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STABILITY OF SYMMETRIC PERIODIC SOLUTIONS WITH SMALL AMPLITUDE OF $\dot{x}(t) = \alpha f(x(t), x(t-1))$

Peter Dormayer

Mathematisches Institut der Universität Giessen
Arndtstrasse 2, 35392 Giessen, Germany

Anatoli F. Ivanov

Department of Mathematics, Pennsylvania State University
P.O. Box PSU, Lehman, PA 18627, USA
and
School of Information Technology and Mathematical Sciences
University of Ballarat, Australia

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Abstract. We study special symmetric periodic solutions of the equation

$$\dot{x}(t) = \alpha f(x(t), x(t-1))$$

where α is a positive parameter and the nonlinearity f satisfies the symmetry conditions $f(-u, v) = -f(u, -v) = f(u, v)$. We establish the existence and stability properties for such periodic solutions with small amplitude.

1. Introduction. Consider the differential delay equation

$$\dot{x}(t) = \alpha f(x(t), x(t-1)) \tag{αf}$$

where $\alpha > 0$ is a parameter and the nonlinearity f satisfies the following conditions:

(1) $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous;
(2) f is symmetrical, i.e., for all $(x, y) \in \mathbb{R} \times \mathbb{R}$,

$$f(-x, y) = -f(x, -y) = f(x, y). \tag{s}$$

(3) $y \cdot f(x, y) < 0$ for $x \in \mathbb{R},\ y \in \mathbb{R}\backslash\{0\}$. (nf)

In the present paper we discuss the existence and stability of so-called special symmetric periodic solutions of equation (αf).

Definition. $x : \mathbb{R} \to \mathbb{R}$ is called a *special symmetric periodic solution* (SSPS) if

- x solves (αf) for some $\alpha \in \mathbb{R}$; -$x(0) = 0,\ x(t) \neq 0$ for $0 < t \leq 1$;
- $x(t+2) = -x(t)$ for all $t \in \mathbb{R}$.

$x(1)$ is called the *amplitude* of x.

Obviously, x is periodic with minimal period 4. The existence of SSPSs for equation (αf) has been proved in [7]. That was a generalization of the well known approach due to Kaplan and Yorke [8] for the equation

$$\dot{x}(t) = g(x(t-1)), \tag{g}$$

where the nonlinearity g satisfies $g(-x) = -g(x), \quad g(x) < 0$ for $x > 0$.

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Our exposition on the existence of SSPSs in section 2 is somewhat different from that in [7]. It also includes new sufficient conditions in the form of Theorem 2.2 below and provides the reader with a detailed proof. [7] is written in Russian and is not easily accessible. Stability of SSPSs with small amplitude of equation (g) has been studied in [4]. The very same idea is applicable to equation (αf). Our section 5 is based on developments in [4]. Some statements from that paper are taken into section 5 without any changes to make the presentation self contained.

2. Existence of special symmetric periodic solutions. Consider the following coupled system of ordinary differential equations

$$\dot{x} = \alpha f(x,y), \dot{y} = -\alpha f(y,x). \tag{$*$}$$

Let (x_a, y_a) denote the solution of $(*)$ with the initial condition $(a,0), a > 0$.

Lemma 2.1. *There is a $T_a > 0$ such that (i) x_a is even, y_a is odd,*

(ii) $x_a(\frac{1}{8}T_a) = y_a(\frac{1}{8}T_a)$,
(iii) $x_a(\frac{1}{4}T_a - t) = y_a(t), x_a(t + \frac{T_a}{2}) = -x_a(t), y_a(t) = x_a(t - \frac{1}{4}T_a)$ *for all* $t \in R$,
(iv) x_a *is decreasing and* y_a *is increasing on* $[0, \frac{1}{4}T_a]$,
(v) (x_a, y_a) *is periodic with period* T_a.

Proof. (i) $u(t) = x_a(-t), v(t) = -y_a(-t)$ solves $(*)$ and coincides with (x_a, y_a) at $t = 0$.

(ii) We have $x_a(0) = a, y_a(0) = 0$ and $\dot{y}_a(0) = -\alpha f(0,a) > 0$. Hence there is a $\tau > 0$ such that $x_a(t) \geq 0$ and $a \geq y_a(t) > 0$ holds for $0 < t \leq \tau$. Now (nf) implies that x_a is decreasing on $[0,\tau]$ while y_a is increasing on $[0,\tau]$. Hence $(x_a(t), y_a(t)) \in \Delta := \{(x,y) \in R^2 | 0 \leq y \leq x \leq a\}$ for $t \in [0,\tau]$. Assume that $(x_a(t), y_a(t)) \in \Delta$ for all $t \geq 0$. Then $x_\infty := \lim_{t\to\infty} x_a(t)$ and $y_\infty := \lim_{t\to\infty} y_a(t)$ exist and $(x_\infty, y_\infty) \in \Delta$. It follows that $x_\infty \geq y_\infty > y_a(0) = 0$, hence $\lim_{t\to\infty} \dot{y}_a(t) = -\alpha f(y_\infty, x_\infty) > 0$. Thus there is a $t^* > 0$ such that $\dot{y}_a(t) \geq c := -\frac{1}{2}\alpha f(y_\infty, x_\infty) > 0$ for $t \geq t^*$. But then $y_a(t) \geq y_a(t^*) + c(t - t^*)$ which contradicts $y_a(t) \to y_\infty$. Consequently there is a $T_a > 0$ with $(x_a(\frac{1}{8}T_a), y_a(\frac{1}{8}T_a)) \in \partial\Delta = ([0,a] \times \{0\}) \cup (\{a\} \times [0,a]) \cup \Delta', \Delta' = \{(x,y) \in R^2 | 0 \leq x \leq a, x = y\}$. Since x_a is decreasing and y_a is increasing on $[0, \frac{1}{8}T_a]$ we have $x_a(\frac{1}{8}T_a) < a, y_a(\frac{1}{8}T_a) > 0$ which implies $(x_a(\frac{1}{8}T_a), y_a(\frac{1}{8}T_a)) \in \Delta'$.

(iii) $u(t) = y_a(\frac{1}{4}T_a - t), v(t) = x_a(\frac{1}{4}T_a - t)$ solves $(*)$ and coincides at $t = \frac{1}{8}T_a$ with (x_a, y_a), hence $x_a(t) = y_a(\frac{1}{4}T_a - t)$. Similarly $u(t) = -x_a(t + \frac{1}{2}T_a), v(t) = -y_a(t + \frac{1}{2}T_a)$ solves (*) and coincides at $t = -\frac{1}{4}T_a$ with (x_a, y_a), since $u(-\frac{1}{4}T_a) = -x_a(\frac{1}{4}T_a) = -y_a(0) = 0 = x_a(\frac{1}{4}T_a) = x_a(-\frac{1}{4}T_a)$ and $v(-\frac{1}{4}T_a) = -y_a(\frac{1}{4}T_a) = y(-\frac{1}{4}T_a)$. Hence $x_a(t) = -x_a(t + \frac{1}{2}T_a)$. Finally $y_a(t) = x_a(\frac{1}{4}T_a - t) = x_a(t - \frac{1}{4}T_a)$.

(iv) Since $(x_a(t), y_a(t)) \in \Delta$ for $t \in [0, \frac{1}{8}T_a]$, x_a is decreasing and y_a increasing on $[0, \frac{1}{8}T_a]$. The relation $x_a(\frac{1}{4}T_a - t) = y_a(t)$ shows that this holds on $[\frac{1}{8}T_a, \frac{1}{4}T_a]$ too.

Solutions of (*) and symmetric solutions of (αf) are related as follows.

Lemma 2.2. *Equation (αf) has a special symmetric periodic solution if and only if $(*)$ has a closed symmetric trajectory of minimal period 4.*

Proof. 1). Let $x(t)$ be a SSPS of equation (αf) and note that $\dot{x}(t) = \alpha f(x(t), y(t))$. On the other hand, $\dot{y}(t) = \dot{x}(t-1) = \alpha f(x(t-1), x(t-2)) = -\alpha f(y(t), x(t))$. Therefore, $(x(t), y(t))$ satisfies system $(*)$. Also it is a closed symmetric trajectory with period 4.

2). Let $(x(t), y(t))$ be a symmetric closed period 4 trajectory of system $(*)$. We can assume that $x(0) > 0, y(0) = 0$. Then $x(1) = 0, y(1) > 0, x(2) < 0, y(2) = 0, x(3) = 0, y(3) < 0$. Direct substitution shows that $(-y(t), x(t))$ is the same trajectory. Therefore, $(x(t), y(t)) = (-y(t+c), x(t+c))$ for all $t \in \mathbb{R}$ and some $c \in [0, 4)$. The symmetry of $(x(t), y(t))$ implies $x(t+2) = -x(t), y(t+2) = -y(t)$ for all $t \in \mathbb{R}$. Therefore, $x(t) = -y(t+c) = -x(t+2c) = x(t+2c+2)$. This implies $2c+2 = 4m$ or $c+1 = 2m$ for some integer m. Since $c \in [0, 4)$ then either $c = 1$ or $c = 3$.

In the case $c = 1$ one has $0 < x(0) = -y(1) < 0$, a contradiction. Therefore, $c = 3$ and thus $y(t) = x(t+3) = x(t-1)$. The first equation of system $(*)$ now implies that $x(t)$ satisfies equation (αf).

We shall give next some sufficient conditions which guarantee the existence of symmetric periodic trajectories of period 4 for the system $(*)$.

Assume that $f(x, y)$ can be represented as

$$f(x, y) = y\bar{f}(x^2, y^2), \tag{$\bar{f}$}$$

where $\bar{f}$ is Lipschitz and C^k with $k \geq 1$, and

$$\bar{f}(u, v) < 0 \quad \text{for} \quad u, v \in \mathbb{R}. \tag{ns}$$

Note that conditions $(\bar{f})$ and (ns) imply (1), (2), (3). In particular we have that $\bar{f}_0 := \bar{f}(0, 0) < 0$.

Assumption $(\bar{f})$ is a technical one and it will be used also in Section 5 to simplify calculations.

Let $w \geq 0$ and define $s_w > 0$ by $s_w^2 := \sup_{0 \leq v \leq u \leq w^2}\{\bar{f}(u, v)/\bar{f}(v, u)\}$. Observe that s_w exists since $\bar{f} < 0$ and that $1 \leq s_w < \infty$. Furthermore define

$$fi(w) := \inf_{u,v \geq w^2} |\bar{f}(u, v)|, \quad fs(w) := \sup_{u,v \geq w^2} |\bar{f}(u, v)|$$

where $fs(w) = \infty$ is allowed; $fi(w)$ and $fs(w)$ describe the behavior of $\bar{f}$ at infinity. Observe that

$$-fi(w) \leq |\bar{f}(u, v)| \leq fs(w) \text{ for } u, v \geq w^2, -fi \text{ is increasing}, fs \text{ is decreasing}.$$

Hence $fi_\infty := \lim_{w\to\infty} fi(w)$ and $fs_\infty := \lim_{w\to\infty} fs(w)$ exist where $fi_\infty = \infty$ and $fs_\infty = \infty$ is allowed. Clearly

$$0 \leq fi(w) \leq fi_\infty \leq fs_\infty \leq fs(w) \quad \text{for} \quad w \geq 0.$$

Now we can give sufficient conditions for the existence of a periodic solution of $(*)$ with period 4.

Theorem 2.1. *Assume that*

(1) $(\bar{f})$ *and* (ns) *hold,*
(2) $\lim_{w\to\infty} \frac{s_w}{w} = 0$,
(3) $\pi/(2|\bar{f}_0|) < \alpha < \frac{\pi}{2fs_\infty}$ *or* $\frac{\pi}{2fi_\infty} < \alpha < \pi/(2|\bar{f}_0|)$ *and* $fi(0) > 0$.

Then $(*)$ *has a periodic solution with period 4.*

Proof. (a) Let $a > 0$ and let us write (x_a, y_a) in polar coordinates, i.e.

$$x_a = \rho_a \cos\theta_a, \quad y_a = \rho_a \sin\theta_a.$$

Then from Lemma 2.1, $\rho_a(0) = a, \theta_a(0) = 0$ and $\theta_a(\frac{1}{8}T_a) = \frac{\pi}{4}$. Rewriting $(*)$ for (ρ_a, θ_a) yields $\dot{\theta}_a = -\alpha[\cos^2\theta_a \cdot \bar{f}(y_a^2, x_a^2) + \sin^2\theta_a \cdot \bar{f}(x_a^2, y_a^2)]$ which results in

$$\theta_a(t) = \alpha \int_0^t [\cos^2\theta_a \cdot |\bar{f}(y_a^2, x_a^2)| + \sin^2\theta_a \cdot |\bar{f}(x_a^2, y_a^2)|]ds$$

(b) Clearly $(0,0)$ is an equilibrium of $(*)$. Linearization gives the matrix

$$\begin{pmatrix} 0 & \alpha\bar{f}_0 \\ -\alpha\bar{f}_0 & 0 \end{pmatrix},$$

where we have used $f_x(0,0) = 0, f_y(0,0) = \bar{f}_0$. Now the Hopf bifurcation theorem implies $\lim_{a\to 0} T_a = 2\pi/(\alpha|\bar{f}_0|) =: T_0$.
(c) Since $0 \le y_a(t) \le x_a(t) \le a$ for $t \in [0, \frac{1}{8}T_a]$ we have

$$\bar{f}(x_a^2(t), y_a^2(t))/\bar{f}(y_a^2(t), x_a^2(t)) \le s_a^2 \quad \text{or} \quad 0 \ge \bar{f}(x_a^2(t), y_a^2(t)) \ge s_a^2\bar{f}(y_a^2(t), x_a^2(t)).$$

It follows that

$$\begin{aligned} \frac{d}{dt}[x_a^2 + s_a^2 y_a^2] &= 2x_a\dot{x}_a + 2s_a^2 y_a\dot{y}_a = 2\alpha x_a f(x_a, y_a) - 2\alpha s_a^2 y_a f(y_a, x_a) \\ &= 2\alpha x_a y_a \bar{f}(x_a^2, y_a^2) - 2\alpha s_a^2 x_a y_a \bar{f}(y_a^2, x_a^2) \\ &= 2\alpha x_a y_a[\bar{f}(x_a^2, y_a^2) - s_a^2\bar{f}(y_a^2, x_a^2)] \ge 0, \end{aligned}$$

i.e. $x_a^2 + s_a^2 y_a^2$ is increasing on $[0, \frac{1}{8}T_a]$. Consequently

$$s_a^2\rho_a^2(t) = s_a^2 x_a^2(t) + s_a^2 y_a^2(t) \ge x_a^2(t) + s_a^2 y_a^2(t) \ge x_a^2(0) + s_a^2 y_a^2(0) = a^2,$$

i.e. $\rho_a(t) \ge \frac{a}{s_a}$ for $t \in [0, \frac{1}{8}T_a]$. The orbit of (x_a, y_a) stays outside the ellipse with the axes a and $\frac{a}{s_a}$.
(d) Let $\vartheta \in]0, \frac{\pi}{4}[$ and $t_a^\vartheta \in]0, \frac{1}{8}T_a[$ with $\theta_a(t_a^\vartheta) = \vartheta$. For $t \in [t_a^\vartheta, \frac{1}{8}T_a]$ we have from the monotonicity properties of x_a and y_a:

$$\begin{aligned} x_a(t) &\ge x_a(\frac{1}{8}T_a) = \rho_a(\frac{1}{8}T_a)\cos\theta_a(\frac{1}{8}T_a) \ge \frac{a}{s_a}\cos\frac{\pi}{4} \ge \frac{a}{s_a}\sin\vartheta, \\ y_a(t) &\ge y_a(t_a^\vartheta) = \rho_a(t_a^\vartheta)\cos\theta_a(\frac{1}{8}T_a) \ge \frac{a}{s_a}\sin\vartheta. \end{aligned}$$

Hence

$$fi(\frac{a}{s_a}\sin\vartheta) \le |\bar{f}(x_a^2(t), y_a^2(t))|, \quad |\bar{f}(y_a^2(t), x_a^2(t))| \le fs(\frac{a}{s_a}\cos\vartheta).$$

Now $\frac{\pi}{4} = \vartheta + \theta_a(\frac{1}{8}T_a) - \theta_a(t_a^\vartheta) = \vartheta + \alpha\int_{t_a^\vartheta}^{T_a/8}[\cos^2\theta_a \cdot |\bar{f}(y_a^2, x_a^2)| + \sin^2\theta_a \cdot |\bar{f}(x_a^2, y_a^2)|]dt$ implies

$$\vartheta + \alpha fi(\frac{a}{s_a}\sin\vartheta)(\frac{1}{8}T_a - t_a^\vartheta) \le \frac{\pi}{4} \le \vartheta + \alpha fs(\frac{a}{s_a}\sin\vartheta)(\frac{1}{8}T_a - t_a^\vartheta).$$

(e) If $\pi/(2|\bar{f}_0|) < \alpha < \frac{\pi}{2fs_\infty}$ holds, then $T_0 < 4$ and $fs_\infty < |\bar{f}_0|$, i.e. $fs_\infty < \infty$. Assume that $T_a < 4$ for all $a > 0$. Then $\frac{1}{8}T_a - t_a^\vartheta < \frac{1}{2}$ and $\frac{\pi}{4} \le \vartheta + \alpha fs(\frac{a}{s_a}\sin\vartheta) \cdot \frac{1}{2}$ holds for all $a > 0$ and $\vartheta \in]0, \frac{\pi}{4}[$. Since $\frac{a}{s_a}\sin\vartheta \to \infty$ for $a \to \infty$ and since fs_∞

is finite it follows that $\frac{\pi}{4} \le \vartheta + \frac{1}{2}\alpha fs_\infty$ holds for all $\vartheta \in]0, \frac{\pi}{4}[$. Hence $\frac{\pi}{4} \le \frac{1}{2}\alpha fs_\infty$ which gives the contradiction $\alpha \ge \frac{\pi}{2fs_\infty}$. Consequently $T_a \ge 4$ for some $a > 0$ and $T_0 < 4 \le T_a$ implies period 4 in this case.
(f) If $\frac{\pi}{2fi_\infty} < \alpha < \pi/(2|\bar{f}_0|)$ and $fi(0) > 0$ holds, then $T_0 > 4$ and $fi_\infty > 0$. Assume that $T_a > 4$ for $a > 0$, then for all $a > 0$ and $\vartheta \in]0, \frac{\pi}{4}[$

$$\frac{\pi}{4} \ge \vartheta + \alpha fi(\frac{a}{s_a}\sin\vartheta)(\frac{1}{8}T_a - t_a^\vartheta) \ge \vartheta + \alpha fi(\frac{a}{s_a}\sin\vartheta)\cdot\frac{1}{2} - \alpha t_a^\vartheta \cdot fi(\frac{a}{s_a}\sin\vartheta).$$

Because of $x_a^2 \ge 0, y_a^2 \ge 0$ we have $|\bar{f}(x_a^2, y_a^2)|, |\bar{f}(y_a^2, x_a^2)| \ge fi(0) > 0$ and thus

$$\vartheta = \theta_a(t_a^\vartheta) = \alpha\int_0^{t_a^\vartheta}[\cos^2\theta_a \cdot |\bar{f}(y_a^2, x_a^2)| + \sin^2\theta_a \cdot |\bar{f}(x_a^2, y_a^2)|]dt \ge \alpha \cdot t_a^\vartheta fi(0).$$

Hence for $a > 0$ and $\vartheta \in]0, \frac{\pi}{4}[$

$$\vartheta + \frac{1}{2}\alpha fi(\frac{a}{s_a}\sin\vartheta) \le \frac{\pi}{4} + \alpha t_a^\vartheta fi(\frac{a}{s_a}\sin\vartheta) \le \frac{\pi}{4} + \vartheta\cdot\frac{fi(\frac{a}{s_a}\sin\vartheta)}{fi(0)}$$

or

$$\frac{1}{2}\alpha fi(\frac{a}{s_a}\sin\vartheta) \le \frac{\pi}{4} + \vartheta\cdot\left[\frac{fi(\frac{a}{s_a}\sin\vartheta)}{fi(0)} - 1\right].$$

If $fi_\infty = \infty$ then $a \to \infty$ yields $\frac{1}{2}\alpha \le \frac{\vartheta}{fi(0)}$ for all $\vartheta \in]0, \frac{\pi}{4}[$ which gives the contradiction $\alpha \le 0$. If $fi_\infty < \infty$ then $a \to \infty$ yields

$$\frac{1}{2}\alpha fi_\infty \le \frac{\pi}{4} + \vartheta[\frac{fi_\infty}{fi(0)} - 1]$$

for all $\vartheta \in]0, \frac{\pi}{4}[$. Since fi is increasing we have $fi_\infty \ge fi(0)$ and $\frac{fi_\infty}{fi(0)} - 1 \ge 0$. Therefore $\frac{1}{2}\alpha fi_\infty \le \frac{\pi}{4}$ which contradicts $\frac{\pi}{2fi_\infty} < \alpha$. Consequently there is an $a > 0$ with $T_a = 4$.

Examples. 1. In the classical case we have $f(x, y) = g(y)$ and $\bar{f}(x, y) = \bar{g}(y^2)$ for some function $\bar{g}$ such that $g(y) = y\bar{g}(y^2)$. Clearly $\bar{g}(0) = g'(0)$. Let us assume that $\bar{g}_\infty = \lim_{v\to\infty}\bar{g}(v) = \lim_{y\to\infty}\frac{g(y)}{y}$ exists and that $0 < \bar{g}_\infty < \infty$.
Then $\bar{f}_0 = \bar{g}(0)$ and $fi(w) = \inf_{u,v\ge w^2}|\bar{g}(v)| = \inf_{v\ge w^2}|\bar{g}(v)|$ which implies $fi_\infty = |\bar{g}_\infty|$. Analogously $fs_\infty = \bar{g}_\infty$. Hence we have period 4 solutions if either $\pi/(2|g'(0)|) < \alpha < \pi/(2|\bar{g}_\infty|)$ or $\pi/(2|\bar{g}_\infty|) < \alpha < \pi/(2|g'(0)|)$.

2. $f(x, y) = y\cdot(A + Bx^2 + Cy^2)$ with $A < 0, B \le 0, C < 0$. Here we have $\bar{f}(u, v) = A + Bu + Cv, \bar{f}_0 = A$. If $u, v \ge w^2$ then $Bu \le Bw^2, Cv \le Cw^2$ and $\bar{f}(u, v) \le A + (B + C)w^2 < 0$ which gives $fi(w) = |A + (B + C)w^2|, fs(w) = \infty$ and thus $fi_\infty = fs_\infty = \infty$. If $C \le B$ then

$$\bar{f}(u, v)/\bar{f}(v, u) - 1 = (B - C)\cdot\frac{u - v}{A + Bv + Cu} \le 0 \quad \text{for} \quad 0 \le v \le u$$

which implies $s_w \le 1$. If $C \ge B$ then

$$\bar{f}(u, v)/\bar{f}(v, u) - \frac{B}{C} = (1 - \frac{B}{C})\frac{A + Bu + Cv}{A + Bv + Cu} \quad \text{for} \quad 0 \le v \le u$$

which implies $s_w^2 \leq \frac{B}{C}$. In both cases we have $\frac{s_w}{w} \to 0$ for $w \to \infty$. Consequently (αf) has special symmetric solutions for $0 < \alpha < -\frac{\pi}{2A}$. In the next section we derive explicit formulas for this example.

3. Primary branch of special symmetric periodic solutions: existence and smoothness. For $z > 0$ let $(X(\cdot, z), Y(\cdot, z)) : \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ denote the solution of

$$\dot{X} = f(X, Y), \quad \dot{Y} = -f(Y, X), \quad X(0) = 0, Y(0) = -z. \qquad (X, Y)$$

From section 2 it follows that X, Y are defined and continuous for $t \in \mathbb{R}$. We shall prove that there exists exactly one $\alpha = \alpha(z) \neq 0$ such that

$$X(t + 2\alpha, z) = -X(t, z),\ Y(t, z) = X(t - \alpha, z), \quad t \in \mathbb{R}.$$

In this case, $x(t) := X(\alpha t, z)$ is a SSPS of (αf) with amplitude z. The proof is almost the same as for the Kaplan–Yorke equations as it is given e.g. in [3]. However some modifications have to be done since in general system (X,Y) is no longer Hamiltonian.

Throughout this section we assume that $f(x, y)$ satisfies conditions (1) - (3).

Lemma 3.1. *Define $h(t, z) = X(t, z) + Y(t, z)$ for $t \in \mathbb{R}$, $z > 0$. Then there exists exactly one $\alpha = \alpha(z)$ with $h(\alpha/2, z) = 0$ and $h(t, z) < 0$ for $0 \leq t < \alpha/2$.*

Proof. Since $h(0, z) = -z < 0$ it is sufficient to show that $h(\cdot, z)$ has positive values; α is then defined by

$$\alpha := 2 \inf\{t \geq 0 |\ h(t, z) \geq 0\}.$$

Suppose first that there exists a $t_* > 0$ with $Y(t_*, z) = 0$ and $Y(t, z) < 0$ for $0 \leq t < t_*$. Then $\dot{X} = f(X, Y)$ implies that $X(\cdot, z)$ is increasing in $[0, t_*]$ which shows that $h(t_*, z) = X(t_*, z) + Y(t_*, z) = X(t_*, z) \geq X(0, z) = 0$. Now assume that $Y(t, z) < 0$ for all $t \geq 0$. Then $X(\cdot, z)$ is increasing on $\mathbb{R}^+$ and thus nonnegative. Hence $\dot{Y} = -f(Y, X)$ implies that $Y(t, z)$ is also increasing on $\mathbb{R}^+$ but nonpositive. Consequently, $c := \lim_{t\to\infty} Y(t, z)$ exists and we have $-z \leq Y(t, z) \leq c \leq 0$ for all $t \geq 0$. If $c = 0$ then $\lim_{t\to\infty} h(t, z) > \lim_{t\to\infty} Y(t, z) = 0$. If $c < 0$ and if $X(t, z)$ is unbounded, then we have again $\lim_{t\to\infty} h(t, z) > 0$.

Finally assume $c < 0$ and $d := \lim_{t\to\infty} X(t, z) < \infty$. Then $(X(t, z), Y(t, z)) \in [\frac{d}{2}, d] \times [-z, c] =: K \subset \mathbb{R}^+ \times \mathbb{R}^-$ for $t > 0$ large enough. By our assumptions, f is positive on K, i.e. $C := \min\{f(u, v) | (u, v) \in K\} > 0$. This implies $\dot{X}(t, z) = f(X(t, z), Y(t, z)) \geq C$, a contradiction to $X(t, z) \to d$.

From the symmetries of $X(t, z), Y(t, z)$ we have for $t \in \mathbb{R}$:

$$X(t, -z) = -X(t, z),\ Y(t, -z) = -Y(t, z),$$

$$X(-t, z) = -X(t, z),\ Y(-t, z) = Y(t, z)$$

and $X(t, 0) = Y(t, 0) = 0$. Differentiation of X, Y with respect to z yields

$$\frac{\partial X}{\partial z}(t, 0) = -\sin(bt), \quad \frac{\partial Y}{\partial z}(t, 0) = \cos(bt)$$

for $t \in \mathbb{R}$ if f is of class C^1 and $b = f_y(0, 0)$. Next one easily derives that $(-Y(\alpha - t), z), -X(\alpha - t, z))$ solves (X, Y) and coincides with $(X(t, z), Y(t, z))$ at $t = \alpha/2$. Thus $X(t, z) = -Y(\alpha - t, z)$. This implies $X(\alpha, z) = z, Y(\alpha, z) = 0$ and one immediately verifies that $(-X(t + 2\alpha, z), -Y(t + 2\alpha, z))$ solves (X,Y) and coincides with $(X(t, z), Y(t, z))$ at $t = -\alpha$. Hence $X(t + 2\alpha, z) = -X(t, z)$ for $t \in \mathbb{R}, z \geq 0$. Now if we set $x(t, z) := X(\alpha t, z), t \in \mathbb{R}, z \geq 0$ we arrive at

Theorem 3.1. *There are continuous maps $\alpha : \mathbb{R}^+ \to \mathbb{R}$ and $x : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ such that for all $z > 0, x(\cdot, z)$ is a special symmetric periodic solution of (αf) with amplitude z.*

Proof. First we show that α and x are continuous. From the preceeding results we know that $\alpha = \alpha(z)$ is the first positive zero of $Y(\cdot, z)$. Because of $\dot{Y}(\alpha, z) = -f(0, z) > 0$, α is a simple zero. Since f is a Lipschitz continuous map, $Y(\cdot, z)$ is differentiable and X and Y are continuous maps. Thus $\dot{Y} = -f(Y, X)$ is continuous. This is enough hypothesis for the application of the implicit function theorem to show that $\alpha(z)$ is continuous.

It remains to prove that $x(t, z) \neq 0$ for $0 < t \leq 1$. This follows immediately from the next lemma.

Note that if f is a C^k-map, $k \in \mathbb{N} \cup \{\infty\} \cup \{\omega\}$, then X, Y and $\dot{Y}$ are C^k- maps and the implicit function theorem implies that α and x are C^k-maps too.

Lemma 3.2. *For all $z > 0$, $x(\cdot, z)$ is strictly increasing in $[0, 1]$.*

Proof. It is sufficient to show that $X(t, z)$ is strictly increasing in $[0, \alpha], \alpha = \alpha(z)$. If $X(\cdot, z)$ is not strictly increasing in $[0, \alpha]$ then there is a $t_* \in [0, \alpha]$ with $\dot{X}(t_*, z) = 0$. We can assume that $\dot{X}(t, z) > 0$ for $0 \leq t < t_*$ since $\dot{X}(0, z) = f(0, -z) > 0$. Thus $X(t, z) > 0$ for $0 < t \leq t_*$ and $\dot{X} = f(X, Y)$ implies $Y(t_*, z) = 0$. This yields $h(t_*, z) > 0$ and by the definition of α we find $t_* > \alpha/2$. Now $\alpha/2 < t_* \leq \alpha$ gives $0 \leq \alpha - t_* < \alpha/2$ and $X(\alpha - t_*, z) = -Y(t_*, z) = 0$ shows that $\alpha - t_* = 0$ or $\alpha - t_* > t_*$. But $\alpha - t_* > t_*$ leads to the contradiction $t_* < \alpha/2$. Thus $\alpha = t_*$ and $\dot{X}(t, z) > 0$ for $0 \leq t < \alpha$.

Definition. Let $x(z) := x(\cdot, z)|_{[0,1]} \in C_0$. *PB(f)*$:= \{(\alpha(z), x(z)) \in \mathbb{R} \times C : z > 0\}$ is called *primary branch* of special symmetric periodic solutions of equation (αf).

PB(f)$\subset \mathbb{R} \times C$ is a curve which has the same differentiability properties as f. If $f \in C^1(\mathbb{R}^2)$ and $b = f_y(0, 0) < 0$, set

$$\alpha(0) := -\frac{\pi}{2b}, \quad x(t, 0) := 0, \quad t \in \mathbb{R}, \quad x(0) := 0.$$

and define, for $z < 0$,

$$\alpha(z) := \alpha(-z),\ x(t, z) = -x(t, -z),\ x(-z) = -x(z), \quad t \in \mathbb{R}.$$

Thus $\alpha(z), x(z)$ are C^{k-1}-maps for $z \in \mathbb{R}$ if f is of class C^k.

It can happen that for a given $\alpha > 0$ there is no SSPS or that there are 2 or more such solutions. Hence, *PB(f)* cannot, in general, be parameterized by the delay parameter α in a unique way. However, this is possible for the amplitude.

Theorem 3.2. *Let $\alpha \in \mathbb{R}$ and $x : \mathbb{R} \to \mathbb{R}$ be a special symmetric periodic solution of (αf) with amplitude $z := x(1) > 0$. Then $\alpha = \alpha(z)$ and $x = x(\cdot, z)$.*

Proof. First of all, $\alpha \neq 0$, since otherwise $\dot{x} = 0$ and $x = x(0) = 0$, i.e. $z = 0$. Define

$$X(t) := x(\frac{t}{\alpha}),\ Y(t) := X(t - \alpha), \quad t \in \mathbb{R}.$$

Then $X(0) = x(0) = 0, Y(0) = X(-\alpha) = x(-1) = -x(-1 + 2) = -x(1) = -z$, and the special symmetry of x implies that $(X(t), Y(t))$ is a solution of (X,Y) with initial value $(0, -z)$. Consequently, $X(t) = X(t, z), Y(t) = Y(t, z), t \in \mathbb{R}$ and thus $x(t, z) = x(\beta t), t \in \mathbb{R}$ where $\beta := \alpha(z)/\alpha$. It remains to prove $\beta = 1$. Note that if x is a SSPS then x is odd, and $x(\tau) = 0, \tau > 0$ implies $\tau \geq 2$. Clearly x is odd

since $x(t) = x(\frac{t}{\beta}, z)$ for all $t \in \mathbb{R}$ and some $\beta \neq 0$. Now $x(t) \neq 0$ for $0 < t \leq 1$, and $x(2-t) = -x(-t) = x(t)$ shows that $x(t) \neq 0$ for $1 \leq t < 2$. Hence $\tau \geq 2$. Analogously, $x(\tau) = 0, \tau < 0$ implies $\tau \leq -2$. Now $x(2, z) = 0$ and $x(t, z) = x(\beta t)$ yield $|\beta| \geq 1$, while $x(2) = 0$ shows $1/|\beta| \geq 1$ and thus $|\beta| = 1$. If $\beta = -1$, then $x(t) = -x(t, z)$ and $z = x(1) = -x(1, z) = -z$, i.e. $z = 0$. This gives $\beta = 1$.

Examples. Let f fulfill our general assumption and let $u : \mathbb{R} \to \mathbb{R}$ be a SSPS of equation (αf). Let $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$ be a Lipschitz map and assume that g is even and symmetric with respect to x and y, i.e.

$$g(-x, y) = g(x, -y) = g(y, x) = g(x, y).$$

Then $\tilde{f} = f \cdot g$ fulfills our general assumptions too. Define

$$G(\tau) = \int_0^\tau \frac{dt}{g(u(t), v(t))}, \quad t \in \mathbb{R},$$

where $v(t) := u(t-1)$. G is continuous, strictly increasing and $G(0) = 0$. Set

$$\alpha := G(1), \quad \tau(t) := G^{-1}(\alpha t), \quad t \in \mathbb{R}.$$

Then $\tau(0) = 0, \tau(1) = 1$, and $\dot{\tau} = \alpha g(u(\tau), v(\tau))$.

One easily verifies that $\tau(t-1) = \tau(t) - 1$ for all $t \in \mathbb{R}$ and that $x := u \circ \tau$ is a SSPS of equation $(\alpha \tilde{f})$. Hence, if we know the primary branch of equation (αf), we can derive the primary branch of the perturbed map $\tilde{f} = f \cdot g$.

Example 1. Let $f(x, y) = -\frac{\pi}{2} y, z > 0, u(t) = z \sin(\frac{\pi}{2} t)$. Then $v(t) = -z \cos(\frac{\pi}{2} t)$ and

$$G(\tau) = \int_0^\tau \frac{dt}{g(z \sin \frac{\pi}{2} t, z \cos \frac{\pi}{2} t)} = \frac{2}{\pi} \int_0^{\frac{\pi\tau}{2}} \frac{ds}{g(z \sin s, z \cos s)}.$$

For the primary branch of $(\tilde{f})$, $\tilde{f}(x, y) = -\frac{\pi}{2} y g(x, y)$, we find

$$\begin{aligned}
\alpha = \alpha(z) &= \frac{2}{\pi} \int_0^{\pi/2} \frac{ds}{g(z \sin s, z \cos s)}, \\
x(t, z) &= z \sin(\frac{\pi}{2} \tau(t, z)), \\
\alpha(z) t &= \frac{2}{\pi} \int_0^{\frac{\pi}{2}\tau(t,z)} \frac{ds}{g(z \sin s, z \cos s)}.
\end{aligned}$$

For the particular equation $\dot{x}(t) = -\alpha x(t-1)(1 + x^4(t) + x^4(t-1))$ we find

$$\alpha(z) = \int_0^{\frac{\pi}{2}} \frac{d\psi}{1 + z^4(\sin^4 \psi + \cos^4 \psi)}, \quad 0 < z < 1.$$

This integral can be evaluated by standard methods, and the result is

$$\alpha(z) = \pi(4 + 6z^4 + 2z^8)^{-1/2} = \pi[2(z^4 + 1)(z^4 + 2)]^{-\frac{1}{2}}.$$

Example 2. Let $f(x, y) = Ay + Bx^2 y + Cy^3, A < 0, (x, y) \in \mathbb{R} \times \mathbb{R}$. This function is the most general cubic f that fulfills our assumptions. For this f we can calculate the primary branch explicitly.

For the case $B = 0$, i.e. $f(x, y) = g(y)$, the result can be found in [2]. The idea of the proof carries over for the case $B \neq 0$.

Consider $\dot{x}(t) = \alpha x(t-1)[A + Bx^2(t) + Cx^2(t-1)]$. We have to solve the system

$$\begin{aligned} \dot{x} &= y(A + Bx^2 + Cy^2), \quad x(0) = 0,\ x(\alpha) = z, \\ \dot{y} &= -x(A + By^2 + Cx^2), \quad y(0) = -z,\ y(\alpha) = 0. \end{aligned}$$

This system is hamiltonian with the Hamilton function

$$H(x, y) = 2A(x^2 + y^2) + 2Bx^2y^2 + C(x^4 + y^4).$$

Hence, the phase portrait consists of algebraic curves of order 4.

Now introduce polar coordinates:

$$x = r\sin\phi, \quad y = -r\cos\phi, \quad r(0) = r(\alpha) = z,\ \phi(0) = 0,\ \phi(\alpha) = \frac{\pi}{2}.$$

The equations for r and ϕ are

$$\begin{aligned} \dot{r} &= -\frac{1}{4}(C - B)r^3 \sin 4\phi, \quad r(0) = z \\ \dot{\phi} &= -A - \frac{1}{4}r^2[(B + 3C) + (C - B)\cos 4\phi], \quad \phi(0) = 0. \end{aligned}$$

Define $h(\phi) = -\frac{1}{4}[(B + 3C) + (C - B)\cos 4\phi]$, $h'(\phi) = (C - B)\sin 4\phi$ for $\phi \in \mathbb{R}$. Then

$$\dot{r} = -\frac{1}{4}r^3 h'(\phi), \quad \dot{\phi} = -A + r^2 h(\phi),$$

or, if $\rho := r^2$,

$$\dot{\rho} = -\frac{1}{2}\rho^2 h'(\phi), \quad \dot{\phi} = -A + \rho h(\phi).$$

This system is still hamiltonian with Hamilton function

$$H(\rho, \phi) := \rho^2 h(\phi) - 2A\rho,$$

and we have $H(\rho(t), \phi(t)) = H(z^2, 0) = -Cz^4 - 2Az^2 =: G(z)$. Set $\psi = 2\phi$, then

$$\begin{aligned} \dot{\psi}^2 = 4\dot{\phi}^2 =& 4A^2 + 4h(\phi)[\rho^2 h(\phi) - 2A\rho] = 4A^2 + 4h(\phi)G \\ =& 4A^2 - G[B + 3C + (C - B)\cos 2\psi] \\ =& 4A^2 - G[B + 3C + (C - B) - 2(C - B)\sin^2\psi] \\ =& 4A^2 - G[4C - 2(C - B)\sin^2\psi] \\ =& 4(A^2 - GC) + 2G(C - B)\sin^2\psi. \end{aligned}$$

Since $x(\alpha/2) = -y(\alpha/2)$, we know that $\phi(\alpha/2) = \frac{\pi}{4}$ or $\psi(\alpha/2) = \frac{\pi}{2}$.

Case $C \neq 0$. Set $m = m(z) = \frac{A}{A + Cz^2}$, $\gamma = \frac{1}{2}(1 - \frac{B}{C})$.

Because of $m\dot{\phi}(0) = m(-A + z^2 h(0)) = -A, 4\dot{\phi}^2(0) = 4(A^2 - GC)$, we see that $\dot{\psi} = 2\dot{\phi}(0)[1 - \gamma(1 - m^2)\sin^2\psi]^{\frac{1}{2}}$. If $F(\psi, n) := \int_0^\psi \frac{d\theta}{\sqrt{1 - n\sin^2\theta}}$ denotes the incomplete elliptic integral of the first kind, we find

$$F(2\phi(t, z), \gamma(1 - m^2(z))) = F(\psi(t, z), \gamma(1 - m^2(z))) = -\frac{2A}{m(z)}t.$$

Now $\psi(\frac{1}{2}\alpha(z), z) = \frac{\pi}{2}$ yields $\alpha(z) = -\frac{m(z)}{A}K(\gamma(1-m^2(z)))$ where $K(n) := F(\frac{\pi}{2}, n)$ denotes the complete elliptic integral of the first kind. $\alpha(z)$ is defined for $z \in [0, z_*)$ where z_* is determined by $z_* = \infty$, or $\gamma(1-m^2(z_*)) = 1$, or $A + Cz_*^2 = 0$.

Case $C = 0$. If $C = 0$, the equation for ψ reduces to

$$\dot{\psi} = 2\dot{\phi}(0)[1 + \frac{B}{A}\, 2z^2 \sin^2\psi]^{\frac{1}{2}},$$

which gives $\alpha(z) = -\frac{1}{A}K(-\frac{B}{A}z^2)$. Observe that Theorem 2.2 applies only for the case $B \leq 0$, $C < 0$. However one can verify by direct calculation that our formulas hold for the general case as long as $0 < z < z_*$.

We mention two other interesting cases:

(i) $A = 0, C \neq 0$. Here we have $G = -Cz^4$ and

$$\dot{\psi} = -2Cz^2[1 - \gamma \sin^2\psi]^{\frac{1}{2}}, \quad \alpha(z) = -\frac{1}{Cz^2}K(\gamma) = -\frac{1}{Cz^2}K(\frac{1}{2}(1 - \frac{B}{C}))$$

which makes sense only if $\frac{B}{C} > -1$. The case $B = -C$ leads to $\gamma = 1$, $\dot{\psi} = -2Cz^2\cos\psi$ and $\psi(t, z) = -\frac{\pi}{2} + 2\arctan\exp(-2Cz^2t)$. The boundary condition $\psi(\alpha/2) = \frac{\pi}{2}$ has no solution, i.e. the equation

$$\dot{x}(t) = \alpha\, Cx(t-1)[x^2(t-1) - x^2(t)]$$

has no SSPSs.

(ii) $B = C$. Here we see that $\dot{\psi} = 2\sqrt{A^2 - GC} = -2(A + z^2C)$ and $\alpha(z) = -\frac{\pi}{2}\frac{1}{A+Cz^2}$. Hence $\dot{\psi} = \frac{\pi}{\alpha}$ or $\psi(t) = \frac{\pi}{\alpha}t$. Since $\dot{r} = 0$ in this case, we find $x(t, z) = z\sin(\frac{\pi}{2}t)$.

4. Characterization of the spectrum. Let $C := C([0, 1])$ be the Banach space of continuous functions $\phi : [0, 1] \to \mathbb{R}$. Let $F : C \to \mathbb{R}$ be a Lipschitz map which is k times continuously differentiable, $k \in \mathbb{N}_0$. Then the initial value problem

$$\dot{x}(t) = F(x_{t-1}), \quad t \geq 1, \quad x_0 = \phi \tag{F}$$

($x_t := x(t + \cdot)|_{[0,1]} \in C$) has a unique solution $x(\cdot, \phi) : \mathbb{R}_0^+ \to \mathbb{R}$ for any $\phi \in C$. Define $X : \mathbb{R}_0^+ \times C \to C$ by $X(t, \phi) := x(\cdot, \phi)_t = x(t + \cdot, \phi)|_{[0,1]}$. Then X is a continuous semiflow on C and a C^k−map on $(k, \infty) \times C$.

Suppose that $x : \mathbb{R} \to \mathbb{R}$ is a periodic solution of (F) with period $\omega > 0$. Let $\phi := x_0 \in C$, then $x|_{\mathbb{R}_0^+} = x(\cdot, \phi)$ and

$$X(\omega, \phi) = x(\omega + \cdot, \phi)|_{[0,1]} = x(\cdot, \phi)|_{[0,1]} = x_0 = \phi.$$

Clearly, if $X(\omega, \phi) = \phi$ holds for some $(\omega, \phi) \in \mathbb{R}^+ \times C$, then $x(\cdot, \phi)$ defines a periodic solution of (F) with period ω.

We can describe periodic solutions of (F) as fixed points of a Poincaré map in the following way. Let $C_0 := \{\phi \in C | \phi(0) = 0\}$. For $\phi \in C_0$ let $\tau(\phi)$ denote the first zero of $x(\cdot, \phi)$. If $x(\cdot, \phi)$ has no such zero, set $\tau(\phi) = \infty$. Let $\Lambda \subset C_0$ be the set of all $\phi \in C_0$ such that

(i) $1 < \tau(\phi) < \infty$
(ii) $\tau(\phi)$ is a simple zero of $x(\cdot, \phi)$
(iii) $x(\cdot, \phi)$ has no zero in $(\tau(\phi), \tau(\phi) + 1)$.

In view of (ii), the implicit function theorem shows that Λ is open and $\tau : \Lambda \to \mathbb{R}^+$ is a C^1-map.

Now define maps $Q : \Lambda \to C_0,\ P : \Omega \to C_0$ by

$$Q(\phi) := X(\tau(\phi), \phi) \in C_0, \phi \in \Lambda, \quad P(\phi) = Q(Q(\phi)), \phi \in \Omega$$

where $\Omega := \{\phi \in \Lambda |\ Q(\phi) \in \Lambda\} = Q^{-1}(\Lambda)$.

Obviously, $\Omega \subset C_0$ is open and Q and P are C^1-maps. Moreover, if $P(\phi) = \phi$ for some $\phi \in \Omega$, then $x(\cdot, \phi)$ yields a periodic solution of equation (F) with period $\omega = \tau(\phi) + \tau(Q(\phi))$.

Definition. $P : \Omega \to C_0$ is called *Poincaré map* of (F), while $Q : \Lambda \to C_0$ is called *semi-Poincaré map* of (F).

If $P(\phi) = \phi$ then $DP(\phi)$ is a compact linear operator. Thus $\sigma(DP(\phi)) = \sigma_p(DP(\phi)) \subset \mathbb{C}$ is compact; $\lambda \in \sigma(DP(\phi))$ is called a *Floquet multiplier* of the periodic solution $x(\cdot, \phi)$. The calculation of $\sigma(DP(\phi))$ can be simplified if F is an odd map and $x = x(\cdot, \phi)$ is symmetric, i.e.

$$x(t + \tau) = -x(t), \quad t \in \mathbb{R}$$

for some $\tau > 0$. Clearly, x is periodic with period $\omega = 2\tau$. If F is odd, then one has for $t \in \mathbb{R}_0^+$ and $\phi \in C$:

$$\begin{aligned} X(t, -\phi) &= -X(t, \phi) \\ -\phi \in \Lambda &\Longleftrightarrow \phi \in \Lambda \\ -\phi \in \Omega &\Longleftrightarrow \phi \in \Omega \\ \tau(-\phi) &= \tau(\phi) \\ Q(-\phi) &= -Q(\phi), P(-\phi) = -P(\phi). \end{aligned}$$

If $Q(\phi) = -\phi$ then $x(\cdot, \phi)$ defines a symmetric solution of (F) with $\tau = \tau(\phi)$.

Clearly $DP(\phi) = DQ(\phi) \circ DQ(\phi)$ and thus

$$\lambda \in \sigma(DP(\phi)) \Longleftrightarrow \text{ there is a } \quad \mu \in \sigma(DQ(\phi)) \quad \text{ with } \quad \lambda = \mu^2.$$

We call $\mu \in \sigma(DQ(\phi))$ *semi-Floquet multiplier* of the symmetric solution $x(\cdot, \phi)$. Obviously, the Floquet multipliers of $x(\cdot, \phi)$ are the squares of the semi-Floquet multipliers.

We relate next the (semi-) Floquet multipliers to the variational operator. Assume that F is an odd C^1-map and that $x = x(\cdot, \phi)$ is a symmetric periodic solution of equation (F) with period $\omega = 2\tau > 2$. The variational equation of F for ϕ is

$$\dot{w}(t) = DF(x_{t-1}(\phi))w_{t-1}, \quad t \geq 1, \quad w_0 = \xi \in C. \qquad (F, \phi, \xi)$$

Let $w(\cdot, \phi, \xi) : \mathbb{R}_0^+ \to \mathbb{R}$ denote the solution of (F, ϕ, ξ) and define $W_\phi \xi := w(\tau(\phi) + \cdot, \phi, \xi)|_{[0,1]} \in C$.

Clearly, $w(\cdot, \phi, \xi)$ is linear in ξ and thus we have $W_\phi \in L(C, C)$.

Lemma 4.1. *Let $U_\phi := DQ(\phi) \in L(C_0, C_0)$ and define $\dot{\phi} \in C$ by $\dot{\phi}(s) := -\dot{x}(\tau + s, \phi)$, $s \in [0, 1]$, i.e. $\dot{\phi} = -\dot{X}(\tau(\phi), \phi)$. Then*

(i) $W_\phi = \frac{\partial X}{\partial \phi}(\tau(\phi), \phi)$

(ii) $U_\phi \eta = W_\phi \eta - \kappa_\phi(\eta) \cdot \dot{\phi}$ *for all* $\eta \in C_0$ *where* $\kappa_\phi(\eta) := \frac{W_\phi \eta(0)}{\dot{\phi}(0)}$.

Proof. First recall that $\dot{\phi}(0) = \dot{x}(\tau) = \dot{x}(\tau(\phi), \phi) \neq 0$ since $\tau = \tau(\phi)$ is assumed to be a simple zero of $x(\cdot, \phi)$. Differentiation of $\dot{x}(t, \phi) = F(X(t-1, \phi)), \quad t \geq 1,\ X(0, \phi) = \phi$ with respect to ϕ yields for $\xi \in C$

$$\frac{d}{dt}[\frac{\partial x}{\partial \phi}(t, \phi)\xi] = DF(X(t-1, \phi))[\frac{\partial X}{\partial \phi}(t-1, \phi)\xi], \quad t \geq 1, \quad [\frac{\partial X}{\partial \phi}(0, \phi)\xi] = \xi.$$

Hence $w(t) := \frac{\partial x}{\partial \phi}(t, \phi)\xi$ solves (F, ϕ, ξ) since $w_t = \frac{\partial X}{\partial \phi}(t, \phi)\xi$. Thus $w(t, \phi, \xi) = \frac{\partial x}{\partial \phi}(t, \phi)\xi$, and $W_\phi \xi = w(\tau + \cdot, \phi, \xi)|_{[0,1]} = w_\tau = \frac{\partial X}{\partial \phi}(\tau, \phi)\xi$, which shows

$$W_\phi = \frac{\partial X}{\partial \phi}(\tau(\phi), \phi).$$

Now let $\eta \in C_0$. The definition of Q yields

$$U_\phi \eta = DQ(\phi)\eta = \dot{X}(\tau(\phi), \phi)[\frac{\partial \tau}{\partial \phi}(\phi)\eta] + \frac{\partial X}{\partial \phi}(\tau(\phi), \phi)\eta = -\dot{\phi}[\frac{\partial \tau}{\partial \phi}(\phi)\eta] + W_\phi \eta.$$

Since $U_\phi \eta \in C_0$, it follows that

$$0 = U_\phi \eta(0) = -\dot{\phi}(0)[\frac{\partial \tau}{\partial \phi}(\phi)\eta] + W_\phi \eta(0)$$

which gives $\frac{\partial \tau}{\partial \phi}(\phi)\eta = \kappa_\phi(\eta)$ and thus (ii).

Now as an easy consequence we can relate the eigenvalues of U_ϕ and W_ϕ.

Corollary 4.1. (i) Suppose that $W_\phi \xi = \lambda \xi$ holds for some $\lambda \in \mathbb{C}$ and $\xi \in C$. Then $U_\phi \eta = \lambda \eta$ where $\eta := \xi - \nu_\phi(\xi)\dot{\phi},\ \nu_\phi(\xi) := \frac{\xi(0)}{\dot{\phi}(0)} \in \mathbb{C}$.

(ii) Suppose that $U_\phi \eta = \lambda \eta$ holds for $\lambda \in \mathbb{C}$ and $\eta \in C_0$. Then $W_\phi \xi = \lambda \xi$ where $\xi := \eta + \frac{1}{1+\lambda}\kappa_\phi(\eta) \cdot \dot{\phi}$.

Since $W_\phi \dot{\phi} = -\dot{\phi}$, we have $-1 \in \sigma(W_\phi)$. Thus the corollary yields

Lemma 4.2. $\sigma(W_\phi) = \sigma(U_\phi) \cup \{-1\}$.

Semi-Floquet multipliers ($\neq -1$) can be found by solving the characteristic equation $W_\phi \xi = \lambda \xi, \quad \xi \neq 0$. Suppose that τ is an integer > 1 and define $V_\phi := \frac{\partial X}{\partial \phi}(1, \phi) \in L(C, C)$. Since X is a semiflow, we have $X(\tau, \phi) = X(1, \phi) \circ \overset{\tau}{\cdots} \circ X(1, \phi)$ and differentiation gives ($x = x(\cdot, \phi)$)

$$W_\phi = \frac{\partial X}{\partial \phi}(\tau, \phi) = \frac{\partial X}{\partial \phi}(1, X(\tau - 1, \phi) \circ \overset{\tau}{\cdots} \circ \frac{\partial X}{\partial \phi}(1, X(0, \phi)) = V_{x_{\tau-1}} \circ \overset{\tau}{\cdots} \circ V_{x_1} \circ V_{x_0}.$$

The characteristic equation for $\lambda \in \mathbb{C}$ can be written in the form ($\xi_0 = \xi$)

$$\begin{aligned} &\xi_1 = V_{x_0}\xi_0 \\ &\cdots \\ &\xi_\tau = V_{x_{\tau-1}}\xi_{\tau-1} \\ &\lambda \xi_0 = \xi_\tau \end{aligned}$$

where $V_\phi \eta = w(1 + \cdot)|_{[0,1]}$ and w is the solution of

$$\dot{w}(t) = DF(X(t-1, \phi))w_{t-1}, \quad t \geq 1, \quad w_0 = \eta.$$

$\lambda \in \mathbb{C} \setminus \{-1\}$ is a semi-Floquet multiplier of ϕ if and only if this system has a nontrivial solution $(\xi_0, \dots, \xi_\tau) \in C^{\tau+1}$.

Let $f \in C^1(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ fulfill the general assumptions. If $ev_s \in C', s \in [0,1]$ denotes the evaluation map $ev_s(\phi) := \phi(s),\ \phi \in C$, equation

$$\dot{x}(t) = f(x(t), x(t-1)), \quad x|_{[0,1]} = \phi \tag{f, ϕ}$$

is equivalent to (F), if $F \in C^1(C)$ is defined by

$$F(\phi) := f(ev_1(\phi), ev_0(\phi)), \quad \phi \in C.$$

Clearly, F is an odd map.

The operator $V_\phi \in L(C, C)$ can be described in the following way

Lemma 4.3. *Let $\phi, \xi, \eta \in C$ and let $x : \mathbb{R}_0^+ \to \mathbb{R}$ be the solution of (f, ϕ). Then $\eta = V_\phi \xi$ holds if and only if η is differentiable, $\eta(0) = \xi(1)$ and if for $t \in [0,1]$*

$$\dot{\eta}(t) = \frac{\partial f}{\partial x}(x(t+1), x(t))\eta(t) + \frac{\partial f}{\partial y}(x(t+1), x(t))\xi(t).$$

Proof. The variational equation (F, ϕ, ξ) is given by

$$\dot{w}(t) = DF(x_{t-1})w_{t-1} = \frac{\partial f}{\partial x}(x(t), x(t-1))w(t) + \frac{\partial f}{\partial y}(x(t), x(t-1))w(t-1),$$
$$t \geq 1,\ w|_{[0,1]} = \xi.$$

If $w : \mathbb{R}_0^+ \to \mathbb{R}$ is the solution, we have $V_\phi \xi = w(1 + \cdot)|_{[0,1]}$. Hence if $\eta = V_\phi \xi$ holds, then $\eta(t) = w(1+t),\ t \in [0,1]$ is differentiable, $\eta(0) = w(1) = \xi(1)$ and differentiation yields the relation stated in the lemma. By uniqueness, we have $\eta(t) = w(1+t),\ t \in [0,1]$ if this relation and $\eta(0) = \xi(1)$ holds for η.

Now let $x : \mathbb{R} \to \mathbb{R}$ be a SSPS. From the proceedings remarks $\lambda \in \mathbb{C} \setminus \{-1\}$ is a semi-Floquet multipliers if and only if (recall $\tau = 2$)

$$\lambda \xi_0 = \xi_2, \quad \xi_1 = V_{x_0}\xi_0, \quad \xi_2 = V x_1 \xi_1$$

has a nontrivial solution $(\xi_0, \xi_1, \xi_2) \in C \times C \times C$. Set $\xi := \xi_0, \eta := \xi_1$, then we have the equivalent system

$$\eta = V_{x_0}\xi, \quad \lambda\xi = V_{x_1}\eta.$$

With our last lemma we can transform this system into a linear ordinary boundary value problem.

Lemma 4.4. *Let $x : \mathbb{R} \to \mathbb{R}$ be a special symmetric periodic solution of equation (f). Then $\lambda \in \mathbb{C} \setminus \{-1\}$ is a semi-Floquet multiplier if and only if the system*

$$\begin{aligned} \dot{\eta} &= -\frac{\partial f}{\partial x}(y, x)\eta + \frac{\partial f}{\partial y}(y, x)\xi, \quad \eta(0) = \xi(1) \\ \lambda\dot{\xi} &= \lambda\frac{\partial f}{\partial x}(x, y)\xi + \frac{\partial f}{\partial y}(x, y)\eta, \quad \lambda\xi(0) = \eta(1) \end{aligned} \tag{λ}$$

has a nontrivial solution (here $y(t) = x(t-1)$).

Proof. Differentiate $\eta = V_{x_0}\xi, \quad \lambda\xi = V_{x_1}\eta$ and use the proceeding lemma and the symmetries of x and f.

For $t \in \mathbb{R}, \lambda \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$ define

$$A(t,\lambda) = \begin{pmatrix} -\frac{\partial f}{\partial x}(y,x) & \frac{\partial f}{\partial y}(y,x) \\ \frac{1}{\lambda}\frac{\partial f}{\partial y}(x,y) & \frac{\partial f}{\partial x}(x,y) \end{pmatrix}(t) \in \mathbb{C}^{(2,2)}$$

and let $S(\cdot,\lambda) : \mathbb{R} \to \mathbb{C}^{(2,2)}$ denote the fundamental matrix solution of

$$\dot{S} = A(\cdot,\lambda)S, \quad S(0) = 1 \in \mathbb{C}^{(2,2)}.$$

If (η,ξ) is a solution of (λ), we have

$$\begin{pmatrix} \eta \\ \xi \end{pmatrix} = S(\cdot,\lambda)\begin{pmatrix} \eta(0) \\ \xi(0) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \eta(1) \\ \xi(1) \end{pmatrix} = S(1,\lambda)\begin{pmatrix} \eta(0) \\ \xi(0) \end{pmatrix}.$$

On the other hand

$$\begin{pmatrix} \eta(1) \\ \xi(1) \end{pmatrix} = \begin{pmatrix} \lambda\xi(0) \\ \eta(0) \end{pmatrix} = \begin{pmatrix} 0 & \lambda \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \eta(0) \\ \xi(0) \end{pmatrix}$$

which gives

$$\left[S(1,\lambda) - \begin{pmatrix} 0 & \lambda \\ 1 & 0 \end{pmatrix}\right]\begin{pmatrix} \eta(0) \\ \xi(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

If (η,ξ) is nontrivial, we have $\begin{pmatrix} \eta(0) \\ \xi(0) \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, otherwise $\begin{pmatrix} \eta(0) \\ \xi(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Hence, if $\lambda \in \mathbb{C} \setminus \{0,-1\}$ is a semi-Floquet multiplier then

$$r(\lambda) := det\left[S(1,\lambda) - \begin{pmatrix} 0 & \lambda \\ 1 & 0 \end{pmatrix}\right] = 0.$$

Clearly, if $r(\lambda) = 0$ for $\lambda \in \mathbb{C} \setminus \{0,-1\}$ then there is a $\begin{pmatrix} \eta_0 \\ \xi_0 \end{pmatrix} \in \mathbb{C}^2$ with $\begin{pmatrix} \eta_0 \\ \xi_0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\left[S(1,\lambda) - \begin{pmatrix} 0 & \lambda \\ 1 & 0 \end{pmatrix}\right]\begin{pmatrix} \eta_0 \\ \xi_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, where $\begin{pmatrix} \eta \\ \xi \end{pmatrix} := S(\cdot,\lambda)\begin{pmatrix} \eta_0 \\ \xi_0 \end{pmatrix}$ defines a solution of (λ). Thus we arrive at

Lemma 4.5. *$\lambda \in \mathbb{C} \setminus \{0,-1\}$ is a semi-Floquet multiplier of x if and only if λ is a zero of the characteristic function r.*

Remarks. 1. r describes the eigenvalues of the variational operator W_{x_0}, i.e. $\sigma(W_{x_0}) = r^{-1}(\{0\})$. Since $-1 \in W_{x_0}$, we have $r(-1) = 0$. If we want to know whether -1 is a semi-Floquet multiplier we need some deeper analysis. The result is that $-1 \in \sigma(U_{x_0})$ holds if and only if $r'(-1) = 0$ (see also [5]).

2. We also expect that the multiplicity of $\lambda \in \mathbb{C}^*$ as a zero of r is equal to the algebraic multiplicity of λ as an eigenvalue of W_{x_0} (compare Corollary 3.3 in [9]).

3. Although the map f may be only of class C^1, r is analytic in $\mathbb{C}^*$ (see [9],[1]).

Lemma 4.6. det $S(1,\lambda) = 1$.

Proof. Since det $S(t,\lambda) = \det S(0,\lambda) \exp\left\{\int_0^t \left(-\frac{\partial f}{\partial x}(y,x) + \frac{\partial f}{\partial x}(x,y)\right) dt\right\}$, it is enough to show that

$$\int_0^1 \left(-f_x(y(t),x(t)) + f_x(x(t),y(t))\right) dt = 0.$$

We can assume further that $x(0) = 0,\ x(t) > 0$ for all $t \in [0,1]$ (as it is defined for SSPSs). Then $y(1) = 0,\ y(t) < 0$ for $t \in [0,1)$, and moreover,

$$x(t) = -y(1-t) \quad \text{and} \quad y(t) = -x(1-t) \quad \text{for all} \quad t \in \mathbb{R}.$$

To show this note that $x(t)$ is odd and $y(t)$ is even in t and $y(t) = x(t-1),\ t \in \mathbb{R}$. This implies $-y(1-t) = -x(-t) = x(t)$ and $-x(1-t) = x(t-1) = y(t)$ for $t \in \mathbb{R}$.

Now using the oddness of f_x with respect to both arguments we find by substituting $t \mapsto 1-t$

$$\int_0^1 \left(-f_x(y,x) + f_x(x,y)\right) dt = 0.$$

5. Stability of special symmetric periodic solutions with small amplitude. The main result of this section is the following

Theorem 5.1. *There exist a C^1-map $s : \mathbb{R} \to \mathbb{R}$ and $\epsilon > 0$ such that for every SSPS of equation (αf) with amplitude $z \in (0,\epsilon)$ $s(\alpha)$ is a nontrivial Floquet multiplier with greatest norm. If additionally $f_{yyy}(0,0) + f_{xxy}(0,0) \neq 0$ then*

$$s(\alpha^*) = 1, \quad s'(\alpha^*) = \frac{16\pi}{4+\pi^2} \cdot f_y(0,0).$$

It will be shown later (see corollary 5.3) that $f_{yyy}(0,0) + f_{xxy}(0,0) > 0$ implies forward bifurcation at $\alpha = \alpha^*$. Since $s'(\alpha^*) < 0$, there is a Floquet multiplier $s(\alpha) < 1$ for $\alpha > \alpha^*$ close to α^*. Since all other Floquet multipliers have norms less than $s(\alpha)$, the corresponding SSPS is stable. In the case $f_{yyy}(0,0) + f_{xxy}(0,0) < 0$ we have backward bifurcation at α^*. Therefore for small $z > 0$ one has $\alpha(z) < \alpha^*$ and thus $s(\alpha(z)) > 1$, which gives unstable SSPSs $x(\cdot, z)$.

Consider again example 2 with $f(x,y) = Ay + Bx^2y + Cy^3$. Here we have forward bifurcation at $\alpha^* = -\frac{\pi}{2A}$ with stable periodic solutions for $B+3C > 0$ and backward bifurcation with unstable periodic solutions for $B + 3C < 0$.

Throughout this section we make the assumption that there exists a $\bar{f} \in C^k(\mathbb{R} \times \mathbb{R}, \mathbb{R}), k \geq 3$ such that

$$f(x,y) = y\bar{f}(x^2,y^2) \quad \text{for all} \quad x,y \in \mathbb{R}^2 \tag{$\bar{f}$}$$

Lemma 5.1. *(i) $\bar{f}(0,0) = f_y(0,0),\ \bar{f}_y(0,0) = \frac{1}{6}f_{yyy}(0,0),\ \bar{f}_x(0,0) = \frac{1}{2}f_{xxy}(0,0)$; (ii) If $\hat{f}(x,y) := \bar{f}(x,y) + 2y\bar{f}_y(x,y)$ then $f_y(x,y) = \hat{f}(x^2,y^2)$ for all $x,y \in \mathbb{R}^2$ and $\hat{f}(0,0) = f_y(0,0),\ \hat{f}_x(0,0) = \frac{1}{2}f_{xxy}(0,0),\ \hat{f}_y(0,0) = \frac{1}{2}f_{yyy}(0,0)$.*

The proof follows by direct calculations.

For the rest of this section denote $b := f_y(0,0) < 0,\ b_1 := f_{yyy}(0,0),\ b_2 := f_{xxy}(0,0)$.

Lemma 5.2. *There are $\epsilon > 0$ and C^k-maps $(X, Y) : \mathbb{R} \times (-\epsilon^2, \epsilon^2) \to \mathbb{R}^2$, $\beta : (-\epsilon^2, \epsilon^2) \to \mathbb{R}^+$ such that for all $(t, w) \in \mathbb{R} \times (-\epsilon^2, \epsilon^2)$:*

(i) $\dot{X}(t, w) = \beta(w)Y(t, w)\bar{f}(wX^2(t, w), wY^2(t, w))$, $X(0, w) = 0$, $X(1, w) = 1$,
$\dot{Y}(t, w) = \beta(w)X(t, w)\bar{f}(wY^2(t, w), wX^2(t, w))$, $Y(0, w) = -1, Y(1, w) = 0$;

(ii) $-1 \le Y(t, w) \le 0 \le X(t, w) \le 1$ *for all* $t \in [0, 1]$;

(iii) $Y(t, w) = X(t - 1, w)$ *and* $X(t + 2, w) = -X(t, w)$;

(iv) $X(t, 0) = \sin \frac{\pi}{2}t$, $Y(t, 0) = -\cos \frac{\pi}{2}t$, *and* $\beta(0) = \alpha^* = -\frac{\pi}{2f_y(0,0)}$.

Proof. Let $(\bar{X}, \bar{Y}) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^2$ be the solution of

$$\begin{aligned} \dot{\bar{X}} &= \bar{Y}\bar{f}(w\bar{X}^2, w\bar{Y}^2), \quad \bar{X}(0, w) = 0 \\ \dot{\bar{Y}} &= -\bar{X}\bar{f}(w\bar{Y}^2, w\bar{X}^2), \quad \bar{Y}(0, w) = -1. \end{aligned} \tag{$\bar{X}, \bar{Y}$}$$

Define $H(t, w) := \bar{X}(t, w) + \bar{Y}(t, w)$ for $t \in \mathbb{R}$, $|w| < \epsilon^2$. Obviously $\bar{X}(t, 0) = -\sin(bt), \bar{Y}(t, 0) = -\cos(bt)$, $t \in \mathbb{R}$ and thus

$$\begin{aligned} H(\frac{\alpha^*}{2}, 0) &= \sin\frac{\pi}{4} - \cos\frac{\pi}{4} = 0, \\ \dot{H}(\frac{\alpha^*}{2}, 0) &= -b\cos\frac{\pi}{4} - b\sin\frac{\pi}{4} = -b\sqrt{2} > 0. \end{aligned}$$

The implicit function theorem yields an $\epsilon > 0$, an open interval $I \subset \mathbb{R}$ with $\alpha^*/2 \in I$, and a C^1-map $\beta : (-\epsilon^2, \epsilon^2) \to I$ with $H^{-1}(\{0\}) \cap (I \times (-\epsilon^2, \epsilon^2)) = \{(\frac{1}{2}\beta(w), w) | \, |w| < \epsilon^2\}$ and $\beta(0) = \alpha^*$. Consequently, $\bar{X}(\frac{1}{2}\beta(w), w) = -\bar{Y}(\frac{1}{2}\beta(w), w)$ for $|w| < \epsilon^2$.

Observe that $(-\bar{X}(-t, w), \bar{Y}(-t, w))$ solves $(\bar{X}, \bar{Y})$ and coincides with $(\bar{X}(t, w), \bar{Y}(t, w))$ at $t = 0$, i.e. $\bar{X}(\cdot, w)$ is odd and $\bar{Y}(\cdot, w)$ is even. Now it is easy to check that $(\bar{Y}(t + \beta(w), w), -\bar{X}(t + \beta(w), w))$ solves $(\bar{X}, \bar{Y})$ and coincides with $(\bar{X}(t, w), \bar{Y}(t, w))$ at $t = -\frac{1}{2}\beta(w)$. Thus

$$\bar{X}(t, w) = \bar{Y}(t + \beta(w), w), \; \bar{Y}(t, w) = -\bar{X}(t + \beta(w), w)$$

which shows

$$\bar{X}(\beta(w), w) = -\bar{Y}(0, w) = 1, \quad \bar{Y}(\beta(w), w) = \bar{X}(0, w) = 0$$

and

$$\bar{Y}(t, w) = \bar{X}(t - \beta(w), w), \quad \bar{X}(t, w) = \bar{Y}(t + \beta(w), w) = -\bar{X}(t + 2\beta(w), w).$$

Define $X(t, w) := \bar{X}(\beta(w)t, w)$, $Y(t, w) := \bar{Y}(\beta(w)t, w)$, then (i), (iii), and (iv) are obvious.

In order to prove (ii) one can assume that $\frac{1}{2}\beta(w)$ is the first zero of $H(\cdot, w)$. Then $\bar{f} < 0$ and $(\bar{X}, \bar{Y})$ imply that $\bar{X}$ and $\bar{Y}$ are increasing on $[0, \beta(w)]$.

X and Y are related to the SSPSs of equation (αf) in the following way:

Corollary 5.1. For $z \in (-\epsilon, \epsilon)$ and $t \in \mathbb{R}$ define $\alpha(z) := \beta(z^2)$, $x(t, z) := z \cdot X(t, z^2), y(t, z) := z \cdot Y(t, z^2)$. Then

(i) $\dot{x}(t, z) = \alpha(z)f(x(t, z), y(t, z))$, $x(0, z) = 0$, $x(1, z) = z$,
$\dot{y}(t, z) = -\alpha(z)f(y(t, z), x(t, z))$, $y(0, z) = -z$, $y(1, z) = 0$;

(ii) $x(0, z) = x(2, z) = 0$, $x(t, z) \neq 0$ for $0 < t < 2$ and
$-z \le y(t, z) \le 0 \le x(t, z) \le z$ for $0 \le t \le 1$.

(iii) $y(t, z) = x(t - 1, z)$, $x(t + 2, z) = -x(t, z)$;

(iv) $x(t, 0) = 0$, $y(t, 0) = 0$, $\alpha(0) = \alpha^*$,
$\frac{\partial x}{\partial z}(t, 0) = \sin\frac{\pi}{2}t$, $\frac{\partial y}{\partial z}(t, 0) = -\cos\frac{\pi}{2}t$, and $\alpha'(0) = 0$.

Definition. Define $g, h, c, d : \mathbb{R} \times (-\epsilon^2, \epsilon^2) \to \mathbb{R}$ by

$$\begin{aligned}
c(t,w) &= -\beta(w) 2wX(t,w)Y(t,w)\bar{f}_u(wY^2(t,w), wX^2(t,w)) \\
d(t,w) &= \beta(w) 2wY(t,w)X(t,w)\bar{f}_u(wX^2(t,w), wY^2(t,w)) \\
g(t,w) &= \beta(w)\hat{f}(wY^2(t,w), wX^2(t,w)) \\
h(t,w) &= \beta(w)\hat{f}(wX^2(t,w), wY^2(t,w)).
\end{aligned}$$

Let

$$A(t,w,\lambda) = \begin{pmatrix} c(t,w) & g(t,w) \\ \frac{1}{\lambda}h(t,w) & d(t,w) \end{pmatrix},$$

and let $S(t,w,\lambda)$ denote the solution of $\dot{S} = A(\cdot,w,\lambda)\,S,\ S(0,w,\lambda) = 1 \in \mathbb{C}^{(2,2)}$ for $t \in \mathbb{R}, |w| < \epsilon^2, \lambda \in \mathbb{C} \setminus \{0\}$. Finally define

$$r(w,\lambda) := \det\left[S(1,w,\lambda) - \begin{pmatrix} 0 & \lambda \\ 1 & 0 \end{pmatrix}\right] \quad \text{for} \quad |w| < \epsilon^2, \lambda \in \mathbb{C} \setminus \{0\}.$$

Note that if $w = z^2$ then A, S and r are the same functions as defined in section 4 for the SSPS $x(\cdot, z)$, e.g. by $(\bar{f})$

$$\begin{aligned}
\alpha\frac{\partial f}{\partial x}(x,y) &= \alpha \cdot 2xy\bar{f}_u(x^2,y^2) \\
&= 2\beta(z^2)\, zX(t,z^2)\, zY(t,z^2)\, \bar{f}_u(z^2X^2(t,z^2), z^2Y^2(t,z^2)) \\
&= 2\beta(w)\, wX(t,w)Y(t,w)\, \bar{f}_u(wX^2(t,w), wY^2(t,w)) = d(t,w).
\end{aligned}$$

Thus, $\lambda \in \mathbb{C} \setminus \{0,-1\}$ is a semi-Floquet multiplier of $x(\cdot,z)$ iff $r(z^2,\lambda) = 0$.

For $\lambda \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$ set $C_\lambda := \begin{pmatrix} 0 & 1 \\ \lambda^{-1} & 0 \end{pmatrix} \in \mathbb{C}^{(2,2)}$, and $C := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Lemma 5.3. $A(t,0,\lambda) = -\frac{\pi}{2}C_\lambda$,
$A_w(t,0,\lambda) = \beta'(0)bC_\lambda + \frac{\beta(0)}{4}b_1(C_\lambda - \cos\pi t C_{-\lambda}) + \frac{\beta(0)}{4}b_2(C_\lambda + \cos\pi t C_{-\lambda})$
$+\frac{\beta(0)}{2}b_2 \sin\pi t\, C.$

Proof. For $t \in \mathbb{R}$ we have $g(t,0) = \beta(0)\hat{f}(0,0) = h(t,0), \quad c(t,0) = d(t,0) = 0$. Now $\beta(0)\hat{f}(0,0) = -\frac{\pi}{2}$ gives $A(t,0,\lambda) = -\frac{\pi}{2}C_\lambda$.

Differentiation of g with respect to w yields

$$\begin{aligned}
g_w(t,w) =& \beta'(w)\hat{f}(wY^2(t,w), wX^2(t,w)) \\
&+ \beta(w)\{\hat{f}_u(wY^2(t,w), wX^2(t,w))[Y^2(t,w) + w\frac{\partial}{\partial w}Y^2(t,w)] \\
&+ \hat{f}_v(wY^2(t,w), wX^2(t,w))[X^2(t,w) + w\frac{\partial}{\partial w}X^2(t,w)]\}.
\end{aligned}$$

By setting $w = 0$ this implies

$$\begin{aligned}
g_w(t,0) &= \beta'(0)\hat{f}(0,0) + \beta(0)[\hat{f}_u(0,0)Y^2(t,0) + \hat{f}_v(0,0)X^2(t,0)] \\
&= \beta'(0)b + \frac{\beta(0)}{2}\left[b_1 \sin^2\frac{\pi}{2}t + b_2\cos^2\frac{\pi}{2}t\right].
\end{aligned}$$

Similarly one finds

$$\begin{aligned}
h_w(t,0) &= \beta'(0)\hat{f}(0,0) + \beta(0)[\hat{f}_u(0,0)X^2(t,0) + \hat{f}_v(0,0)Y^2(t,0)] \\
&= \beta'(0)b + \frac{\beta(0)}{2}\left[b_1 \cos^2\frac{\pi}{2}t + b_2\sin^2\frac{\pi}{2}t\right].
\end{aligned}$$

Differentiation of c and d with respect to w shows that

$$\frac{\partial c(t,w)}{\partial w}|_{w=0} = \frac{\beta(0)}{2} b_2 \sin \pi t, \quad \frac{\partial d(t,w)}{\partial w}|_{w=0} = -\frac{\beta(0)}{2} b_2 \sin \pi t.$$

By substitution into $A_w(t,0,\lambda)$ the result follows.

Corollary 5.2. Let $t \in \mathbb{R}, \lambda \in \mathbb{C}^*$ and $\mu \in \mathbb{C}$ with $\lambda\mu^2 = -1$. Then

(i) $S(t,0,\lambda) = \begin{pmatrix} \cos \frac{\pi}{2}\mu t & -\mu^{-1} \sin \frac{\pi}{2}\mu t \\ \mu \sin \frac{\pi}{2}\mu t & \cos \frac{\pi}{2}\mu t \end{pmatrix} = \cos \frac{\pi}{2}\mu t \cdot 1 - \frac{1}{\mu} \sin \frac{\pi}{2}\mu t \cdot C_\lambda.$

(ii) $S(t,0,-1) = \begin{pmatrix} \cos \frac{\pi}{2} t & -\sin \frac{\pi}{2} t \\ \sin \frac{\pi}{2} t & \cos \frac{\pi}{2} t \end{pmatrix} = \cos \frac{\pi}{2} t \cdot 1 - \sin \frac{\pi}{2} t \cdot C_{-1}.$

(iii) $S(1,0,\lambda) = \cos \frac{\pi}{2}\mu \cdot 1 - \frac{1}{\mu} \sin \frac{\pi}{2}\mu \cdot C_\lambda.$

(iv) $S(1,0,-1) = -C_{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$

Lemma 5.4. *Let*

$$C^- := C_{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad C^+ := C_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad C := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then $C^-C^- = -1, \quad C^+C^- = -C, \quad C^-C^+ = C, \quad C^-C = -C^+, \quad CC^- = C^+.$

Lemma 5.5. *There are holomorphic maps* $K, L : \mathbb{C} \to \mathbb{C}$ *with* $\lambda K(\lambda^2) = \sin \frac{\pi}{2}\lambda$, $L(\lambda^2) = \cos \frac{\pi}{2}\lambda$ *for all* $\lambda \in \mathbb{C}$. *Furthermore,* $K(1) = 1$, $K'(1) = -\frac{1}{2}, K''(1) = \frac{3}{4} - \pi^2/16$ *and* $L(1) = 0$, $L'(1) = -\frac{\pi}{4}$.

Proof. Since $\sin \frac{\pi}{2}\lambda = \frac{\pi}{2}\lambda \left[1 - \frac{1}{3!}\left(\frac{\pi}{2}\right)^2 (\lambda)^2 + \frac{1}{5!}\left(\frac{\pi}{2}\right)^4 (\lambda)^4 + \cdots\right],$

$$\cos \frac{\pi}{2}\lambda = 1 - \frac{\lambda^2}{2!}\left(\frac{\pi}{2}\right)^2 + \frac{\lambda^4}{4!}\left(\frac{\pi}{2}\right)^4 - \frac{\lambda^6}{6!}\left(\frac{\pi}{2}\right)^6 + \cdots,$$

for $\mu \in \mathbb{C}$ define $K(\mu) := \frac{\pi}{2}\left(1 - \frac{1}{3!}\left(\frac{\pi}{2}\right)^2 \mu + \frac{1}{5!}\left(\frac{\pi}{2}\right)^4 \mu^2 + \cdots\right)$ and

$$L(\mu) := 1 - \frac{\mu}{2!}\left(\frac{\pi}{2}\right)^2 + \frac{\mu^2}{4!}\left(\frac{\pi}{2}\right)^4 - \frac{\mu^3}{6!}\left(\frac{\pi}{2}\right)^6 + \cdots$$

Then K and L are holomorphic and $\lambda K(\lambda^2) = \sin \frac{\pi}{2}\lambda$, $L(\lambda^2) = \cos \frac{\pi}{2}\lambda$ for all $\lambda \in \mathbb{C}$.

Lemma 5.6. $S_t(1,0,-1) = -\frac{\pi}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, S_\lambda(1,0,-1) = \frac{1}{2}\begin{pmatrix} -\frac{\pi}{2} & 1 \\ 1 & -\frac{\pi}{2} \end{pmatrix}.$

$$S_w(1,0,-1) = \begin{pmatrix} \beta'(0)b + \frac{3}{8}\beta(0)(b_1+b_2) & 0 \\ 0 & \beta'(0)b + \frac{1}{8}\beta(0)(b_1+b_2) \end{pmatrix}$$

Proof. $S_t(1,0,-1) = \dot{S}(1,0,-1) = A(1,0,-1)S(1,0,-1) = -\frac{\pi}{2}C_{-1}(-C_{-1}) = \frac{\pi}{2}C^-C^- == -\frac{\pi}{2} \cdot 1$. By Corollary 5.2 (iii) and Lemma 5.5 we have $S(1,0,\lambda) = L(-\frac{1}{\lambda}) \cdot 1 - K(-\frac{1}{\lambda}) \cdot C_\lambda$. Therefore,

$$S_\lambda(1,0,\lambda) = \frac{1}{\lambda^2}L'(-\frac{1}{\lambda}) \cdot 1 - \frac{1}{\lambda^2}K'(-\frac{1}{\lambda})C_\lambda - K(-\frac{1}{\lambda})(-\frac{1}{\lambda^2})\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

and

$$\begin{aligned} S_\lambda(1,0,-1) &= L'(1)\cdot 1 - K'(1)C_{-1} - K(1)\cdot(-1)\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ &= -\frac{\pi}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -\frac{\pi}{4} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\pi}{4} \end{pmatrix}. \end{aligned}$$

For the calculation of $S_w(1,0,-1)$ we differentiate

$$\dot{S}(t,w,\lambda) = A(t,w,\lambda)S(t,w,\lambda), \quad S(0,w,\lambda) = 1.$$

This gives

$$\dot{S}_w(t,w,\lambda) = A(t,w,\lambda)S_w(t,w,\lambda) + A_w(t,w,\lambda)S(t,w,\lambda), \quad S_w(0,w,\lambda) = 0.$$

Therefore,

$$\begin{aligned} S_w(t,w,\lambda) &= S(t,w,\lambda)\int_0^t S^{-1}(s,w,\lambda)A_w(s,w,\lambda)S(s,w,\lambda)\,ds, \\ S_w(1,0,-1) &= S(1,0,-1)\int_0^1 S^{-1}(t,0,-1)A_w(t,0,-1)S(t,0,-1)\,dt, \end{aligned}$$

where $S(t,0,-1) = \cos\frac{\pi}{2}t\cdot 1 - \sin\frac{\pi}{2}t\cdot C^-$, $S(1,0,-1) = -C^-$, $S^{-1}(t,0,-1) = \cos\frac{\pi}{2}t\cdot 1 + \sin\frac{\pi}{2}t\cdot C^-$. The matrix $A_w(t,0,-1)$, which is given by Lemma 5.3, can be represented as $A_w(t,0,-1) = a^-C^- + a^+C^+\cos\pi t + aC\sin\pi t$, where $a^- = \beta'(0)b + \frac{\beta(0)}{4}(b_1+b_2)$, $a^+ = \frac{\beta(0)}{4}(b_2-b_1)$, $a = \frac{\beta(0)}{2}b_2$. The calculation of

$$A_1 := -C^-\int_0^1 (\cos\frac{\pi}{2}t\cdot 1 + \sin\frac{\pi}{2}t\cdot C^-)a^-C^-(\cos\frac{\pi}{2}t\cdot 1 - \sin\frac{\pi}{2}t\cdot C^-)\,dt$$

is easy if we identify C^- with the complex number $i\in\mathbb{C}$ and $\cos\frac{\pi}{2}t\cdot 1 + \sin\frac{\pi}{2}t\cdot C^-$ with $e^{i(\pi/2)t}$. Then we have

$$A_1 \cong -a^-i\int_0^1 e^{i(\pi/2)t}\cdot i\cdot e^{-i(\pi/2)t}\,dt = a^-\cdot 1 \in \mathbb{C}^{(2,2)}.$$

Direct calculation similar to Lemma 12 in [4] shows that

$$\begin{aligned} B :=& -a^+C^-\int_0^1 \cos\pi t(\cos\frac{\pi}{2}t\cdot 1 + \sin\frac{\pi}{2}t\cdot C^-)C^+(\cos\frac{\pi}{2}t\cdot 1 - \sin\frac{\pi}{2}t\cdot C^-)\,dt \\ =& -\frac{1}{2}a^+C. \\ A_2 :=& -aC^-\int_0^1 \sin\pi t(\cos\frac{\pi}{2}t\cdot 1 + \sin\frac{\pi}{2}t\cdot C^-)C(\cos\frac{\pi}{2}t\cdot 1 - \sin\frac{\pi}{2}t\cdot C^-)\,dt \\ =& \frac{\beta(0)}{4}b_2\,C = \frac{1}{2}aC. \end{aligned}$$

Altogether we have,

$$\begin{aligned} S_w(1,0,-1) &= a^-\cdot 1 - \frac{1}{2}a^+C + \frac{1}{2}aC \\ &= \begin{pmatrix} \beta'(0)b + \frac{3}{8}\beta(0)(b_1+b_2) & 0 \\ 0 & \beta'(0)b + \frac{1}{8}\beta(0)(b_1+b_2) \end{pmatrix}. \end{aligned}$$

Let $S(t,w,\lambda) := \begin{pmatrix} p & u \\ q & v \end{pmatrix}$ be the fundamental matrix solution of $\dot{S} = A(t,w,\lambda)\,S$ satisfying $S(0,w,\lambda) = 1 \in \mathbb{C}^{(2,2)}$ (see the proof of Lemma 4.4).

Lemma 5.7. $S(1,w,-1) = \begin{pmatrix} p(1,w,-1) & -1 \\ 1 & 0 \end{pmatrix}$ *for* $|w| < \epsilon^2$.

Proof. Differentiation of

$$\dot{X}(t,w) = \beta(w)Y(t,w)\bar{f}(wX^2(t,w), wY^2(w,t))$$
$$\dot{Y}(t,w) = -\beta(w)X(t,w)\bar{f}(wY^2(t,w), wX^2(w,t))$$

shows that

$$\frac{d}{dt}\begin{pmatrix} \dot{Y}(t,w) \\ -\dot{X}(t,w) \end{pmatrix} = A(t,w,-1)\begin{pmatrix} \dot{Y}(t,w) \\ -\dot{X}(t,w) \end{pmatrix}.$$

Therefore,

$$\begin{pmatrix} \dot{Y}(1,w) \\ -\dot{X}(1,w) \end{pmatrix} = S(1,w,-1)\begin{pmatrix} \dot{Y}(0,w) \\ -\dot{X}(0,w) \end{pmatrix},$$

and thus,

$$\begin{pmatrix} -\beta(w)\bar{f}(0,w) \\ 0 \end{pmatrix} = S(1,w,-1)\begin{pmatrix} 0 \\ \beta(w)\bar{f}(0,w) \end{pmatrix} = \beta(w)\bar{f}(0,w)\begin{pmatrix} u(1,w,-1) \\ v(1,w,-1) \end{pmatrix}.$$

This shows that $u(1,w,-1) = -1,\ v(1,w,-1) = 0$.

For $q(1,w,-1)$ we find

$$q(1,w,-1) = \det\begin{pmatrix} p(1,w,-1) & -1 \\ q(1,w,-1) & 0 \end{pmatrix} = \det\ S(1,w,-1) = 1.$$

Corollary 5.3. $\alpha(0) = \alpha^*, \alpha'(0) = 0, \alpha''(0) = 2\beta'(0) = \frac{\pi}{8f_y^2(0,0)}[f_{yyy}(0,0) + f_{xxy}(0,0)]$.

Proof. Since $\alpha(z) = \beta(z^2)$ then $\alpha(0) = \beta(0) = \alpha^* = -\frac{\pi}{2f_y(0,0)}$, $\alpha'(z) = 2z\beta'(z^2)$, $\alpha'(0) = 0$, and $\alpha''(0) = 2\beta'(0)$. Lemmas 5.6 and 5.7 show that

$$\begin{aligned} S_w(1,0,-1) &= \begin{pmatrix} \beta'(0)b + \frac{3\beta(0)}{8}(b_1+b_2) & 0 \\ 0 & \beta'(0)b + \frac{\beta(0)}{8}(b_1+b_2) \end{pmatrix} \\ &= \begin{pmatrix} p_w(1,0,-1) & 0 \\ 0 & 0 \end{pmatrix}, \end{aligned}$$

hence, $\beta'(0)b + \frac{\beta(0)}{8}(b_1+b_2) = 0$. Since $\beta(0) = \alpha^* = -\frac{\pi}{2b}$, we find $\beta'(0) = \frac{\pi}{16b^2}(b_1 + b_2)$, and therefore,

$$\alpha''(0) = \frac{\pi}{8f_y^2(0,0)}[f_{yyy}(0,0) + f_{xxy}(0,0)].$$

Corollary 5.4. $S_w(1,0,-1) = \frac{-\pi(b_2+b_1)}{8b}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

Corollary 5.5. $r(w,-1) = 0$ for all $|w| < \epsilon^2$.

Proof.

$$\begin{aligned} r(w,-1) &= \det\left[S(1,w,-1) - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right] \\ &= \det\left[\begin{pmatrix} p(1,w,-1) & -1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right] = \det\begin{pmatrix} p(1,w,-1) & 0 \\ 0 & 0 \end{pmatrix} = 0. \end{aligned}$$

Lemma 5.8 *(i)* $r(0,\lambda) = 1-\lambda-2K(-\frac{1}{\lambda})$, $r_\lambda(0,\lambda) = -1-\frac{2}{\lambda^2}K'(-\frac{1}{\lambda})$, $r_{\lambda\lambda}(0,\lambda) = \frac{4}{\lambda^3}K'(-\frac{1}{\lambda}) - \frac{2}{\lambda^4}K''(-\frac{1}{\lambda})$ *for* $\lambda \in \mathbb{C}^*$*;*
(ii) $r(0,-1)=0$, $r_\lambda(0,-1)=0$, $r_w(0,-1)=0$, $r_{\lambda\lambda}(0,-1) = \frac{1}{2}+\frac{1}{8}\pi^2$, $r_{\lambda w}(0,-1) = \frac{\pi^2}{32}[f_{xxy}(0,0)+f_{yyy}(0,0)]/f_y(0,0)$;
(iii) -1 *is a zero of* $r(0,\cdot)$ *with multiplicity 2. If* $\lambda \in \mathbb{C}^*$ *is a zero of* $r(0,\cdot)$ *and* $\lambda \neq -1$*, then* $|\lambda| < 1$.

Proof. (i) follows by differentiating r and Lemma 5.5.

(ii). By using Lemma 5.5 $r(0,-1)$, $r_\lambda(0,-1)$, and $r_{\lambda\lambda}(0,-1)$ can be calculated from part (i). From Lemma 4.6 we have

$$\begin{aligned} r(w,\lambda) &= \det\left[\begin{pmatrix} p & u \\ q & v \end{pmatrix}(1,w,\lambda) - \begin{pmatrix} 0 & \lambda \\ 1 & 0 \end{pmatrix}\right] = pv - qu + \lambda q + u - \lambda \\ &= 1-\lambda+u(1,w,\lambda)+\lambda q(1,w,\lambda). \end{aligned}$$

Hence $r_w(w,\lambda) = u_w(1,w,\lambda)+\lambda q_w(1,w,\lambda)$, and Corollary 5.4 gives

$$r_w(0,-1) = u_w(1,0,-1) - q_w(1,0,-1) = 0.$$

Analogously, $r_{w\lambda}(w,\lambda) = u_{w\lambda}(1,w,\lambda)+\lambda q_{w\lambda}(1,w,\lambda)+q_w(1,w,\lambda)$ and

$$r_{w\lambda}(0,-1) = u_{w\lambda}(1,0,-1) - q_{w\lambda}(1,0,-1).$$

Differentiation of $pv-qu=1$ with respect to λ yields $p_\lambda v + pv_\lambda - q_\lambda u - qu_\lambda = 0$, and differentiation with respect to w gives

$$p_{\lambda w}v + p_\lambda v_w + p_w v_\lambda + pv_{\lambda w} - q_{\lambda w}u - q_\lambda u_w - q_w u_\lambda - qu_{\lambda w} = 0.$$

If we evaluate this expression at $(1,0,-1)$ and use Corollary 5.2 (iv) and Corollary 5.4 we arrive at $(p_w v_\lambda + q_{\lambda w} - u_{\lambda w})(1,0,-1) = 0$. Now we find from Lemma 5.6

$$r_{\lambda w}(0,-1) = u_{\lambda w}(1,0,-1) - q_{\lambda w}(1,0,-1) = p_w(1,0,-1)\cdot v_\lambda(1,0,-1) = \frac{\pi^2}{32}\cdot\frac{b_1+b_2}{b}.$$

(iii). Lemma 14 (iii) in [4].

A deeper analysis of the equation $r(0,\lambda)=0$ shows

Corollary 5.6. $|\lambda| < \frac{1}{2}$ for all $\lambda \in \mathbb{C}\setminus\{0,-1\}$ with $r(0,\lambda)=0$.

In fact one can show that $r(0,\cdot)^{-1}(\{0\}) = \{-1\}\cup\{\lambda_k,\bar{\lambda}_k \mid k\in\mathbb{N}\}$ with $Re\ \lambda_k < 0$ and $|\lambda_k| \approx 1/(4k+1)^2$.

Proof of Theorem 5.1. Set $r(w,\lambda) = (\lambda+1)\cdot R(w,\lambda)$, where

$$R(w,\lambda) := \int_0^1 r_\lambda(w,-1+s+s\lambda)\,ds$$

is a C^1-map. Clearly, $R_\lambda(0,-1) = \int_0^1 s r_{\lambda\lambda}(0,-1)\,ds = \frac{1}{2}\,r_{\lambda\lambda}(0,-1) \neq 0$, and

$$R_w(0,-1) = \int_0^1 r_{\lambda w}(0,-1)\,ds = r_{\lambda w}(0,-1),$$

and thus the implicit function theorem yield a neighbourhood of $(0,-1)\in\mathbb{R}\times\mathbb{C}^*$ and a C^1-map $\lambda(w)$ such that the only zeros of r in this neighbourhood are -1 and

$\lambda(w)$. Moreover,

$$\lambda(0) = -1, \quad \lambda'(0) = -\frac{R_w(0,-1)}{R_\lambda(0,-1)} = -2 \cdot \frac{r_{\lambda w}(0,-1)}{r_{\lambda\lambda}(0,-1)}.$$

Now let $z > 0$ be small. Then $r(z^2, \lambda(z^2)) = 0$, and since $\lambda'(0) \neq 0$, we can assume $\lambda(z^2) \neq -1$. Hence $\lambda^2(z^2)$ is a Floquet multiplier of $x(\cdot, z)$. From Corollary 5.6 we can further assume that $\lambda^2(z^2)$ is the nontrivial Floquet multiplier of $x(\cdot, z)$ which has the greatest norm. Thus, $\lambda^2(z^2)$ determines the stability of $x(\cdot, z)$, i.e. $x(\cdot, z)$ is stable (unstable) if $\lambda^2(z^2) < 1$

From Corollary 5.3 we have

$$\alpha(z) = \beta(z^2),\ \beta(0) = \alpha^* = -\frac{\pi}{2b},\ \beta'(0) = \frac{\pi}{16b^2}\left(f_{yyy}(0,0) + f_{xxy}(0,0)\right).$$

Assume that $\beta'(0) \neq 0$. Then β is locally invertible and for α close to α^*,

$$s(\alpha) := \lambda^2(\beta^{-1}(\alpha))$$

exists and defines a C^1-map. Obviously

$$s(\alpha(z)) = \lambda^2(z^2)),\ s(\alpha^*) = \lambda^2(0) = 1,$$
$$s'(\alpha^*) = 2\lambda(0) \cdot \lambda'(0) \cdot \frac{1}{\beta'(0)} = \frac{4}{\beta'(0)} \frac{r_{\lambda w}(0,-1)}{r_{\lambda\lambda}(0,-1)} = \frac{16\pi}{4+\pi^2} \cdot b.$$

Thus, if x is a SSPS of (αf) with a small amplitude $z > 0$, then $x = x(\cdot, z), \alpha = \alpha(z)$, and $s(\alpha)$ is the nontrivial Floquet multiplier of x with the greatest norm.

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Peter.Dormayer@math.uni-giessen.de; afi1@psu.edu