\begin{aligned} \dot p(t) &= \alpha\dot x(\alpha t)=\alpha f(x(\alpha t),y(\alpha t))=\alpha f(p(t),x(\alpha t-\alpha))=\alpha f(p(t),x(\alpha(t-1)))\\ &= \alpha f(p(t),p(t-1)). \end{aligned}$$

We have shown that p is a KY solution of (2.1). □

3. The variational equation and stability

Assume that $p : \mathbb{R} \to \mathbb{R}$ is a KY solution of $\dot{x}(t) = \alpha f(x(t), x(t-1))$ for some $\alpha > 0$. We write $\vec{p}(t)$ for $(p(t), p(t-1))$ $(t \in \mathbb{R})$, and we set

$$\mathcal{O} := \left\{ \vec{p}(t) \mid t \in \mathbb{R} \right\} \subset \mathbb{R}^2.$$

With $z := p(1)$ one has

$$\vec{p}(0) = (0, -z), \quad \vec{p}(1) = (z, 0).$$

The variational equation along p is

$$\dot{v}(t) = a(t)v(t) + b(t)v(t-1), \tag{v}$$

where $a(t) := \alpha \partial_1 f(\vec{p}(t)),\ b(t) := \alpha \partial_2 f(\vec{p}(t))$.

The symmetries of f and p imply that a and b are two-periodic, and the function B defined by $B(t) := b(t) \exp[-\int_{t-1}^{t} a(s)\, ds]$ is also two-periodic [7].

If v solves the variational equation (v), then the function w defined by

$$w(t) := v(t) \exp[-\int_0^t a(s)\, ds]$$

satisfies the equation

$$\dot{w}(t) = B(t)w(t-1), \tag{w}$$

and vice versa. Thus, studying the time-2-operator of equation (v) is equivalent to studying the time-2-operator defined by solutions of equation (w).

The following condition on the derivative of f with respect to the second argument is crucial for our approach:

$$\partial_2 f(x, y) < 0 \text{ for all } (x, y) \in \mathcal{O}. \tag{nd}$$

It implies that $b < 0$ and $B < 0$, so that equation (w) has a negative coefficient.

We assume now that both (nd) and the negative feedback condition

$$y f(x, y) < 0 \text{ for all } (x, y) \in \mathcal{O} \text{ with } y \neq 0 \tag{nf}$$

hold for the periodic solution p. Note that (nf) implies that $\dot{p} > 0$ on $[0, 1)$, and that the parameter $z = p(1)$ from above is the amplitude of p, i.e.,

$$z = \max \left\{ p(t) \mid t \in \mathbb{R} \right\}. \tag{3.1}$$

It follows from (nf) that the zeros of $\dot{p}$ are the same as the zeros of p shifted by one, in particular, the solution $\dot{p}$ of the variational equation is slowly oscillating (i.e., has zeroes spaced at distances larger than the delay 1).

The time-two operator $V : C^0([0, 1], \mathbb{C}) \to C^0([0, 1], \mathbb{C})$ associated with complex valued solutions of equation (w) was studied in [2, 12, 7, 8], and we recall some results from these references (sometimes, $[-1, 0]$ is used in place of $[0, 1]$).

$\dot{p}\big|_{[0,1]}$ is an eigenvector of V with eigenvalue -1, and the following additional facts are known:

1) There is an analytic function $r : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ (with an essential singularity at zero) with $r(\mathbb{R} \setminus \{0\}) \subset \mathbb{R}$, $r(-1) = 0$, and such that

$$\forall \lambda \in \mathbb{C} \setminus \{-1, 0\} : \ \lambda \text{ is an eigenvalue of } V \Longleftrightarrow r(\lambda) = 0.$$

(The squares of the zeroes of r are precisely the Floquet multipliers of the periodic solution.) The function r is called the characteristic function.

2) The space of solutions of (v) associated with eigenvalues λ, where $|\lambda| \geq 1$, has at most dimension two ([2], p. 139, [12], p. 304). It contains the periodic solution $\dot{p}$ (associated with $\lambda = -1$). In particular, such λ must be real, and r has at most one zero in $(-\infty, -1)$ (counted with multiplicity).

3) $\lim\limits_{\lambda \to -\infty} r(\lambda)/\lambda = -1$ (Lemma 11 [2]).

4) r has no zero in $[1, \infty)$ ([2], p. 139).

We call the periodic solution p **simply unstable** if it has precisely one Floquet multiplier outside the unit circle, and this multiplier is simple and lies in $(1, \infty)$. (It is then the square of a zero of r in $(-\infty, -1)$.

One sees from 2) and 3) that if $r'(-1) < 0$ then r has no zero in $(-\infty, -1)$, and if $r'(-1) > 0$ then r has exactly one zero in $(-\infty, -1)$. Hence we obtain the implications

$$\text{(nd), (nf) and } r'(-1) > 0 \Longrightarrow p \text{ is simply unstable;}$$
$$\text{(nd), (nf) and } r'(-1) < 0 \Longrightarrow p \text{ is exponentially stable.}$$

Under assumptions (nd) and (nf), expressions for $r'(-1)$ were calculated in [2], [7], [8], and the following relations were shown: If B is decreasing/increasing on $[0, 1]$ then $r'(-1)$ is negative/positive (see [8], second last page, and [2], subsection 7.V, and [12], Section 3, formula (12)). In view of the significance of $r'(-1)$ for stability, and assuming differentiable B, we conclude

$$\text{(nd), (nf) and } \dot{B} \leq 0 \text{ on } [0, 1] \Longrightarrow p \text{ is exponentially stable,}$$
$$\text{(nd), (nf) and } \dot{B} \geq 0 \text{ on } [0, 1] \Longrightarrow p \text{ is simply unstable.}$$

This connection between the monotonicity of B and the stability was exploited only for the so-called 'product case' $f(x, y) = G(x)H(y)$ in [8]. We show now how it can be used for more general functions f.

From the definitions of a, b and B, we have

$$B(t) = \alpha \partial_2 f(\vec{p}(t)) \exp[-\alpha \int_{t-1}^{t} \partial_1 f(\vec{p}(s))\, ds].$$

Assuming that f is C^2, we calculate

$$\dot{B}(t) =$$
$$\alpha \left\{ \partial_1 \partial_2 f(\vec{p}(t)) \dot{p}(t) + \partial_2^2 f((\vec{p}(t)) \dot{p}(t-1) - \alpha \partial_2 f(\vec{p}(t)) [\partial_1 f(\vec{p}(t)) - \partial_1 f(\vec{p}(t-1))] \right\}$$
$$\times \exp[...].$$

Only the $\{...\}-$bracket is relevant for the sign of $\dot{B}$. Note that the symmetries of f and p imply

$$\dot{p}(t-1) = \alpha f(\vec{p}(t-1)) = \alpha f(p(t-1), p(t-2)) = -\alpha f(p(t-1), p(t)),$$

and

$$\partial_1 f(\vec{p}(t-1)) = \partial_1 f(p(t-1), -p(t)) = -\partial_1 f(p(t-1), p(t)).$$

Thus, writing (x, y) for $(p(t), p(t-1))$, we obtain

$$\begin{aligned}\{...\} &= \partial_1\partial_2 f(x,y)\alpha f(x,y) - \partial_2^2 f(x,y)\alpha f(y,x) - \alpha\partial_2 f(x,y)[\partial_1 f(x,y) + \partial_1 f(y,x)] \\ &= \alpha\{\partial_1\partial_2 f(x,y) f(x,y) - \partial_2^2 f(x,y) f(y,x) - \partial_2 f(x,y)[\partial_1 f(x,y) + \partial_1 f(y,x)]\}.\end{aligned}$$

(In the product case $f(x, y) = G(x)H(y)$, the first term in this expression cancels with the term $\partial_2 f(x, y)\partial_1 f(x, y)$, leading to a simpler form.)

We summarize the result of this section:

Lemma 3.1. *Let p and $\mathcal{O}$ be as above. For $(x, y) \in \mathbb{R}$, set*

$$S(x,y) := \partial_1\partial_2 f(x,y) f(x,y) - \partial_2^2 f(x,y) f(y,x) - \partial_2 f(x,y)[\partial_1 f(x,y) + \partial_1 f(y,x)].$$

Then one has the implications

(nf) *and* (nd) *hold and $S \le 0$ on $\mathcal{O} \Longrightarrow p$ is exponentially stable,*
(nf) *and* (nd) *hold and $S \ge 0$ on $\mathcal{O} \Longrightarrow p$ is simply unstable.*

Remarks. 1) While (nd) is crucial for all our methods, condition (nf) can probably be relaxed in the case of instability results. However, in all our concrete statements on solutions in specific parameter ranges, it turns out that (nf) is automatically implied by other restrictions. Therefore we decided to keep it from the beginning, also for the sake of simplicity.

2) In Lemma 3.1 above, it would be sufficient to have the inequalities involving S only on the set $\tilde{\mathcal{O}} := \left\{\vec{p}(t) \;\middle|\; t \in [0,1]\right\}$ – but the symmetries of f then imply that they hold on all of $\mathcal{O}$.

4. Cubic nonlinearities

The most general delay equation with at most cubic right hand side satisfying our symmetry assumptions (see Section 2) and also condition (nf), at least in a neighborhood of $(0, 0)$, has the form

$$\dot{x}(t) = \alpha f(x(t), x(t-1)), \tag{4.1}$$

where $\alpha > 0$ and

$$f(x,y) = y(-1 + \beta x^2 + \gamma y^2), \quad \beta, \gamma \in \mathbb{R}. \tag{4.2}$$

Note that in the notation of Example 2 from [7], our form for the cubic function f corresponds to $A = -1, B = \beta, C = \gamma$. The plane system (2.2) in this case has the form

$$\begin{aligned} \dot{x} &= -y + \beta x^2 y + \gamma y^3, \\ \dot{y} &= x - \beta y^2 x - \gamma x^3, \end{aligned} \tag{4.3}$$

and is Hamiltonian with first integral (and Hamilton function)

$$H(x,y) = -\frac{1}{2}(x^2+y^2) + \frac{\beta}{2}(x^2y^2) + \frac{\gamma}{4}(x^4+y^4).$$

The level curves of H are bounded, which implies that all maximal solutions of (4.3) exist on $\mathbb{R}$, and the level curves are invariant under the transformations R_j, $j = 1, ..., 4$.
We focus on periodic solutions:

Proposition 4.1. *If $z > 0, \beta \in \mathbb{R}$ and $\gamma \in \mathbb{R}$ are such that*

$$l := l(z,\beta,\gamma) := \frac{z^2(2-\gamma z^2)(\gamma-\beta)}{2(1-\gamma z^2)^2} > -1 \text{ and } (1-\gamma z^2) > 0, \tag{4.1.1}$$

then there is a unique positive $\alpha = \alpha^*(z,\beta,\gamma)$ such that the solution (x^z, y^z) of (4.3) *with initial value $x^z(0) = 0,\ y^z(0) = -z$ satisfies the conditions of Lemma 2.2,b) with this α. Hence, the function defined by $p(t) := x^z(\alpha t)$ is a KY solution of eq.* (4.1).

Proof. Assume that (4.1.1) holds and that $(x,y) = (x^z, y^z) : \mathbb{R} \to \mathbb{R}^2$ is a solution of (4.3) as in the proposition. Setting $r(t) := [x^2(t) + y^2(t)]^{1/2}$ for $t \in \mathbb{R}$, uniqueness of solutions implies that $r(t) \neq 0$ for all $t \in \mathbb{R}$. Further, there exists a unique continuous function $\phi : \mathbb{R} \to \mathbb{R}$ (which actually is C^1) such that $x(t) = r(t)\sin(\phi(t)),\ y(t) = -r(t)\cos(\phi(t))$ $(t \in \mathbb{R})$, and we have $r(0) = z,\ \phi(0) = 0$. (See [7], p. 69; note that this choice of ϕ corresponds to assigning the angle $\phi = 0$ to the negative $y-$axis.) Introducing $\psi := 2\phi,\ G := G(z) := 2z^2 - \gamma z^4 = z^2(2-\gamma z^2)$, it was shown in [7] that

$$\left(\dot{\psi}\right)^2 = 4(1-G\gamma) + 2G(\gamma-\beta)\sin^2(\psi).$$

Now $(1-G\gamma) = (1-2\gamma z^2+\gamma^2 z^4) = (1-\gamma z^2)^2$, so we conclude

$$\left(\dot{\psi}\right)^2 = 4(1-\gamma z^2)^2 + 2z^2(2-\gamma z^2)(\gamma-\beta)\sin^2(\psi).$$

Assume now that (4.1.1) holds. Since $1-\gamma z^2 > 0$, we can rewrite the last equality as

$$\left(\dot{\psi}\right)^2 = 4(1-\gamma z^2)^2[1 + \frac{z^2(2-\gamma z^2)(\gamma-\beta)}{2(1-\gamma z^2)^2}\sin^2(\psi)].$$

Using (4.1.1) again, we conclude that

$$|\dot{\psi}| = 2(1-\gamma z^2)\sqrt{1 + l\sin^2(\psi)}. \tag{4.1.2}$$

In particular, setting $\omega := 2(1-\gamma z^2)\sqrt{1+\min\{0,l\}} > 0$ we then have $|\dot{\psi}| \geq \omega > 0$. Further, from (4.3) and (4.1.1) we get $\dot{x}(0) = z + \gamma(-z)^3 = z(1-\gamma z^2) > 0$, and $\dot{y}(0) = 0$, which implies $\dot{\psi}(0) > 0$. It follows that we have

$$\dot{\psi}(t) \geq \omega > 0 \text{ for all } t \in \mathbb{R}, \tag{4.1.3}$$

and there exists a unique $\alpha = \alpha^*(z,\beta,\gamma) > 0$ such that $\psi(\alpha/2) = \pi/2$, or $\phi(\alpha/2) = \pi/4$. It follows that $x(\alpha/2) = -y(\alpha/2)$. Now (recall the passage preceding Lemma 2.2) both $t \mapsto (x,y)(\alpha/2+t)$ and $t \mapsto R_4 \circ (x,y)(\alpha/2-t) = [-y(\alpha/2-t), -x(\alpha/2-t)]$ define solutions of (4.3) which coincide for $t = 0$ and hence for all t. Thus,

$$x(\alpha/2+t) = -y(\alpha/2-t) \quad (t \in \mathbb{R}). \tag{4.1.4}$$

In particular,

$$x(\alpha) = -y(0) = z \text{ and } y(\alpha) = -x(0) = 0, \tag{4.1.5}$$

Since ϕ is strictly increasing from 0 to $\pi/4$ on $[0,\alpha/2]$, we have $y(t) = -r(t)\cos(\phi(t)) < 0$ for $t \in [0,\alpha/2]$, and $x(t) = r(t)\sin(\phi(t)) > 0$ for $t \in (0,\alpha/2]$. Using (4.1.4) for $t \in (-\alpha/2, 0]$ and for $t \in [0,\alpha/2]$, one sees that

$$y(t) < 0 \text{ for } t \in [0,\alpha) \text{ and } x(t) > 0 \text{ for } t \in (0,\alpha]. \tag{4.1.6}$$

In view of (4.1.5) and (4.1.6), we have that (x,y) satisfies the restrictions in (2.2.1) with $\alpha = \alpha^*(z,\beta,\gamma)$. Uniqueness of α with these properties is clear, since α has to be the first zero of y. The assertion on p follows from Lemma 2.2,b). □

Next, we give more detailed information on the $z-$values for which (4.1.1) holds, given the values of β and γ.

Proposition 4.2. *Let $\beta,\gamma \in \mathbb{R}$, $(\beta,\gamma) \neq (0,0)$. Define*

$$u^* := u^*(\beta,\gamma) := \sup\Big\{u > 0 \;\Big|\; \textit{Condition } (4.1.1) \textit{ holds for all } z > 0 \textit{ with } z^2 < u\Big\}.$$

Then u^* is given explicitly as follows:
1) If $\gamma \neq 0$ then

$$u^* = \begin{cases} \dfrac{\beta+\gamma-\sqrt{\beta^2-\gamma^2}}{\gamma(\beta+\gamma)} & \textit{if} \quad \beta+\gamma > 0 \textit{ and } \beta-\gamma \geq 0, \\ 1/\gamma & \textit{if} \quad \gamma > 0 \textit{ and } \beta-\gamma < 0, \\ \infty & \textit{if} \quad \gamma < 0 \textit{ and } \beta+\gamma \leq 0. \end{cases}$$

2) If $\gamma = 0, \beta \neq 0$ then $u^* = \begin{cases} 1/\beta & \textit{if} \quad \beta > 0, \\ \infty & \textit{if} \quad \beta < 0. \end{cases}$

If we set $z^*(\beta,\gamma) := \sqrt{u^*(\beta,\gamma)}$ ($z^* := \infty$, if $u^* = \infty$) then condition (4.1.1) holds for all z,β,γ such that $z \in (0, z^*(\beta,\gamma))$, and hence $\alpha^*(z,\beta,\gamma)$ is well defined and $t \mapsto x^z(\alpha^*(z,\beta,\gamma)\cdot t)$ is a KY solution of (4.1) for all such z,β,γ.

γ	$\beta+\gamma$	$\beta-\gamma$	u_-	u_+	u_1	u_2	$u^* = \min\{u_1, u_2\}$
$+$	$+$	≥ 0	$+$	$+$	u_-	$1/\gamma$	u_-
$+$	$+$	$-$	complex	complex	∞	$1/\gamma$	$1/\gamma$
$+$	$-$	$-$	$+$	$-$	u_-	$1/\gamma$	$1/\gamma$
$-$	$+$	$+$	$+$	$-$	u_-	∞	u_-
$-$	$-$	≤ 0	$-$	$-$	∞	∞	∞
$-$	$-$	$+$	complex	complex	∞	∞	∞

Table 1. Values of $u^* = \min\{u, u_2\}$ in dependence of β and γ

Proof. Let β and γ be given. Writing u for z^2, condition (4.1.1) is equivalent to

$$u(2-\gamma u)(\gamma-\beta) > -2(1-\gamma u)^2 \quad \text{and} \quad 1-\gamma u > 0. \tag{4.2.1}$$

Both conditions in (4.2.1) are satisfied for all sufficiently small $u > 0$. Define u_1 (resp. u_2) as the supremum of all values of u such that the first (resp. the second) condition in (4.2.1) holds; then we have $u^* = \min\{u_1, u_2\}$. In each case, we shall show that u^* is given by the value stated in the present proposition. The remaining assertions then follow from Proposition 4.1.

It is obvious that $u_2 = \infty$ if $\gamma \leq 0$ and $u_2 = 1/\gamma$ if $\gamma > 0$. In order to determine u_1, and thus u^*, we distinguish several cases.

The case $\gamma \neq 0 \neq \beta+\gamma$: The first condition in (4.2.1) holds if and only if

$$\gamma(\beta+\gamma)u^2 - 2(\beta+\gamma)u + 2 > 0. \tag{4.2.2}$$

The (possibly complex) zeroes of the left hand side of this inequality are given by

$$u_\pm := \frac{\beta+\gamma \pm \sqrt{\beta^2-\gamma^2}}{\gamma(\beta+\gamma)}.$$

Note that the left hand side of (4.2.2) has the value $2 > 0$ for $u = 0$. One sees that u_1 is given by the smallest positive number in $\{u_-, u_+\}$ (if one of the two is positive), and $u_1 = \infty$ otherwise.

The values of $u_\pm, u_1$ and u_2 have to be compared in all subcases to obtain u^*; the result of this discussion is summarized in Table 1, where '$+$' and '$-$' stand for positive/negative.

Comparing the values of u^* from the statement of the present proposition in the different subcases with the table, one obtains the assertion (for the case $\gamma \neq 0 \neq \beta+\gamma$).

The case $\beta+\gamma = 0 \neq \gamma$: Then (4.2.2) shows that $u_1 = \infty$, so $u^* = \min\{u_1, u_2\} = u_2$, and hence $u^* = 1/\gamma$ if $\gamma > 0$, and $u^* = \infty$ if $\gamma < 0$.

The case $\beta \neq 0 = \gamma$: Then we have $u_2 = \infty$ and, in view of (4.2.2), u_1 is given by the condition $-2\beta u + 2 > 0$. Hence $u_1 = \infty$ if $\beta < 0$, and $u_1 = 1/\beta$ if $\beta > 0$, and u^* has the same values as u_1, in agreement with the statement above. $\square$

Case	$\bar{u}_1(\beta,\gamma)$
$\beta,\gamma>0$	$1/(\beta+\gamma)$
$\beta,\gamma<0$	∞
$\beta>0>\gamma$	$1/\beta$
$\gamma>0>\beta$	$1/\gamma$

Table 2. Values of $\bar{u}_1$ in dependence of β and γ

Remarks. 1. Note that the case $\gamma=\beta=0$ corresponds to a linear equation and is therefore not of interest. Note also that the different cases in the above proposition are mutually disjoint and exhaust all possibilities.

2. We do not pursue to prove this here but it seems that the solutions (x^z,y^z), where z ranges in the interval $(0,z^*)$ from the above proposition, describe in all cases the maximal neighborhood of $(0,0)$ which is filled with periodic orbits of (4.3).

3. Consider now f of possibly higher order than 3, satisfying the symmetry properties. In order for the vector field (2.2) to have a neighborhood of (0,0) filled by periodic orbits, it is necessary that the vector field have zero divergence in that neighborhood (as is easily seen from the Gauß-Green Theorem). This property is automatically guaranteed in the cubic case. For higher order f, this is not so. For example, if one considers f with a local Taylor expansion $f(x,y)=y(-1+\beta x^2+\gamma y^2+\delta x^4+\varepsilon x^2y^2+...)$ at $(0,0)$, then zero divergence of the field in (2.2) in a neighborhood of $(0,0)$ implies that $\varepsilon=2\delta$. Thus, for the higher order f, the existence of a branch of KY solutions could be guaranteed only under restrictive conditions on the Taylor coefficients.

Also, the proof of Proposition 4.1 above, and the description of the function α^* in terms of elliptic integrals (Section 5 below), are methods specific for the cubic case.

It is of interest to find conditions on $z=p(1)$ and the parameters β,γ in system (4.3) which guarantee that z is the amplitude (the maximum) of the KY solution corresponding to the solution (x^z,y^z) of (4.3). The following result gives an answer.

Lemma 4.3. *Assume $\beta,\gamma\in\mathbb{R}$, $\beta\gamma\neq 0$. Define $\bar{u}_1=\bar{u}_1(\beta,\gamma)$ by Table 2. Then $\bar{u}_1\le u^*(\beta,\gamma)$. Further, if $z^2<\bar{u}_1$, then the corresponding KY solution p satisfies*

$$z=\max\Big\{|p(t)|\;\Big|\;t\in\mathbb{R}\Big\},$$

i.e., has amplitude z.

Proof. Claim: $\bar{u}_1\le u^*(\beta,\gamma)$. Proof: In case $\beta,\gamma>0$ we have $u^*=u_-$ or $u^*=1/\gamma$,

depending on whether $\beta \geq \gamma$ or $\beta < \gamma$. Further, from Table 2, one has in case $\beta \geq \gamma$

$$\bar{u}_1 = 1/(\beta+\gamma) = \frac{\gamma}{\gamma(\beta+\gamma)} \leq \frac{\beta+\gamma-\sqrt{\beta^2-\gamma^2}}{\gamma(\beta+\gamma)} = u_- = u^*,$$

and in case $\beta < \gamma$ one has $\bar{u}_1 = 1/(\beta+\gamma) < 1/\gamma = u^*$.

In case $\beta, \gamma < 0$, we have $u^*(\beta,\gamma) = \infty = \bar{u}_1$.

In case $\beta > 0 > \gamma$, we also have $u^*(\beta,\gamma) = \infty$ if $\beta \leq |\gamma|$. If $\beta > |\gamma|$, we have $u^* = u_-$, and $\bar{u}_1 = 1/\beta$, so it suffices to show that $1/\beta \leq u_-$. This is true since $\beta > \sqrt{\beta^2-\gamma^2}$ implies $\beta\sqrt{\beta^2-\gamma^2} > \beta^2-\gamma^2$, and hence

$$\beta u_- = \frac{\beta(\sqrt{\beta^2-\gamma^2}-\beta-\gamma)}{|\gamma|(\beta-|\gamma|)} > \frac{\beta^2-\gamma^2-\beta^2-\beta\gamma}{|\gamma|(\beta-|\gamma|)} = 1.$$

In case $\gamma > 0 > \beta$, we have $u^* = 1/\gamma$, and Table 2 shows $\bar{u}_1 = u^*$. The claim is proved.

Assume now $z^2 < \bar{u}_1$ (then $z < z^*(\beta,\gamma)$). Consider the periodic solution (x^z, y^z) of system (4.3) with initial value $(0,-z)$, as in Proposition 4.1, and the corresponding KY solution p. We have $p(\mathbb{R}) = x^z(\mathbb{R})$ and, from the symmetries of (4.3),

$$P := \max\Big\{|x^z(t)| \;\Big|\; t \in \mathbb{R}\Big\} = \max\Big\{\max\{|x^z(t)|, |y^z(t)|\} \;\Big|\; t \in [0,\alpha/2]\Big\},$$

where $\alpha = \alpha^*(z,\beta,\gamma)$. Further, in the notation of the proof of Proposition 4.1, we have $\phi(\alpha/2) = \pi/4$, so $x^z(\alpha/2) = -y^z(\alpha/2)$, and (4.1.3) implies that $\forall t \in [0,\alpha/2) : |y^z(t)| > |x^z(t)|$. It follows that $P = \max\Big\{|y^z(t)| \;\Big|\; t \in [0,\alpha/2]\Big\}$. Recall (from Lemma 2.2) that $y^z < 0 < x^z$ on $(0,\alpha)$. Hence it suffices to prove

$$y^z \geq -z \text{ on } [0,\alpha/2]. \tag{4.3.1}$$

Claim: $[1-\gamma z^2] > 0$, and $1-\beta z^2 > 0$. Proof: The first assertion is trivial for $\gamma < 0$ and follows from $z^2 < \bar{u}_1 \leq 1/\gamma$ if $\gamma > 0$ (see Table 2). The second assertion is trivial for $\beta < 0$. If $\beta > 0$, we see from Table 2 that $\bar{u}_1 \leq 1/\beta$, which implies the second assertion.

Note now that $\dot{x}^z(0) = -(-z)[1-\gamma z^2] > 0$, and $\dot{y}^z(t) = x^z(t)[1-\beta(y^z(t))^2 - \gamma(x^z(t))^2]$, and the last square bracket equals $1-\beta z^2 > 0$ for $t = 0$. Hence there exists $t_1 \in (0,\alpha/2)$ with

$$x^z > 0 \text{ on } (0,t_1] \text{ and } -z < y^z < 0 \text{ on } (0,t_1].$$

Assume that there exists $t_0 \in (t_1,\alpha/2]$ with $y^z(t_0) = -z$. Then, from the Theorem of Rolle, there exists $t_2 \in (t_1,t_0)$ with $\dot{y}(t_2) = 0$. Set $(x,y) := (x^z(t_2), y^z(t_2))$, and note that (from Lemma 2.2) $x > 0$. We have $x(1-\beta y^2-\gamma x^2) = 0$, and hence

$$1 = \beta y^2 + \gamma x^2. \tag{4.3.2}$$

Note that $t_2 < \alpha/2$ implies $|x| < |y|$, so we have

$$x^2 < y^2 = z^2.$$

We derive a contradiction from (4.3.2) for all cases of the values of β and γ.

If $\beta, \gamma > 0$, then $\beta y^2 + \gamma x^2 \le (\gamma + \beta) z^2 < (\gamma + \beta)\bar{u}_1 = 1$, which contradicts (4.3.2).

If $\beta, \gamma < 0$ then (4.3.2) is clearly impossible.

If $\beta > 0 > \gamma$ then $\bar{u}_1 = 1/\beta$ and $\beta y^2 + \gamma x^2 \le \beta z^2 < \beta \bar{u}_1 = 1$, contradicting (4.3.2).

If $\gamma > 0 > \beta$ then $\bar{u}_1 = 1/|\gamma|$ and $\beta y^2 + \gamma x^2 \le \gamma z^2 < \gamma \bar{u}_1 = 1$, again contradicting (4.3.2).

We have proved that there can be no t_0 in $(t_1, \alpha/2]$ with $y^z(t_0) = -z$. With the properties of t_1, we conclude from the Intermediate Value Theorem that $y^z > -z$ on the interval $(0, \alpha/2]$, i.e., (4.3.1) is proved. □

We turn to stability questions now. For the expression $S(x, y)$ from Lemma 3.1, we obtain for cubic f as in (4.2)

$$\partial_2 f(x,y) = -1+\beta x^2+3\gamma y^2,\ \partial_1 f(x,y) = 2\beta xy,\ \partial_1\partial_2 f(x,y) = 2\beta x,\ \partial_2^2 f(x,y) = 6\gamma y,$$

and

$$\begin{aligned} S(x,y) &= 2\beta x f(x,y) - 6\gamma y f(y,x) - [-1+\beta x^2+3\gamma y^2]4\beta xy \\ &= 2\beta xy[-1+\beta x^2+\gamma y^2] - 6\gamma yx[-1+\beta y^2+\gamma x^2] \\ &\qquad - 4\beta xy[-1+\beta x^2+3\gamma y^2] \\ &= 2xy\left\{-\beta+\beta^2x^2+\beta\gamma y^2+3\gamma-3\gamma\beta y^2-3\gamma^2x^2+2\beta-2\beta^2x^2-6\beta\gamma y^2\right\} \\ &= -2xy\left\{(\beta^2+3\gamma^2)x^2+8\beta\gamma y^2-(\beta+3\gamma)\right\}. \end{aligned} \tag{4.4}$$

Depending on the signs of β and γ, the regions

$$Q^- := \left\{(x,y)\in\mathbb{R}^2 \;\middle|\; (\beta^2+3\gamma^2)x^2+8\beta\gamma y^2 \le (\beta+3\gamma)\right\},$$
$$Q^+ := \left\{(x,y)\in\mathbb{R}^2 \;\middle|\; (\beta^2+3\gamma^2)x^2+8\beta\gamma y^2 \ge (\beta+3\gamma)\right\}.$$

are bounded by an ellipse, or by the branches of a hyperbola.

Note that for a KY solution p of eq. (4.1), we have with $x(t) := p(t)$, $y(t) := p(t-1)$ that $xy \le 0$ on $[0,1]$ (even if f is not cubic). For cubic f, it follows from (4.4) that for $t \in \mathbb{R}$, the sign of $S(x(t), y(t))$ is the same as the sign of the bracket in the last expression of (4.4) (with $x := x(t), y := y(t)$), and hence depends on whether $(x(t), y(t))$ is in Q^+ or in Q^-.

Note also that conditions (nd) and (nf) from Section 3 are in the present cubic case equivalent to $\beta x^2+3\gamma y^2 < 1$ and $\beta x^2+\gamma y^2 < 1$ for all (x, y) in $\mathcal{O}$, respectively. (Here $\mathcal{O}$ is as in Section 3.) We introduce the sets ND and NF by

$$ND := \Big\{(x, y) \in \mathbb{R}^2 \;\Big|\; \beta x^2 + 3\gamma y^2 < 1\Big\}, \quad NF := \Big\{(x, y) \in \mathbb{R}^2 \;\Big|\; \beta x^2 + \gamma y^2 < 1\Big\}.$$

Combining the above observations with Lemma 3.1, we arrive at the following theorem, which is the basis for more specialized statements concerning specific parameter ranges. Recall the definition of the term 'simply unstable' from Section 3.

Theorem 4.4. *Let p be a KY solution of $\dot{x}(t) = \alpha f(x(t), x(t-1))$, where the cubic function f is as in* (4.2)*, and define $\mathcal{O} := \Big\{(p(t), p(t-1)) \;\Big|\; t \in \mathbb{R}\Big\} \subset \mathbb{R}^2$. Then the following implications hold:*

$$\mathcal{O} \subset Q^- \cap ND \cap NF \Longrightarrow \; p \text{ is exponentially stable};$$
$$\mathcal{O} \subset Q^+ \cap ND \cap NF \Longrightarrow \; p \text{ is simply unstable.}$$

The elliptic case. By this we mean the case $\beta\gamma > 0$, so that the coefficients of x^2 and of y^2 in the expression defining $Q^{\pm}$ are both positive. Here we have the following result.

Corollary 4.5. *a) Assume $\beta, \gamma > 0$, and set*

$$\bar{u} := \min\left\{\frac{1}{\beta + 3\gamma}, \frac{\beta + 3\gamma}{\beta^2 + 3\gamma^2 + 8\beta\gamma}\right\}.$$

Then $\bar{u} < u^*(\beta, \gamma)$. If $z^2 < \bar{u}$ then the corresponding solution (x^z, y^z) of (4.3) *(as in Prop. 4.1) defines a KY solution p of* (4.1) *(with $\alpha = \alpha^*(z, \beta, \gamma)$) with amplitude z, which is exponentially stable.*

b) If β and γ are negative then for all $z > 0$ the corresponding solution (x^z, y^z) of (4.3) *defines a KY solution p of* (4.1) *with amplitude z, which is simply unstable.*

Proof. Ad a): Let $\beta, \gamma > 0$, and assume $z^2 < \bar{u}$. It is obvious from the definition of $\bar{u}$ and from Table 2 that $\bar{u} \le 1/(\beta + \gamma) = \bar{u}_1(\beta, \gamma)$, and hence it follows from Lemma 4.3 that (x^z, y^z) defines a KY solution p with amplitude z. (With this p, define $\mathcal{O}$ as in Theorem 4.4.) Since $\max\{x^2, y^2\} \le z^2$ for $(x, y) \in \mathcal{O}$, we conclude from $z^2 < \bar{u}$ that for all such (x, y) one has

$$\begin{aligned}(\beta^2 + 3\gamma^2)x^2 + 8\beta\gamma y^2 \le (\beta^2 + 3\gamma^2 + 8\beta\gamma)z^2 < \beta + 3\gamma, \text{ and}\\ \beta x^2 + 3\gamma y^2 \le (\beta + 3\gamma)z^2 < 1,\end{aligned} \tag{4.5.1}$$

so $\mathcal{O} \subset Q^- \cap ND$. Finally, we have $ND \subset NF$, since $\beta, \gamma > 0$. The stability assertion now follows from Theorem 4.4.

Ad b): If $\beta, \gamma < 0$ then $u^*(\beta, \gamma) = \infty, Q^- = \emptyset,\ Q^+ = ND = NF = \mathbb{R}^2$. We see from Lemma 4.3 that for every $z > 0$ the corresponding solution (x^z, y^z) defines

a KY solution, and that this solution has amplitude z. The instability assertion follows from Theorem 4.4. □

Remark. If $\beta, \gamma > 0$, the case of an orbit $\mathcal{O}$ as in Theorem 4.4 lying in ND, but entirely in Q^+ is impossible: Then $\mathcal{O}$ would have to be outside the ellipse-shaped region Q^- but inside the ellipse-shaped region ND, which requires $Q^- \subset ND$. This inclusion is equivalent to

$$\frac{\beta + 3\gamma}{\beta^2 + 3\gamma^2} \leq \frac{1}{\beta} \text{ and } \frac{\beta + 3\gamma}{8\beta\gamma} \leq \frac{1}{3\gamma},$$

and these two inequalities are easily seen to be equivalent to $\beta \leq \gamma$ and (simultaneously) $5\beta \geq 9\gamma$, which is impossible. Hence, we cannot use Theorem 4.4 to obtain an instability result in this case.

The hyperbolic case. This is the case $\beta\gamma < 0$.
For the periodic solutions that we consider, the amplitude P equals the distance z of the intersection points with the coordinate axes from the origin. Consider, for example, the case $\beta > 0 > \gamma$. Demanding that the orbit $\mathcal{O}$ of such a solution is contained in $ND = \left\{(x, y) \in \mathbb{R}^2 \;\middle|\; \beta x^2 + 3\gamma y^2 < 1\right\}$ (a region bounded by a hyperbola) is equivalent to the property $\beta z^2 < 1$. (Similarly with NF.) This is why in the hyperbolic case it causes no loss of information if we systematically replace regions bounded by hyperbolas with corresponding strips in the plane (e.g., in the case above the set ND with $\left\{(x, y) \in \mathbb{R}^2 \;\middle|\; \beta x^2 < 1\right\}$.

Note the following implications:

$$\begin{aligned} \beta + 3\gamma > 0 &\Longrightarrow \left\{(x, y) \;\middle|\; x^2 \leq \frac{\beta + 3\gamma}{\beta^2 + 3\gamma^2}\right\} \subset Q^-, \\ \beta + 3\gamma < 0 &\Longrightarrow \left\{(x, y) \;\middle|\; y^2 \leq \frac{|\beta + 3\gamma|}{8|\beta\gamma|}\right\} \subset Q^+. \end{aligned} \tag{4.5}$$

Further note that

$$\beta > 0 > \gamma \Longrightarrow \left\{(x, y) \;\middle|\; x^2 < 1/\beta\right\} \subset ND \cap NF, \tag{4.6}$$

$$\gamma > 0 > \beta \Longrightarrow \left\{(x, y) \;\middle|\; y^2 < 1/(3\gamma)\right\} \subset ND \subset NF. \tag{4.7}$$

Motivated by (4.5)–(4.7), we define

$$\bar{u}_2 := \bar{u}_2(\beta, \gamma) := \begin{cases} \dfrac{\beta + 3\gamma}{\beta^2 + 3\gamma^2} & \text{if} \quad \beta + 3\gamma > 0, \\ \dfrac{|\beta + 3\gamma|}{8|\beta\gamma|} & \text{if} \quad \beta + 3\gamma < 0; \end{cases}$$

$$\bar{u}_3 := \bar{u}_3(\beta, \gamma) := \begin{cases} 1/\beta & \text{if} \quad \beta > 0 > \gamma, \\ 1/(3\gamma) & \text{if} \quad \gamma > 0 > \beta. \end{cases}$$

Case	Subcase	$\bar{u}(\beta,\gamma)$
$\beta+3\gamma>0$	$\beta>0>\gamma$	$\dfrac{\lvert\beta+3\gamma\rvert}{\beta^2+3\gamma^2}$
	$\gamma>0>\beta$	$\min\left\{\dfrac{\lvert\beta+3\gamma\rvert}{\beta^2+3\gamma^2},\dfrac{1}{3\gamma}\right\}$
$\beta+3\gamma<0$		$\dfrac{\lvert\beta+3\gamma\rvert}{8\lvert\beta\gamma\rvert}$

Table 3. Values of $\bar{u}$ in dependence of β and γ in the hyperbolic case

We now determine the minimum of $\bar{u}_1, \bar{u}_2, \bar{u}_3$, which will help us to obtain solutions (x^z, y^z) of (4.3) with amplitude z, with orbit in $ND \cap NF$, and in Q^- or Q^+.

Proposition 4.6. $\bar{u} := \bar{u}(\beta,\gamma) := \min\{\bar{u}_1, \bar{u}_2, \bar{u}_3\}$ *has the values indicated in Table 3, and* $\bar{u} < \bar{u}_1 \le u^*(\beta,\gamma)$.

Proof. The inequality $\bar{u}_1 \le u^*$ is known from Lemma 4.3. Therefore it suffices to prove in all cases that $\bar{u}$ has the asserted value, and that $\bar{u} < \bar{u}_1$.

Case 1: $\beta+3\gamma>0$. Then $\bar{u}_2 = \dfrac{\beta+3\gamma}{\beta^2+3\gamma^2}$.

Case 1a: $\beta>0>\gamma$. Then $\bar{u}_1 = 1/\beta$, $\bar{u}_3 = 1/\beta$, and $\dfrac{\beta+3\gamma}{\beta^2+3\gamma^2} \le \dfrac{\beta}{\beta^2+3\gamma^2} < 1/\beta$, so $\bar{u} = \dfrac{\beta+3\gamma}{\beta^2+3\gamma^2} < \bar{u}_1$.

Case 1b: $\gamma>0>\beta$. Then $\bar{u}_1 = 1/\gamma$, $\bar{u}_3 = 1/(3\gamma)$, and $\bar{u} = \min\{\dfrac{\beta+3\gamma}{\beta^2+3\gamma^2}, \dfrac{1}{3\gamma}\} < \bar{u}_1$.

Case 2: $\beta+3\gamma<0$. Then $\bar{u}_2 = \dfrac{\lvert\beta+3\gamma\rvert}{8\lvert\beta\gamma\rvert}$.

Case 2a: $\beta>0>\gamma$. Then $\bar{u}_1 = 1/\beta = \bar{u}_3$, and $\dfrac{\lvert\beta+3\gamma\rvert}{8\lvert\beta\gamma\rvert} \le \dfrac{\lvert 3\gamma\rvert}{8\lvert\beta\gamma\rvert} < 1/\beta$, so $\bar{u} = \dfrac{\lvert\beta+3\gamma\rvert}{8\lvert\beta\gamma\rvert} < \bar{u}_1$.

Case 2b: $\gamma>0>\beta$. Then $\bar{u}_1 = \dfrac{1}{\gamma}$, $\bar{u}_3 = \dfrac{1}{3\gamma}$, and $\dfrac{\lvert\beta+3\gamma\rvert}{8\lvert\beta\gamma\rvert} \le \dfrac{\lvert\beta\rvert}{8\lvert\beta\gamma\rvert} < \dfrac{1}{3\gamma} < \bar{u}_1$. □

Using (4.5)–(4.7), we now obtain the following counterpart to Corollary 4.5 on stability/instability for the hyperbolic case.

Corollary 4.7. *Assume* $\beta\gamma<0$, $\beta+3\gamma \ne 0$, *and define* $\bar{u}$ *by Table 3. If* $0 < z^2 \le \bar{u}$

then the solution (x^z, y^z) *of* (4.3) *defines a KY solution* p *of* (4.1) *with amplitude* z.

Further, one has the implications

a) $\beta + 3\gamma > 0 \Longrightarrow p$ *is exponentially stable;*
b) $\beta + 3\gamma < 0 \Longrightarrow p$ *is simply unstable.*

Proof. The first assertion follows from $\bar{u} < \bar{u}_1$ (Proposition 4.6) and from Lemma 4.3.

Ad a): Assume $\beta+3\gamma > 0$ and $z^2 \leq \bar{u}$. The corresponding solution p has amplitude z, so for all $(x, y) \in \mathcal{O}$ one has

$$\max\{x^2, y^2\} \leq z^2 \leq \bar{u} \leq \min\{\bar{u}_2, \bar{u}_3\}.$$

The definitions of $\bar{u}_2$ and $\bar{u}_3$ and (4.5), (4.6) and (4.7) together now show that $\mathcal{O} \subset Q^- \cap ND \cap NF$. The stability assertion follows from Theorem 4.4.

Ad b): Assume now $\beta + 3\gamma < 0$ and $z^2 \leq \bar{u}$. Analogous to the proof of a), one obtains now $\mathcal{O} \subset Q^+ \cap ND \cap NF$, and the instability statement follows from Theorem 4.4. □

Remarks. 1. Note that Corollary 4.7 is in accordance with the local result from Corollary 5.3, p. 80 of [7], which described stability of KY solutions in dependence of the sign of $\beta + 3\gamma$.

2. We do not consider the cases $\beta = 0 \neq \gamma$ and $\beta \neq 0 = \gamma$ (which would be covered by our methods as well), since in both cases f has the product structure $f(x, y) = G(x)H(y)$ which is treated in Section 3 of [8].

3. We comment on some cases not covered by the above corollary: If $\beta+3\gamma = 0$ then the hyperbolas bounding Q^+ and Q^- degenerate to lines through $(0, 0)$, and it is impossible for the orbit $\mathcal{O}$ (which winds once around $(0, 0)$) to be contained in one of these sets. Hence Theorem 4.4 cannot be applied in this case. Also, if $\beta + 3\gamma$ is positive (negative), then Q^+ (Q^-) is disconnected, and therefore we cannot derive an instability (stability) result from Theorem 4.4, as $\mathcal{O} \subset Q^+$ (Q^-) is impossible.

The conditions of Corollary 4.5 and Corollary 4.7 still involve the amplitude z of periodic solutions and are thus not completely explicit in terms of the parameters α, β, γ. We want to obtain statements of the type 'inequalities on α, β, γ imply existence and stability/instability of periodic solutions'. To this end it is necessary to at least estimate the function $z \mapsto \alpha^*(z, \beta, \gamma)$, and then to replace inequalities on z from the above statements by inequalities on α. We pursue this in Section 5.

5. Explicit criteria in terms of parameters

For $l \in (-1, \infty)$, set

$$\tilde{K}(l) := \int_0^{\pi/2} \frac{d\psi}{\sqrt{1 + l\sin^2(\psi)}}.$$

The relation of $\tilde{K}$ to the complete elliptic integral K of the first kind is that if $l = -k^2$ then $\tilde{K}(l) = K(k)$.

Proposition 5.1. *Let $\beta, \gamma \in \mathbb{R}$, $(\beta, \gamma) \neq (0, 0)$. With $l = l(z, \beta, \gamma)$ as in Prop. 4.1 and for $z \in (0, z^*(\beta, \gamma))$, one has*

$$\alpha^*(z, \beta, \gamma) = \frac{1}{1 - \gamma z^2}\tilde{K}(l). \tag{5.1.1}$$

Further, $\lim_{z \to 0} \alpha^*(z, \beta, \gamma) = \pi/2$.

Proof. In the notation from the proof of Proposition 4.1, the function α^* satisfies $\psi(\alpha^*(z, \beta, \gamma)/2) = \pi/2$ (for $z \in (0, z^*(\beta, \gamma))$). Using formula (4.1.2) and recalling that $\dot{\psi} > 0$ for the solutions in question, we see that

$$\frac{\alpha^*(z, \beta, \gamma)}{2} = \int_0^{\alpha^*(z,\beta,\gamma)/2} 1\, dt = \int_0^{\alpha^*(z,\beta,\gamma)/2} \frac{\dot{\psi}(t)}{2(1 - \gamma z^2)\sqrt{1 + l\sin^2(\psi(t))}}\, dt$$

$$= \int_0^{\pi/2} \frac{d\psi}{2(1 - \gamma z^2)\sqrt{1 + l\sin^2(\psi)}} = \frac{1}{2(1 - \gamma z^2)}\tilde{K}(l).$$

Formula (5.1.1) follows. The last assertion is a consequence of $\tilde{K}(0) = \pi/2$ and of $l(z, \beta, \gamma) \to 0$ as $z \to 0$. □

We extend the definition of α^* continuously by setting $\alpha^*(0, \beta, \gamma) := \pi/2$.

Next, we recall the notion of the arithmetic-geometric mean $\mathrm{agm}(a, b)$ of two positive numbers a, b:

$$\mathrm{agm}(a, b) = \lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n,$$

where the sequences (a_n) and (b_n) are recursively defined by (see, e.g., [1]).

$$a_0 := a,\ b_0 := b,\ a_{n+1} := \frac{a_n + b_n}{2},\ b_{n+1} := \sqrt{a_n b_n}. \tag{5.1}$$

One has

$$b_n \leq b_{n+1} \leq \mathrm{agm}(a, b) \leq a_{n+1} \leq a_n \text{ for } n \geq 1, \tag{5.2}$$

as follows easily from $b_1 \leq a_1$ by induction. The homogeneity property $\mathrm{agm}(\lambda a, \lambda b) = \lambda\,\mathrm{agm}(a, b)$ is obvious.

Proposition 5.2. $\lim_{x\to 0} \operatorname{agm}(1,x) = 0$ *and* $\lim_{x\to\infty} \dfrac{\operatorname{agm}(1,x)}{x} = 0$.

Proof. Define the sequences $(a_n) = (a_n(x)), (b_n) = (b_n(x))$ corresponding to $a = 1, b = x$ as above. It is easy to see inductively that, for all $n \in \mathbb{N}$, $b_n(x) \to 0$ and $a_n(x) \to 2^{-n}$ as $x \to 0$. Let $\varepsilon > 0$. There exists $n \in \mathbb{N}$ with $2^{-n} < \varepsilon/2$. There exists $\bar{x} > 0$ such that $\forall x \in (0, \bar{x}] : a_n(x) < 2^{-n} + \varepsilon/2$. For such x, we conclude from (5.2) that $\operatorname{agm}(1,x) \le a_n(x) < \varepsilon/2 + \varepsilon/2 = \varepsilon$, which establishes the first limit. The second follows from $\operatorname{agm}(1,x) = x \cdot \operatorname{agm}(1/x, 1) = x \cdot \operatorname{agm}(1, 1/x) = \dfrac{\operatorname{agm}(1,1/x)}{1/x}$. □

The following relation between the integral $\tilde{K}$ and the agm, which was essentially already known to Gauß (see [3]), will be helpful for us to approximate the function α^*.

Proposition 5.3. *For* $l \in (-1,\infty)$, *one has* $\tilde{K}(l) = \dfrac{\pi/2}{\operatorname{agm}(1,\sqrt{1+l})}$.

Proof. For $l \in (-1,0), l = -k^2$ with $k \in (0,1)$, this follows, e.g., from formula (1.7) in [3]. For $l \ge 0$, the formula then follows from the fact that both sides of the asserted equation are (real) analytic functions on $(-1,\infty)$. We give a proof that does not depend on this fact: For $l \ge 0$ one has $\operatorname{agm}(1,\sqrt{1+l}) = \sqrt{1+l}\operatorname{agm}(1, 1/\sqrt{1+l})$, and $1/\sqrt{1+l} = \sqrt{1+\tilde{l}}$, where $\tilde{l} \in (-1,0)$ satisfies $1+\tilde{l} = 1/(1+l)$, so $\tilde{l} = -l/(1+l)$. Now, using the result of the proposition for $\tilde{l}$, observing that $\tilde{l}(1+l) = -l$, and substituting ψ by $\pi/2 - \psi$, one obtains

$$\begin{aligned}
\frac{\pi/2}{\operatorname{agm}(1,\sqrt{1+l})} &= \frac{\pi/2}{\sqrt{1+l}\operatorname{agm}(1,\sqrt{1+\tilde{l}})} = \frac{1}{\sqrt{1+l}} \int_0^{\pi/2} \frac{d\psi}{\sqrt{1+\tilde{l}\sin^2(\psi)}} \\
&= \int_0^{\pi/2} \frac{d\psi}{\sqrt{1+l+\tilde{l}(1+l)\sin^2(\psi)}} = \int_0^{\pi/2} \frac{d\psi}{\sqrt{1+l\cos^2(\psi)}} \\
&= \int_0^{\pi/2} \frac{d\psi}{\sqrt{1+l\sin^2(\psi)}} = \tilde{K}(l).
\end{aligned}$$

□

In order to estimate α^*, it is necessary to estimate $\tilde{K}$. Here the relation of $\tilde{K}$ to the agm, together with (5.1) and (5.2), provides a simple method:

Proposition 5.4. *For* $l \in (-1,\infty)$, *one has with* $w := (1+l)^{1/4}$ *the estimates*

$$\frac{2\pi}{(w+1)^2} \le \tilde{K}(l) \le \frac{\pi}{\sqrt{2w(1+w^2)}}.$$

Proof. Applying the agm iteration (see (5.1)), we obtain from (5.2) for $n = 1$ that

$$(1+l)^{1/4} \le \operatorname{agm}(1, \sqrt{1+l}) \le \frac{1+\sqrt{1+l}}{2}.$$

With the above definition of w, this reads as

$$w \le \operatorname{agm}(1, \sqrt{1+l}) \le \frac{1+w^2}{2}.$$

For $n = 2$, we obtain

$$\sqrt{w\frac{1+w^2}{2}} \le \operatorname{agm}(1, \sqrt{1+l}) \le \frac{2w+1+w^2}{4} = \frac{(w+1)^2}{4}.$$

The assertion now follows from Proposition 5.3. □

Now we estimate α^*:

Corollary 5.5. *Let $\beta, \gamma \in \mathbb{R} \setminus \{0\}$, and set $q := \beta/\gamma$. Then for $u \in (0, u^*(\beta,\gamma))$ one has $\frac{1+q}{2} + \frac{1-q}{2(1-\gamma u)^2} > 0$. Setting*

$$w := w(u,\beta,\gamma) := \left[\frac{1+q}{2} + \frac{1-q}{2(1-\gamma u)^2}\right]^{1/4},$$

one has for $z \in (0, z^*(\beta,\gamma))$ with $u := z^2$ and $w := w(u,\beta,\gamma)$ the inequalities

$$\frac{2\pi}{(1-\gamma u)(w+1)^2} \le \alpha^*(z,\beta,\gamma) \le \frac{\pi}{(1-\gamma u)\sqrt{2w(1+w^2)}}. \tag{5.5.1}$$

Proof. Assume $z \in (0, z^*(\beta,\gamma))$. With $l = l(z,\beta,\gamma)$ as in Proposition 4.1 and $q := \beta/\gamma$, $u := z^2$, $v := 1-\gamma u$ we then have $l \in (-1,\infty)$, $v > 0$, $u = (1-v)/\gamma$, and

$$l = \frac{u(2-\gamma u)(\gamma-\beta)}{2(1-\gamma u)^2} = \frac{(1-v)(1+v)(1-q)}{2v^2} = \frac{(1-q)(1-v^2)}{2v^2}.$$

Now $1 + l = \dfrac{(1+q)v^2 + 1 - q}{2v^2} = \dfrac{1+q}{2} + \dfrac{1-q}{2v^2}$, hence the last expression is positive, and the definition of w means that $w = (1+l)^{1/4}$. Estimate (5.5.1) is a consequence of Proposition 5.4 and (5.1.1). □

With $\Lambda(u) := \dfrac{u(2-\gamma u)}{(1-\gamma u)^2}$ (for all $u \in \mathbb{R}$ with $\gamma u \ne 1$), the function l from Prop. 4.1 takes the form $l(z,\beta,\gamma) = \frac{\gamma-\beta}{2}\Lambda(z^2)$. Thus, setting

$$A_1(u) := \frac{1}{1-\gamma u},\ A_2(u) := \tilde{K}\left(\frac{\gamma-\beta}{2}\Lambda(u)\right),\ A(u) := A_1(u)A_2(u),$$

we obtain from Prop. 5.1 that

$$\alpha^*(z,\beta,\gamma) = A(z^2).$$

Note also that with $v := 1 - \gamma u$ we have for $\gamma \neq 0 \neq v$, and setting $q := \beta/\gamma$, the identities

$$A_1(u) = \frac{1}{v}\,, \quad A_2(u) = \tilde{K}[\frac{(1-q)(1-v^2)}{2v^2}]. \tag{5.3}$$

It is convenient to discuss the function A instead of α^*. We collect some properties of the functions $\tilde{K}$ and A in the following statement:

Proposition 5.6. *a) The function $\tilde{K}$ satisfies*

$$\tilde{K}'(l) = -\frac{1}{2}\int_0^{\pi/2} \frac{\sin^2(\psi)}{[1 + l\sin^2(\psi)]^{3/2}}\, d\psi < 0 \text{ for } l \in (-1, \infty), \tag{5.6.1}$$

$$\tilde{K}'(0) = -\pi/8. \tag{5.6.2}$$

$$\forall\, c \geq 0: \quad \lim_{v\to 0} \frac{1}{v}\,\tilde{K}[c\frac{1-v^2}{v^2}] = \infty; \quad \forall\, c \in [0,1]: \quad \lim_{v\to\infty} \frac{1}{v}\,\tilde{K}[c\frac{1-v^2}{v^2}] = 0. \tag{5.6.3}$$

$$\lim_{l\to -1} \tilde{K}(l) = \infty. \tag{5.6.4}$$

b) Assume that $u > 0$ is such that $1 - \gamma u > 0$, and $l = \frac{\gamma-\beta}{2}\Lambda(u) > -1$. Then

$$\begin{aligned} A'(u) &= \frac{1}{(1-\gamma u)^2}\left[\frac{(\gamma-\beta)\tilde{K}'(l)}{(1-\gamma u)^2} + \gamma\tilde{K}(l)\right] \\ &= \frac{1}{(1-\gamma u)^2}\int_0^{\pi/2} \{\sigma(\psi) + \gamma\}\,\frac{1}{\sqrt{1 + l\sin^2(\psi)}}\, d\psi, \end{aligned} \tag{5.6.5}$$

where $\sigma(\psi) := \dfrac{(\beta-\gamma)\sin^2(\psi)}{2(1-\gamma u)^2(1 + l\sin^2(\psi))}$. *Further, one has*

$$A(0) = \pi/2,\ A'(0) = \frac{3\gamma+\beta}{8}\pi. \tag{5.6.6}$$

c) If $0 \neq |\gamma| \geq |\beta|$ then

$$\forall u \in [0, u^*):\ 1 - \gamma u > 0,\ \textit{and}\ \operatorname{sign}(A'(u)) = \operatorname{sign}(\gamma). \tag{5.6.7}$$

Proof. Ad a): The formula for $\tilde{K}'(l)$ is obvious, and $\tilde{K}'(0) = -\frac{1}{2}\int_0^{\pi/2} \sin^2(\psi)\, d\psi = -\pi/8$. Next,

$$|\tilde{K}'(-1/2)| = \frac{1}{2}\int_0^{\pi/2} \frac{\sin^2(\psi)}{[1 - \frac{1}{2}\sin^2(\psi)]^{3/2}}\, d\psi \leq \frac{1}{2}\int_0^{\pi/2} \frac{\sin^2(\psi)}{(1/2)^{3/2}}\, d\psi = \frac{\pi\sqrt{2}}{4}.$$

Assume now $c \geq 0$. The first assertion in (5.6.3) is clear if $c = 0$. Assume now $c > 0$, and note that $\tilde{K}$ is decreasing. For $v \in (0, 1]$ sufficiently small so that $1 \leq c/v^2$, one obtains from Proposition 5.3 that

$$\begin{aligned}\tilde{K}[c\frac{1-v^2}{v^2}] &\geq \tilde{K}(c/v^2) \\ &= \frac{\pi/2}{\operatorname{agm}(1, \sqrt{1+(c/v^2)})} \geq \frac{1}{\operatorname{agm}(1, \sqrt{2c/v^2})} = \frac{1}{\operatorname{agm}(1, \sqrt{2c}/v)}\end{aligned}$$

We conclude (for such v)

$$\frac{1}{v}\,\tilde{K}[c\frac{1-v^2}{v^2}] \geq \frac{1}{v\operatorname{agm}(1, \sqrt{2c}/v)} = \frac{1}{\sqrt{2c}} \cdot \frac{1}{\operatorname{agm}(v/\sqrt{2c}, 1)}.$$

Now the first limit in (5.6.3) follows from Proposition 5.2.

Assume now $c \in [0, 1]$. If $c < 1$ then $\tilde{K}[c\frac{1-v^2}{v^2}] \to \tilde{K}(-c)$ as $v \to \infty$, and the limit assertion follows. In case $c = 1$, one has

$$\begin{aligned}\frac{1}{v}\tilde{K}[\frac{1-v^2}{v^2}] &= \frac{1}{v}\frac{\pi/2}{\operatorname{agm}(1, \sqrt{1+(1/v^2)-1})} = \frac{\pi/2}{v\operatorname{agm}(1, 1/v)} \\ &= \frac{\pi/2}{\operatorname{agm}(1, v)} \to 0 \ (v \to \infty),\end{aligned}$$

since $\operatorname{agm}(1, v) \geq \sqrt{v}$. The second limit assertion in (5.6.3) is proved. The assertion of (5.6.4) follows from Proposition 5.3 and from the first limit in Proposition 5.2.

Ad b): First,

$$\Lambda'(u) = \frac{2(1-\gamma u)(1-\gamma u)^2 - u(2-\gamma u)2(1-\gamma u)(-\gamma)}{(1-\gamma u)^4} = \frac{2}{(1-\gamma u)^3}.$$

Now we calculate

$$\begin{aligned}A'(u) &= A_1'(u)A_2(u) + A_1(u)A_2'(u) \\ &= \frac{\gamma}{(1-\gamma u)^2}A_2(u) + A_1(u)\frac{\gamma-\beta}{2}\tilde{K}'(\tfrac{\gamma-\beta}{2}\Lambda(u))\Lambda'(u) \\ &= \frac{\gamma}{(1-\gamma u)^2}\tilde{K}(l) + \frac{1}{(1-\gamma u)}\frac{\gamma-\beta}{2}\tilde{K}'(l)\frac{2}{(1-\gamma u)^3} \\ &= \frac{1}{(1-\gamma u)^2}\left[\frac{(\gamma-\beta)\tilde{K}'(l)}{(1-\gamma u)^2} + \gamma\tilde{K}(l)\right],\end{aligned}$$

which is the first equality in (5.6.5). The bracket in the above expression satisfies

$$\begin{aligned}
[...] &= \frac{\beta-\gamma}{2(1-\gamma u)^2}\int_0^{\pi/2}\frac{\sin^2(\psi)}{[1+l\sin^2(\psi)]^{3/2}}\,d\psi+\gamma\int_0^{\pi/2}\frac{1}{\sqrt{1+l\sin^2(\psi)}}\,d\psi\\
&= \int_0^{\pi/2}\left\{\frac{(\beta-\gamma)\sin^2(\psi)}{2(1-\gamma u)^2(1+l\sin^2(\psi))}+\gamma\right\}\frac{1}{\sqrt{1+l\sin^2(\psi)}}\,d\psi\\
&= \int_0^{\pi/2}\{\sigma(\psi)+\gamma\}\,\frac{1}{\sqrt{1+l\sin^2(\psi)}}\,d\psi,
\end{aligned}$$

which proves the second equality in (5.6.5). We have $A(0)=\tilde{K}(0)=\pi/2,\ A'(0)=(\gamma-\beta)(-\pi/8)+\gamma\pi/2=(\beta+3\gamma)\pi/8$, which proves (5.6.6).

Ad c): Assume $0\neq|\gamma|\geq|\beta|$, and $u\in[0,u^*)$. The property $1-\gamma u>0$ follows from the definition of u^* in Prop. 4.2 and from the second inequality in (4.1.1).

In view of the second equality in (5.6.5), it suffices to prove the following result:

$$|\sigma(\psi)|\leq|\gamma|,\ \text{with strict inequality for } \psi\in[0,\pi/2). \tag{5.6.8}$$

Recall that $l>-1$ always, and note that $\dfrac{w}{1+lw}\leq\dfrac{1}{1+l}$ for $w\in[0,1]$, with strict inequality except for $w=1$. With $v:=1-\gamma u$ and $q:=\beta/\gamma$ we have $u(2-\gamma u)=\frac{1}{\gamma}(1-v)(1+v)=(1-v^2)/\gamma$, and we conclude that for $\psi\in[0,\pi/2]$

$$\begin{aligned}
|\sigma(\psi)| &\leq \left|\frac{(\beta-\gamma)}{2v^2(1+l)}\right| = \frac{|\beta-\gamma|/2}{\left|v^2+\frac{\gamma-\beta}{2}u(2-\gamma u)\right|} = \frac{|q-1|/2}{\left|v^2+\frac{1-q}{2}(1-v^2)\right|}|\gamma|\\
&= \frac{|q-1|/2}{\left|v^2(1+\frac{q-1}{2})+\frac{1-q}{2}\right|}|\gamma|.
\end{aligned} \tag{5.6.9}$$

Now $|\gamma|\geq|\beta|$ implies $|q|\leq 1$ and $|q-1|/2\leq 1$, so that the coefficient of v^2 in (5.6.9) and the term $\frac{1-q}{2}$ are both nonnegative. It follows that $|\sigma(\psi)|\leq|\gamma|$. This inequality is strict except if $\sin\psi=1$, i.e., $\psi=\pi/2$. Claim (5.6.8) is proved. □

It is common to describe the Kaplan–Yorke branch of periodic solutions (the set of all KY solutions) by a curve in $\mathbb{R}^2$ which in our situation is the set

$$\Big\{(\alpha(z,\beta,\gamma),z)\ \Big|\ z\in[0,z^*(\beta,\gamma))\Big\}$$

(see, e.g., [5]). In the following two theorems (one for the elliptic and one for the hyperbolic case), we describe this 'branch' globally.

Theorem 5.7. *a) If $\beta,\gamma<0$ then $\alpha^*(\cdot,\beta,\gamma)$ is defined and strictly decreasing on $[0,\infty)$,*

with $\lim_{z\to\infty} \alpha^*(z,\beta,\gamma) = 0$.
b) If $\beta,\gamma > 0$ *then* $\alpha^*(\cdot,\beta,\gamma)$ *is strictly increasing on* $[0, z^*(\beta,\gamma))$*, with* $\lim_{z\to z^*} \alpha^*(z,\beta,\gamma) = \infty$.

Proof. Ad a): Assume $\beta,\gamma < 0$. Then, from Prop. 4.2, we have $z^*(\beta,\gamma) = \infty$. Since $\gamma < 0$, we have $\lim_{u\to\infty} A_1(u) = 0$. Further, $\frac{\gamma-\beta}{2}\Lambda(u) \to \frac{\gamma-\beta}{2}(-1/\gamma) = -\frac{1}{2} + \frac{\beta}{2\gamma} > -1/2$ as $u \to \infty$, and hence $A_2(u)$ remains bounded as $u \to \infty$. It follows that $\lim_{z\to\infty} \alpha^*(\cdot,\beta,\gamma) = 0$. For the strict monotonicity, it suffices to prove that $A' < 0$ on $[0,\infty)$.

First case: $0 > \gamma \geq \beta$. Then, since $\tilde{K}' < 0$ (see (5.6.1)) and $\tilde{K} > 0$ on $(-1,\infty)$, one sees from the first equality in (5.6.5) that $A' < 0$.

Second case: $0 > \beta > \gamma$. Then $|\gamma| > |\beta|$, and (5.6.7) shows $A' < 0$.

Ad b): If $\beta,\gamma > 0$ then A is defined on $[0,u^*)$, where $u^* = u_-$ if $\beta \geq \gamma$ and $u^* = 1/\gamma$ if $\beta < \gamma$.
First case: $\beta \geq \gamma > 0$. Using $\tilde{K}' < 0$, $\tilde{K} > 0$, it follows immediately from (5.6.5) that $A' > 0$ on $(0,u^*)$, which implies the monotonicity assertion. If $\beta = \gamma$ then A_2 is constant, $u^* = 1/\gamma$, and $A_1(u) \to \infty$ as $u \to 1/\gamma$, which implies the limit assertion. If $\beta > \gamma$ then $z^* = \sqrt{u_-}$ and $1/(1-\gamma z^2) > 1$ for $z \in (0,z^*)$. Further, $l(z,\beta,\gamma) \to -1$ as $z \to z^*$, and we know from (5.6.4) that $\tilde{K}(l) \to \infty$ as $l \to -1$. The limit assertion follows from (5.1.1).

Second case: $0 < \beta < \gamma$. Then $u^*(\beta,\gamma) = 1/\gamma$ (see Prop. 4.2), and (5.6.7) shows $A' > 0$, which proves the monotonicity. Further, we have $q = \beta/\gamma < 1$, and hence $c := (1-q)/2 > 0$. In the notation of (5.3), one sees that $A(u) = \dfrac{1}{v}\tilde{K}[c\dfrac{1-v^2}{v^2}]$, and $v \to 0$ as $u \to u^* = 1/\gamma$. The limit assertion follows from (5.6.3). □

Theorem 5.8. *Assume* $\beta\gamma < 0$.

a) $\alpha^*(\cdot,\beta,\gamma)$ *is strictly increasing or decreasing on a subinterval* $[0,z_1)$ *of* $[0,z^*)$*, depending on whether* $\beta + 3\gamma > 0$ *or* $\beta + 3\gamma < 0$.

b) If $\gamma > 0$ *or* $\gamma < 0$ *and* $\beta > |\gamma|$ *then* $z^* = z^*(\beta,\gamma)$ *is finite and* $\lim_{z\to z^*} \alpha^*(z,\beta,\gamma) = \infty$.

c) If $\gamma < 0$, $|\gamma| \geq |\beta|$ *then* $z^* = \infty$ *and* $\lim_{z\to\infty} \alpha^*(z,\beta,\gamma) = 0$.

d) If $|\gamma| \geq \beta$ *then* $\alpha^*(\cdot,\beta,\gamma)$ *is strictly monotone (increasing to* ∞ *if* $\gamma > 0$*, decreasing to zero if* $\gamma < 0$*).*

Proof. Assertion a) follows from the formula for $A'(0)$ in (5.6.6).

Ad b): Assume $\gamma > 0$ first, so $\beta < 0 < \gamma$ and (see Prop. 4.2) $u^* = 1/\gamma$, $z^* = \sqrt{1/\gamma}$. We have $q = \beta/\gamma < 0$, and $c := (1-q)/2 > 0$. Setting $v = 1 - \gamma u$, we know that $A(u) = \dfrac{1}{v}\tilde{K}[c\dfrac{1-v^2}{v^2}]$ for $u \in (0,u^*)$. Since $v \to 0$ as $u \to u^*$, the limit assertion follows from the first limit in (5.6.3).

Assume now $\gamma < 0$ and $|\gamma| < \beta$. Then (see Prop. 4.2) $u^* = u_- < 1/\gamma$, so z^* is finite, and $\dfrac{1}{1-\gamma z^2} \to \dfrac{1}{1-\gamma u_-} > 0$ as $z \to z^*$. Further, the construction of u_- and the equality $u^* = u_-$ imply $l(z,\beta,\gamma) \to -1$ as $z \to z^*$. Again, from (5.6.4) we have $\tilde{K}(l) \to \infty$ as $l \to -1$, and the limit assertion follows from (5.1.1).

Ad c): Assume $\gamma < 0$, $|\gamma| \geq \beta$. Then, from Prop. 4.2, $u^* = \infty = z^*$, and $v = 1-\gamma u$ satisfies $v \to \infty$ as $u \to \infty$. The limit assertion follows from the second limit in (5.6.3).

Ad d): The monotonicity statements follow from part c) of Proposition 5.6, and the limits are clear from b) and c) of the present theorem.

□

Remarks 1. In case that $\gamma > 0 > \beta$ and $\beta + 3\gamma < 0$, we have backward bifurcation (i.e., $\alpha^*(\cdot,\beta,\gamma)$ is decreasing first), according to Theorem 5.8, a), but $\alpha^*(z,\beta,\gamma) \to \infty$ as $z \to z^*$ (according to part b)), so the branch turns forward again. It seems that in this case there is always precisely one minimum of $\alpha^*(\cdot,\beta,\gamma)$ (corresponding to a saddle-node-bifurcation, where a stable and an unstable KY solution annihilate each other), but we have not proved this.

2. In the cases of forward bifurcation ($\beta + 3\gamma > 0$) where $|\gamma| < |\beta|$, i.e., $-3\gamma < \beta < -\gamma < 0 < \gamma$ or $\beta > -3\gamma > 0 > \gamma$, the convergence of $\alpha^*(\cdot,\beta,\gamma)$ to ∞ seems to be always monotone, but we have not proved this.

Finally, we state two theorems on stability (for the 'elliptic' and the 'hyperbolic' case).

Theorem 5.9. *a) If $\beta,\gamma < 0$ then for every $\alpha \in (0,\pi/2)$ there exists a unique KY solution p of eq.* (4.1) *with the properties that $p(1) = z$ and that $\alpha = \alpha^*(z,\beta,\gamma)$, and this solution is simply unstable.*

b) Assume $\beta,\gamma > 0$. Then with $\bar{u}$ from Corollary 4.5 and $w := w(\bar{u},\beta,\gamma)$ as in Corollary 5.5, and with

$$\bar{\alpha} := \frac{2\pi}{(1-\gamma\bar{u})(w+1)^2},$$

the following is true: For every $\alpha \in (\pi/2,\bar{\alpha})$ there exists a unique KY solution p of eq. (4.1) *with the properties that $p(1) = z \in (0,\sqrt{\bar{u}})$ and $\alpha = \alpha^*(z,\beta,\gamma)$, and this solution is exponentially stable.*

Proof. Ad a): Assume $\beta,\gamma < 0$. From Theorem 5.7a), we see that $\alpha^*(\cdot,\beta,\gamma)$ decreases strictly from $\pi/2$ to 0 on $[0,\infty)$. Thus, for $\alpha \in (0,\pi/2)$ there exists a unique $z \in [0,\infty)$ with $\alpha^*(z,\beta,\gamma) = \alpha$. Proposition 4.1 provides a corresponding KY solution p with $p(1) = z$, obtained from the solution (x^z, y^z) of (4.3) with initial value $(0,-z)$. Corollary 4.5, b) shows that p is simply unstable.

Proof of uniqueness: Every KY solution $\tilde{p}$ of (4.1) with the same α and with $\tilde{p}(1) = z$ defines, according to Lemma 2.2,a), a solution $(\tilde{x},\tilde{y})$ of (4.3) with initial value $(0,-z)$, and then necessarily $(\tilde{x},\tilde{y}) = (x,y)$. It follows that $\tilde{p} = p$.

Ad b): Assume now $\beta, \gamma > 0$. We know from Corollary 4.5 that $\bar{u} < u^*(\beta, \gamma)$, so with $\bar{z} := \sqrt{\bar{u}}$ we have that $\alpha^*(\cdot, \beta, \gamma)$ is defined on $[0, \bar{z}]$, and Corollary 5.5 shows that $\alpha^*(\bar{z}, \beta, \gamma) \geq \bar{\alpha}$. It follows from the intermediate value theorem that the values of $\alpha^*(\cdot, \beta, \gamma)$ on $[0, \bar{z})$ contain the interval $[\pi/2, \bar{\alpha})$. Further, we get from Theorem 5.7, b) that $\alpha^*(\cdot, \beta, \gamma)$ is strictly increasing. Now, for $\alpha \in [\pi/2, \bar{\alpha})$ existence and uniqueness of the KY solutions with amplitude z such that $\alpha = \alpha^*(\cdot, \beta, \gamma)$ follows as in the proof of a). Corollary 4.5, a) shows the exponential stability, since the $z-$values of these solutions satisfy $z^2 < \bar{u}$. □

In the hyperbolic case, the 'branch' of pairs (α, z) mentioned before Theorem 5.7 describes a backward bifurcation if $\beta + 3\gamma < 0$, but can then turn in the 'forward' direction again (i.e., $\alpha(z, \beta, \gamma) < \pi/2$ for small $z-$values, but grows beyond $\pi/2$ for larger z). This is why in the following counterpart to Theorem 5.9 we have in general no uniqueness statement for the KY solutions. (The proof of Theorem 5.10 uses arguments similar to the ones used above.)

Theorem 5.10. *Assume $\beta\gamma < 0$, define $\bar{u}$ by Table 3, and set $w := w(\bar{u}, \beta, \gamma)$ (see Corollary 5.5).*
a) If $\beta + 3\gamma < 0$ then set $\bar{\alpha} := \dfrac{\pi}{(1 - \gamma\bar{u})\sqrt{2w(1 + w^2)}}$.
If $\pi/2 > \bar{\alpha}$ then for every $\alpha \in [\bar{\alpha}, \pi/2)$ there exists a KY solution p of eq. (4.1) with amplitude $z \in (0, \sqrt{\bar{u}}]$ such that $\alpha = \alpha^*(z, \beta, \gamma)$. This solution is simply unstable.
b) If $\beta + 3\gamma > 0$ then set $\bar{\alpha} := \dfrac{2\pi}{(1 - \gamma\bar{u})(w + 1)^2}$. If $\pi/2 < \bar{\alpha}$ then for every $\alpha \in (\pi/2, \bar{\alpha}]$ there exists a KY solution p of eq. (4.1) with amplitude $z \in (0, \sqrt{\bar{u}})$ such that $\alpha = \alpha^*(z, \beta, \gamma)$, and this solution is exponentially stable.
c) If $|\gamma| \geq |\beta|$ then, for given α, the values $z \in (0, \sqrt{\bar{u}})$ with alpha $= \alpha^*(z, \beta, \gamma)$ and the KY solutions described in b) and c) with amplitude z are unique.

Proof. Ad a): Assume that $\beta + 3\gamma < 0$ and that $\pi/2 > \bar{\alpha}$. It follows from the second estimate in (5.5.1) that with $\bar{z} := \sqrt{\bar{u}}$ one has $\alpha^*(\bar{z}, \beta, \gamma) \leq \bar{\alpha}$, and hence $\alpha^*(0, \beta, \gamma) = \pi/2$ implies that the values of $\alpha^*(\cdot, \beta, \gamma)$ on $[0, \bar{u}]$ contain the interval $[\bar{\alpha}, \pi/2]$. The statement on the existence of the KY solutions follows. Since these solutions have amplitudes z with $z^2 \leq \bar{u}$, the instability follows from $\beta + 3\gamma < 0$ and from Corollary 4.7, b).

The proof of b) is analogous, using the first estimate in (5.5.1), the inequality $\beta + 3\gamma > 0$, and Corollary 4.7, a).

Finally, c) follows from Theorem 5.8,d). □

6. Numerical observations and examples

We briefly describe numerical observations in connection with the results of the previous sections. The approximation of the function α^*, as given by (5.5.1), seems

to be quite satisfactory for all parameters β and γ. It was based on the second iterate of the agm recursion scheme. A corresponding formula based on the first iterate gave a satisfactory result only for certain parameter ranges; the second iterate appears to be a good combination of simplicity and accuracy.

In the elliptic case $\beta\gamma > 0$, the stability criterion of Corollary 4.5, a) does not describe the maximal value of $u = z^2$ for which the solution (x^z, y^z) is still contained in the 'stability region' $Q^- \cap ND \cap NF$ too exactly. The reason lies in the 'worst case estimates' of the type $c_1x^2 + c_2y^2 \leq (c_1 + c_2)z^2$ (if $\max\{x^2, y^2\} \leq z^2$, compare (4.5.1)). Geometrically, these estimates mean that the requirement $\mathcal{O} \subset Q^-$ is replaced by a stronger condition that the square with corners $(\pm z, \pm z)$ be contained in Q^-. (Compared to the 'loss' in these estimates, the inaccuracy of the approximation of α^* is negligible.)

There is no obvious possibility to improve this in general, because the solutions (x^z, y^z) can have orbits close to this square, in which case the mentioned estimate becomes sharp.

As a consequence, the value of $\bar{\alpha}$ in Theorem 5.9 is, in general, significantly below the maximal value of the parameter α at which the condition $\mathcal{O} \subset Q^- \cap ND \cap NF$ fails to hold. For example, if $\beta = 3$ and $\gamma = 1$, one has $\bar{\alpha} \approx 2.17$, and the $\alpha-$value where the KY solution stops being contained in the region $ND \cap Q^-$ is approximately 2.79.

In the hyperbolic case, the situation is somewhat better: Recall that our solutions (x^z, y^z) always pass through the points $(0, -z), (z, 0)$, etc., and satisfy $\max\{(x^z(t))^2, (y^z(t))^2\} \leq z^2$ for all t. In view of the position of the hyperbola branches bounding Q^- (in case $\beta + 3\gamma > 0$) or Q^+ (in case $\beta + 3\gamma < 0$), namely, symmetric to the $y-$axis or to the $x-$ axis, the requirement $\mathcal{O} \subset Q^-$ (or $\subset Q^+$) is equivalent to having the already mentioned square with corners $(\pm z, \pm z)$ in this region, this time with no loss of information. In combination with the rather accurate approximation of α^* from (5.5.1), the value of $\bar{\alpha}$ in Theorem 5.10 then describes essentially the exact $\alpha-$value up to which the property $\mathcal{O} \subset Q^{\pm} \cap ND \cap NF$ holds. Concrete examples are $\beta = 3$ and $\gamma = -0.5$ where $\bar{\alpha}$ from Theorem 5.10 b) satisfies $\bar{\alpha} \approx 1.71$, or $\beta = 3$ and $\gamma = -2$, where $\bar{\alpha}$ (this time from Theorem 5.10 a) satisfies $\bar{\alpha} \approx 1.50$.

The numbers from these examples also give an impression of the degree to which our results go beyond a local stability analysis at the bifurcation value $\alpha(0, \beta, \gamma) = \pi/2 \approx 1.57$.

Recall that all stability and instability results are based on Lemma 3.1, the conditions of which are sufficient, but not necessary. Therefore, even in the hyperbolic case, we cannot expect to be close to actual borders of stability. In the above example with $\beta = 3$ and $\gamma = 1$, where the KY solution is in the 'stability region' up to $\alpha \approx 2.79$, it actually seems to remain stable up to $\alpha \approx 4.2$ (this was found by simple forward integration of the equation, starting with a slightly perturbed

initial segment of the KY solution). The corresponding values of z^2 are $z^2 \approx 0.25$ and $z^2 \approx 0.28$, so one sees that (in this case) an actual stability loss happens at values where the solution is not too far outside the region which gives a sufficient condition for stability.

For $\beta = -3$ and $\gamma = 2$, where Theorem 5.10 gives $\bar{\alpha} \approx 1.78$, application of the same simple test gives stability up to $\alpha \approx 2.16$. Apparently chaotic (e.g., $\alpha \approx 3.2$) and finally unbounded oscillations are seen for larger $\alpha-$values.

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