On a High Order Differential Delay Equation

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1. Introduction

Consider the differential delay equation

\[
\left( e_m \frac{d}{dt} + 1 \right) \cdots \left( e_0 \frac{d}{dt} + 1 \right) y(t) = f(y(t-1)),
\]

where \( f \in C(\mathbb{R}, \mathbb{R}) \) and \( e = (e_0, \ldots, e_m) \in \mathbb{R}^{m+1}_+ = (0, \infty)^{m+1} \). Equation (1.1) is equivalent to the system

\[
\begin{align*}
e_0 \dot{x}_0(t) + x_0(t) & = x_1(t) \\
\vdots & \\
e_m \dot{x}_m(t) + x_m(t) & = x_{m+1}(t)
\end{align*}
\]

(1.2)

Our objective is to give some conditions on the nonlinear function \( f \) which will either ensure the stability of an equilibrium solution of (1.1) or the existence of a slowly oscillating periodic solution of (1.1).

To describe the results, we need some notation. Let \( x = \text{col}(x_0, x_1, \ldots, x_m) = \text{col}(y, z) \in \mathbb{R}^{m+1} \), where \( z = \text{col}(x_1, \ldots, x_m) \in \mathbb{R}^m \). If we define \( X = C([-1, 0], \mathbb{R}) \times \mathbb{R}^m \), then, for any \( \psi = (\varphi, \xi) \in X \), there is a unique solution \( x(t) = x(t, e, \psi) = \text{col}(y(t), z(t)) \) which exists for all \( t \geq 0 \) and satisfies the initial condition \( y(\theta) = \varphi(\theta), \ \theta \in [-1, 0], \ z(0) = z_0 \). If we define \( y_1(\theta) = y(t + \theta), \ \theta \in [-1, 0] \), and

\[
T_i(t) \psi = \text{col}(y_1, z(t)),
\]

then \( T_i(t) : X \to X, \ t \geq 0 \), is a \( C^0 \)-semigroup.

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For any interval \( I \subseteq \mathbb{R} \) (closed or open), let \( X_I = C([-1, 0], I) \times I^m \). Our first result is

**Theorem 1.1 (Positive Invariance).** If \( I \) is an interval such that \( f(I) \subseteq I \), then \( T_{\varepsilon}(t) X_I \subseteq X_I \) for \( t \geq 0 \).

If \( x_0 \) is a fixed point of \( f \), then \( x_0^* = \text{col}(x_0, \ldots, x_0) \in X \) is an equilibrium point of \((1.2)\) and conversely. If \( x_0 \) is an attracting fixed point of \( f \), we say that an interval \( J \) is the maximal interval of attraction of \( x_0 \) if \( x_0 \in J \), \( f(J) \subseteq J \), \( f^n(x) \to x_0 \) as \( n \to \infty \) for each \( x \in J \) and there is no interval \( J' \supseteq J \) with this property. We remark that the maximal interval of attraction is open.

**Theorem 1.2 (Stability).** If \( x_0 \) is an attracting fixed point of \( f \) with maximal interval of attraction \( J \), then the equilibrium solution \( x_0^* \) of \((1.2)\) is asymptotically stable and, for each \( \psi \in X_J \) and every \( \varepsilon \in \mathbb{R}_{+}^{m+1} \), we have

\[
\lim_{t \to \infty} T_{\varepsilon}(t) \psi = x_0^*.
\]

Theorems 1.1 and 1.2 are extensions of results of Ivanov and Sharkovsky [4] for the scalar case \( m = 0 \).

We say that a continuous scalar function \( u: [t_0, \infty) \to \mathbb{R} \) oscillates if it has arbitrarily large zeros. We say that \( u \) oscillates with respect to a constant function \( u_0 \) if \( u - u_0 \) oscillates. A continuous vector function \( x: [t_0, \infty) \to \mathbb{R}^{m+1} \) is said to be slowly oscillating if each component of \( x \) oscillates and the distance between zeros is greater than 1.

**Theorem 1.3 (Existence of a Slowly Oscillating Periodic Solution).** Suppose that \( I \) is a bounded interval such that \( f(I) \subseteq I \), \( x_0 \in I \) is a fixed point of \( f \) with \( f'(x_0) < -1 \), and \( (y - x_0)[f(y) - x_0] < 0 \) for \( y \neq x_0 \) (negative feedback). Then there exists \( \delta > 0 \) such that, for each \( \varepsilon \in (0, \delta)^{m+1} \), system \((1.2)\) has a slowly oscillating periodic solution.

For \( m = 0 \), Theorem 1.3 has been given by Hadeler and Tomiuk [2] and, for \( m = 1 \), by an der Heiden [1] without restrictions on the parameters \( \varepsilon \) except those, of course, which imply that the origin is unstable. We prove the result for arbitrary \( m \), but require that \( |\varepsilon| \) is small.

We remark that the above results hold true if we consider the equation

\[
\left( \tilde{\varepsilon}_m \frac{d}{dt} + x_m \right) \cdots \left( \tilde{\varepsilon}_0 \frac{d}{dt} + x_0 \right) y(t) = f(y(t-1)),
\]

where each \( \tilde{\varepsilon}_i \), \( x_i \) is positive. It is simply a scaled version of \((1.1)\) with \( \varepsilon_i = \tilde{\varepsilon}_i / x_i \), and \( f \) replaced by \( f / x_0 \ldots x_m \).
2. Proof of Theorem 1.1

We need the following auxiliary result.

**Lemma 2.1.** Suppose that $I$ is an interval (open or closed) and $a : [0, \infty) \to \mathbb{R}$ is a continuous function with values in $I$. If $\sigma > 0$ is a constant and $u(t)$, $t \geq 0$, is a solution of the equation

$$\sigma \ddot{u}(t) + u(t) = a(t) \tag{2.1}$$

satisfying $u(0) \in I$, then $u(t) \in I$ for $t \geq 0$.

**Proof.** Let $\bar{I} = [\alpha, \beta]$ and suppose that the conclusion of the lemma is not true. Then there is a first time $t_0 \geq 0$ at which the solution leaves $I$. To be specific, suppose that $u(t_0) = \beta$ and every interval $(t_0, t_0 + \delta)$, $\delta > 0$, contains a point $\tau$ such that $u(\tau) > \beta$. This interval also must contain a point $s$ such that $u(s) > \beta$ and $\ddot{u}(s) > 0$. On the other hand, it follows from (2.1) that $\ddot{u}(s) < 0$, which is a contradiction. The case $u(t_0) = \alpha$ is discussed in a similar way to complete the proof.

To prove Theorem 1.1, we use the assumptions that $f(I) \subseteq I$ and $\psi \in X_I$ together with Lemma 2.1 to observe that $x_{m_i}(t) \in I$ for $0 \leq t \leq 1$. Using this fact and the $m$th equation in (1.2), we see that $x_{m-1}(t) \in I$ for $0 \leq t \leq 1$. Proceeding in this way, we observe that $x_i(t) \in I$ for $0 \leq t \leq 1$, $i = 0, 1, \ldots, m$. This implies that $T_i(t)\psi \in X_I$ for $0 \leq t \leq 1$. The proof is completed by an induction argument.

3. Proof of Theorem 1.2

We need the following auxiliary result.

**Lemma 3.1.** Suppose that $K$, $L$ are intervals in $\mathbb{R}$ with $K \subseteq L$ and consider Eq. (2.1) with $a(t) \in K$ for $t \geq 0$. Let $L_i$ be any interval satisfying $K \subseteq L_i \subseteq L$ and $L \neq L_i \neq K$ if such an interval exists. Otherwise, let $L_1 = K$. If $u(t)$ is the solution of (2.1) with $u(0) = u_0 \in L$, then, there is a time $t_0 = t_0(u_0, L_1)$ such that $u(t) \in L_i$ for all $t \geq t_0$.

**Proof.** If there is a time $t_0$ such that $u(t_0) \in K$, then Lemma 2.1 implies that $u(t) \in K$ for $t \geq t_0$ and Lemma 3.1 is proved. Therefore, we may assume that $u(t) \notin K$ for all $t \geq 0$. To be specific, let us assume that $u(t) > \sup K = \sup \{b \in K\}$ for $t \geq 0$. Then (2.1) implies that $\ddot{u}(t) < 0$ for all $t \geq 0$ and, thus, $u(t) \to u_* \text{ as } t \to \infty$. If $u_* = \sup K$, the lemma is proved. If $u_* > \sup K$, then $\ddot{u}(t) = [-u(t) + a(t)]/\sigma \leq [\sup K - u_*]/\sigma < 0$ for sufficiently large $t$. This implies that $u(t) \to -\infty$ as $t \to \infty$, which
contradicts the fact that $u(t) > \sup K$ for $t \geq 0$. The case where $u(t) < \inf K$ is treated in a similar way.

We now turn to the proof of Theorem 1.2. Fix $\epsilon$. For any $\psi \in X_f$, we have $T_s(t)\psi \in X_f$ from Theorem 1.1.

Let $\beta = \max\{\min_{1 \leq i \leq 0} \varphi(s), x_i^0, \ldots, x_n^0\}$, $\beta = \max\{\min_{1 \leq i \leq 0} \varphi(s), x_i^0, \ldots, x_n^0\}$, let $L$ be the minimal interval containing $[\alpha, \beta]$ such that $f(L) \subset L$, and define $\alpha = f(L)$.

Our first objective is to prove the following Claim: For any $\delta > 0$, there is a set $L_1 \subset [\inf K - \delta, \sup K + \delta]$ satisfying the conditions of Lemma 3.1 and a time $t_0$ such that $T_s(t)\psi \in X_{t_0}$ for $t \geq t_0$.

Let $\tilde{L}_1$ be any interval satisfying the conditions of $L_1$ in Lemma 3.1 and consider the $(m+1)$st equation of system (1.2). From Lemma 3.1, there exists a $t^0_m \geq 0$ such that $x_m(t) \in \tilde{L}_1$ for all $t \geq t^0_m$. Now redefine $K = \tilde{L}_1$ and choose $\tilde{L}_2$ with $L \supset \tilde{L}_2 \supset K$ satisfying the conditions of $L_1$ in Lemma 3.1. If we consider the $m$th equation of system (1.2), then Lemma 3.1 implies that there exists $t^0_{m-1} \geq 0$ such that $x_{m-1}(t) \in \tilde{L}_2$ for all $t \geq t^0_{m-1}$. We continue in this way through the first equation of system (1.2) to obtain $x_m(t) \in \tilde{L}_m$ for all $t \geq t^0_m$. We can obviously choose the intervals $\tilde{L}_m$ so that $\tilde{L}_1 \subset \tilde{L}_2 \subset \cdots \subset \tilde{L}_{m+1} \subset [\inf K - \delta, \sup K + \delta]$ for any fixed $\delta > 0$. If we let $t^0 = \max\{t^0_m, \ldots, t^0_0\}$, then we have proved the claim.

With the initial $L$ as above, the attractivity of the fixed point $x_0$ of $f$ implies that $L \supset f(L) \supset f^2(L) \supset \cdots$ and $\bigcap_{n \geq 0} f^n(L) = \{x_0\}$. We may repeat the above argument with $L, K$ replaced by $f^k(L), f^{k+1}(L)$ to obtain an interval $L_k \subset f^k(L)$ and $T_s(t)\psi \in X_{t_k}$ for all $t \geq t_k$. This obviously completes the proof.

4. PROOF OF THEOREM 1.3

The proof of Theorem 1.3 follows the standard procedure of obtaining a mapping of a cone into itself with an ejective fixed point with some special modifications due to Hadeleer and Tomiuk [2]. We recall these concepts.

DEFINITION 4.1. Suppose that $X$ is a Banach space, $U$ is a subset of $X$, and $x$ is a given point in $U$. Given a map $A: U \setminus \{x\} \to X$, the point $x \in U$ is said to be an ejective point of $A$ if there is an open neighborhood $G \subset X$ of $x$ such that, for every $y \in G \cap U$, $y \neq x$, there is an integer $m = m(y)$ such that $A^m y \notin G \cap U$.

We need also the following theorem of Nussbaum [5] (see also [3]).

THEOREM 4.2. If $K$ is a closed, bounded, convex infinite-dimensional set in $X$, $A: K \setminus \{x_\ast\} \to K$ is completely continuous, and $x_\ast \in K$ is an ejective point of $A$, then there is a fixed point of $A$ in $K \setminus \{x_\ast\}$.
We first construct the set $K$. Let $I$ be an interval such that $f(I) \subseteq I$, let $x_0 \in I$ be a fixed point of $f$ with $f'(x_0) < -1$, and $f$ satisfies the negative feedback condition: $(x-x_0)[f(x)-x_0] < 0$ for $x \neq x_0$. Without loss of generality, we may assume that $x_0 = 0$. We now define

$$K = \{ \psi \in X_j; \varphi(-1) = 0, \varphi(s) > 0, \varphi(s) \in I, \varphi(s) e^{-\alpha s} \text{ nondecreasing for } s \in [-1, 0], x_i \geq 0, i = 1, 2, ..., m \}. $$

**Lemma 4.3.** Let $\psi \in K$ be arbitrary and suppose that the solution $T_t \psi$ of (1.2) has the property that its first component $x_0(t)$ oscillates. Then

(i) all zeros of $x_0(t)$ are simple and distances between successive zeros are larger than 1;

(ii) between each two successive zeros $z$ and $\beta$, $z < \beta$, of $x_0(t)$, there exists only one zero $\gamma_i$ of $x_i(t)$, $i = 1, 2, ..., m$, and $\gamma_i - z > 1$ and $\gamma_i - z < \gamma_{i+1}$;

(iii) if $x$ is a zero of $x_0(t)$, then $|x_0(t) \exp(t/\varepsilon_0)|$ is nondecreasing for $t \in [z, z + 1]$.

**Proof.** Let $T > 0$ be fixed and consider Eq. (2.1). If $u(0) > 0$ and $a(t) \geq 0$ for $t \in [0, T]$, then it is obvious that $u(t) > 0$ for all $t \in [0, T]$. From the form of system (1.2) and the assumption that $x_0(t)$ oscillates, it follows that each $x_i(t)$, $i = 1, ..., m$, must have a zero. Furthermore, if $t_1 > 0$ is the first zero of $x_0(t)$, then it is easy to see that there exist $0 < t_1 < \cdots < t_m < t_1$ such that $x_i(t_{m+1}) = 0$, $i = 1, 0, 1, ..., m$, and these are the first such zeros.

Since $f(x_0(t-1)) < 0$ for $t \in (-1, t_1)$, it follows that $t_1$ is a simple zero of $x_m(t)$ and $x_m(t) < 0$ for $t \in (t_1, t_1)$. Using the same type of reasoning, we see that $x_i(t) < 0$ for $t \in (t_i, t_i+1)$ for each $i = 0, 1, ..., m$ and each of the zeros is simple. Since $x_i(t) < 0$ for $t \in (t_i, t_i+1)$, the first equation of system (1.2) implies that $(d/dt)[x_0(t) \exp(t/\varepsilon_0)] < 0$, which implies that the function $|x_0(t) \exp(t/\varepsilon_0)|$ is nondecreasing on $[t_i, t_i+1]$. This completes the proof for the first two successive zeros of $x_0(t)$. It is clear that the other situations can be treated in the same way.

**Lemma 4.4.** If $f'(0) > 0$ and $yf(y) < 0$ for $y \neq 0$, then there exists a $\delta > 0$ such that, for every $\varepsilon$ satisfying $0 < \varepsilon < \delta$, $i = 0, 1, ..., m$, all solutions of Eq. (1.1) oscillate.

**Proof.** Suppose that $x(t) = \text{col}(x_0(t), x_1(t), ..., x_m(t))$ is a nonoscillatory solution of system (1.2). Then there must be one component of $x(t)$, say $x_\delta(t)$, that is nonoscillatory.

Claim 1. There is a $t_0$ such that every component $x_i(t)$ of $x(t)$ has a fixed sign for $t \geq t_0$. 


To prove this, we make the following observation: If \( v(t) \) is a given function of constant sign for large \( t \) and \( u(t) \) is a nonzero solution of \( \dot{u}(t) + u(t) = v(t), \delta > 0 \), then there is a \( t_0 \) such that \( u(t) \) is of constant sign for \( t > t_0 \). In fact, if \( v(t) \) is of constant sign for \( t > t_0 \) and either \( u(t_0) v(t_0) > 0 \) or \( u(t_0) = 0 \), then the variation of constants formula implies that \( u(t) v(t) > 0 \) for \( t > t_0 \). If \( u(t_0) v(t_0) < 0 \), then \( u(t) \) is strictly monotone near \( t_0 \) and, therefore, either \( u(t) v(t) < 0 \) for \( t > t_0 \) or \( u(t_0) v(t_0) \) eventually becomes positive and remains so.

We can now apply this remark to (1.2) to see that \( x_{k+1}(t) \) is non-oscillatory. Proceeding in this way, we observe that \( x_0(t) \) is nonoscillatory. From the \((m+1)\)st equation in (1.2), it follows that \( x_m(t) \) is nonoscillatory. The proof that the other components of \( x(t) \) are nonoscillatory is completed using the same argument as before. This completes the proof of the claim.

From Claim 1, we know that every component of \( x(t) \) is nonoscillatory. For definiteness, suppose that \( x_0(t) > 0 \) for \( t \in (t_0, \infty) \) (the case where \( x_0(t) < 0 \) is analogous). Let \( x_k(t), k \geq 0 \), be the last component of \( x(t) \) which is positive for large \( t \). Therefore, \( x_j(t) < 0 \) for \( j > k \).

Claim 2. Every component \( x_j(t) \) of \( x(t) \) approaches 0 as \( t \to \infty \) and there is a \( t_0 \) such that each \( x_j(t) \) is strictly monotone for \( t \geq t_0 \).

To prove this, let us first suppose that \( k = m \). From (1.2) and the fact that \( f(x_0(t-1)) < 0 \) for \( t > t_0 + 1 \), it follows that \( x_m(t) \) is a strictly monotone decreasing function. This implies that there is a constant \( \xi \) such that \( x_m(t) \to \xi \) as \( t \to \infty \). From (1.2), this implies that all \( x_j(t) \to \xi \) as \( t \to \infty \) for all \( 0 \leq j \leq m \). On the other hand, using the last equation in (1.2), we see that \( x_m(t) \to f(\xi) \) as \( t \to \infty \). Thus, \( f(\xi) = \xi \), which implies that \( \xi = 0 \).

Since \( x_m(t) \) is strictly monotone for large \( t \), we see that \( x_{m-1}(t) \) is strictly monotone from the following fact: if the function \( v(t) \) has fixed sign and is strictly monotone, then every solution of the equation \( \dot{u}(t) + u(t) = v(t) \) is eventually strictly monotone. If this statement were not true, then we would have two distinct points \( t_1 > t_0 \) such that \( u(t_1) = v(t_1), u(t_0) = v(t_0) \). For definiteness, suppose that \( v(t) > 0 \) and \( v(t) < v(s) \) for all \( t > s \). Using the variation of constants formula, we have

\[
e^\delta \int_{t_0}^{t_1} v(t_1) = v(t_0) + \frac{1}{\delta} \int_{t_0}^{t_1} e^\delta \int_{s_0}^{s} v(s) \, ds
\]

and this last relation leads to the contradiction \( v(t_1) > v(t_0) \).

We can now use the same argument to see that each component of \( x(t) \) is eventually strictly monotone.
If \( k < m \), then \( x_k(t) \) is eventually strictly monotone since \( x_k(t) x_{k+1}(t) < 0 \) for large \( t \). Now we can use the same argument as in the previous case to see that \( x_j(t) \) is eventually strictly monotone for \( 0 \leq j \leq k \). Since \( x_0(t) \to 0 \) as \( t \to \infty \) and \( f'(0) < 0 \), it follows that the function \( f(x_0(t-1)) \) is eventually strictly monotone. Therefore, we will have that \( x_m(t) \) is eventually strictly monotone. Now we can continue the argument to complete the proof of Claim 2.

**Claim 3.** If \( x_i(t) \) and \( x_{i+1}(t) \) eventually have the same sign, then there is a \( t_0 \) such that \( |x_i(t)| > |x_{i+1}(t)| \) for \( t \geq t_0 \).

To be specific, suppose that \( x_i(t) > 0, x_{i+1}(t) > 0 \), and that both are strictly monotone decreasing for \( t \geq t_1 \). Then the variation of constants formula for \( x_i(t) \) implies that

\[
x_i(t) = x_i(t_1) e^{-\frac{\varepsilon_i}{\varepsilon_i} (t - t_1)} + \frac{1}{\varepsilon_i} \int_{t_1}^{t} e^{-\frac{\varepsilon_i}{\varepsilon_i} (t - s)} x_{i+1}(s) \, ds
\geq x_i(t_1) e^{-\frac{\varepsilon_i}{\varepsilon_i} (t - t_1)} + \frac{x_{i+1}(t)}{\varepsilon_i} \int_{t_1}^{t} e^{-\frac{\varepsilon_i}{\varepsilon_i} (t - s)} \, ds
= x_{i+1}(t) + [x_i(t_1) - x_{i+1}(t)] e^{-\frac{\varepsilon_i}{\varepsilon_i} (t - t_1)}
> x_{i+1}(t)
\]

for large \( t \) since \( x_{i+1}(t) \to 0 \) as \( t \to \infty \) and \( x_i(t_1) > 0 \). This completes the proof of Claim 3.

Claim 3 implies that \( x_0(t) > \ldots > x_{k+1}(t) > 0 > x_m(t) > \ldots > x_k(t) \) for all \( t \geq t_1 \) with \( t_1 \) sufficiently large. Therefore,

\[
\varepsilon_k \dot{x}_k(t) + x_k(t) = x_{k+1}(t) < x_m(t) = f(x_0(t-1)) - \varepsilon_m \dot{x}_m(t)
< f(x_0(t-1)) < f(x_{k+1}(t-1)).
\]

Therefore, there is an \( \eta > 0 \) and a \( t_2 \) such that \( a = f'(0) + \eta < 0 \) and, for \( t \geq t_2 \), we have

\[
\varepsilon_k \dot{x}_k(t) + x_k(t) \leq ax_k(t - 1).
\]

Using the variation of constants formula on the interval \([t, t+1]\), we have

\[
x_k(t+1) \leq x_k(t) e^{-1/\varepsilon_k} + \frac{1}{\varepsilon_k} \int_{t+1}^{t+1/\varepsilon_k} e^{1/\varepsilon_k s} x_k(s) \, ds,
\]

which clearly contradicts the fact that \( x_k(t) > 0 \) if \( \varepsilon_k \) is small enough. This completes the proof of Lemma 4.4.
Our next objective is to define a map $A: K \to K$. From Lemma 4.4, we know that $x_0(t)$ oscillates. From Lemma 4.3, we have $-T_x(t_0^0 + 1)\psi \in K$. If $t_2^0$ is the second zero of $x_0(t)$, then $T_x(t_2^0 + 1)\psi \in K$. Therefore, we define
\[ A\psi = T_x(t_2^0 + 1)\psi \quad \text{for} \quad \psi \in K. \] (4.1)
It is natural to extend the definition of $A$ to the closure $\overline{K}$ of $K$ by setting $A0 = 0$.

The set $\overline{K}$ is closed bounded and convex. Following the same proof as in Hadeler and Tomiuk [2], we see that $A$ is a completely continuous map on $\overline{K}$. As a first step in showing that $0$ is an ejective fixed point of $A$, we need specific information about the characteristic equation
\[ (\varepsilon_m \lambda + 1) \cdots (\varepsilon_0 \lambda + 1) - ae^{-\lambda} = 0, \quad a = f'(0), \] (4.2)
corresponding to the linear variational equation of (1.1) about the origin:
\[ \left( \varepsilon_m \frac{d}{dt} + 1 \right) \cdots \left( \varepsilon_0 \frac{d}{dt} + 1 \right) y(t) = ay(t - 1). \] (4.3)

**Lemma 4.5.** There exists a $\delta > 0$ such that, for every $\varepsilon$ with $0 < \varepsilon_i \leq \delta$, $i = 0, 1, ..., m$, the characteristic equation (4.2) has a solution $\lambda_0 = \mu + iv$ with $\mu > 0$, $0 < v < \pi$.

**Proof.** For $\varepsilon = 0$, there is a solution of (4.2) with $\mu = \ln |a|$, $v = \pi$. This solution also is simple. By Rouché's Theorem, there exists a $\delta > 0$ and a neighborhood $U$ of $\mu + iv$ in the complex plane such that, for $0 < \varepsilon_i \leq \delta$, $i = 0, 1, ..., m$, there is a unique solution $\mu(\varepsilon) + iv(\varepsilon)$ of (4.2) in $U$. To show that $v(\varepsilon) < \pi$ for $\delta$ sufficiently small, it is enough to show that $(\partial v/\partial \varepsilon_i)(0) < 0$ for all $i$. Setting $\lambda = \mu(\varepsilon) + iv(\varepsilon)$ in (4.2), differentiating with respect to $\varepsilon_i$, and setting $\varepsilon = 0$, we deduce that the imaginary part of the resulting expression satisfies
\[ v(0) + ae^{-\mu(0)}(\cos v(0)) \frac{\partial v}{\partial \varepsilon_i}(0) = 0. \]
Since $ae^{-\mu(0)}(\cos v(0)) = 1$, this implies that $(\partial v/\partial \varepsilon_i)(0) = -\pi < 0$, $i = 0, 1, ..., m$. This completes the proof.

**Lemma 4.6.** There exists a $\delta > 0$ such that, for every $\varepsilon$ with $0 < \varepsilon_i \leq \delta$, $i = 0, 1, ..., m$, the fixed point $0$ of $A$ is ejective.

**Proof.** To prove the ejectivity of the fixed point $0$ of $A$, we plan to use some general results from [3, Chap. 11]. To do this, we need some specific but elementary facts about the linear equation (4.3).
We may write (4.3) as a system of equations

$$\dot{x}(t) = Cx(t) + Dx(t-1),$$

(4.4)

where

$$C = \begin{bmatrix}
-\frac{1}{\epsilon_0} & 1 & \cdots & 0 & 0 \\
0 & \frac{1}{\epsilon_0} & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & \cdots & \cdots \\
\vdots & \vdots & \ddots & \ddots & \cdots \\
0 & 0 & \cdots & 0 & -\frac{1}{\epsilon_m}
\end{bmatrix}, \quad D = \begin{bmatrix}
0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & \cdots & \cdots \\
\vdots & \vdots & \ddots & \ddots & \cdots \\
0 & 0 & \cdots & 0 & 0 \\
\frac{a}{\epsilon_m} & 0 & \cdots & 0 & 0
\end{bmatrix}.$$  

The equation adjoint to (4.4) (see [3]) is

$$\frac{d}{ds}w(s) = -w(s)C - w(s + 1)D,$$

(4.5)

where \(w \in (\mathbb{R}^{m+1})^*\) is an \((m + 1)\)-dimensional row vector. From the form of the matrix \(D\), the initial data for (4.5) is taken in the space \(X^* = (\mathbb{R}^m)^* \times C([0, 1], \mathbb{R})\).

For any \(\psi^* = (\eta, \zeta) \in X^*\) and any \(\psi = (\varphi, \xi) \in X\), there is an associated bilinear form

$$(\psi^*, \psi) = \psi^*(0)\psi(0) + \int_0^1 \zeta(z)(\epsilon_m^{-1} \varphi(z - 1) \, dz.$$  

(4.6)

Let \(\lambda_0\) be the solution of (4.2) guaranteed by Lemma 4.5. A few computations show that a corresponding eigenfunction of (4.5) is given by

$$\psi^*_{\lambda_0}(s) = e^{-\lambda_0 s}[\epsilon_0(\epsilon_1^{1/2} + 1) \cdots (\epsilon_m^{1/2} + 1), \epsilon_1(\epsilon_2^{1/2} + 1) \cdots (\epsilon_m^{1/2} + 1), \ldots, \epsilon_m].$$

The results in [3, Lemma 4.4, Chap. 11] imply that the fixed point 0 of \(A\) is ejective if

$$\inf\{|(\psi^*, \psi) |: \psi \in \mathcal{K}, |\psi| = 1\} > 0,$$

(4.7)

where, for \(\psi = (\varphi, \xi), \xi = \text{col}(x_1, \ldots, x_m)\), we define

$$|\psi| = \max\left\{\max_{1 \leq i < 0} |\varphi(s)|, |x_1|, \ldots, |x_m|\right\}.$$
For any \( \psi \in \bar{K} \), we let \((\psi^*_{n_0}, \psi) = R(\psi) + iL(\psi)\), where \(R(\psi)\) and \(L(\psi)\) are real. Using (4.6) and the expression for \(\psi^*_{n_0}\), we see that (we have put \(\varphi(0) = x_0\))

\[
L(\psi) = [e_0(e_1 + \ldots + e_m)x_0 + e_1(e_2 + \ldots + e_m)x_1 + \ldots + e_{m-1}e_mx_{m-1}] \nu u(0) \\
+ a \int_0^1 e^{\nu(0)x} \sin(\varphi(0)x) \varphi(x - 1) \, dx + O(|\varepsilon|).
\]

We claim next that there exists an index \(i \in \{0, 1, \ldots, m\}\) such that \(x_i \geq \exp\{-1/e_i\}\). In fact, since \(|\psi| = 1\), it follows that either \(x_i = 1\) for some \(i \in \{1, 2, \ldots, m\}\) or \(\max_{1 \leq s \leq 0} |\varphi(s)| = 1\). In the first case, the claim is obvious. In the second one, since \(\varphi(s) \exp\{s/e_0\}\) is increasing, we have \(x_0 = \varphi(0) \geq \max_{1 \leq s \leq 0} \varphi(s) \exp\{-1/e_0\}\). This proves the claim.

Finally, since the kernel in the integral term of \(L(\psi)\) is positive, we have \(L(\psi) \geq c = c(\varepsilon) > 0\), for \(|\varepsilon| \leq \delta\) and \(|\psi| = 1\). This obviously implies that (4.7) is satisfied and completes the proof of the lemma.

Theorem 1.3 is now a consequence of Theorem 4.2.

REFERENCES