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PERIODIC SOLUTIONS AND THEIR STABILITY OF A DIFFERENTIAL-DIFFERENCE EQUATION

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Abstract. Existence, stability, and shape of periodic solutions are derived for the differential-difference equation $\varepsilon\dot{x}(t) + x(t) = f(x([t-1]))$, $0 < \varepsilon \ll 1$, where $[\cdot]$ is the integer part function. The equation can be viewed as a special discretization (discrete version) of the singularly perturbed differential delay equation $\varepsilon\dot{x}(t)+x(t) = f(x(t-1))$. The principal analysis is based on reduction to the two-dimensional map $F : (u, v) \to (v, f(u) + [v - f(u)]e^{-1/\varepsilon})$, many relevant properties of which follow from those of the one-dimensional map f.

1. **Introduction.** This paper concerns the existence and stability of periodic solutions of the differential-difference equation

$$\varepsilon\dot{x}(t) + x(t) = f(x([t-1])), \tag{1}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, $[\cdot]$ is the integer part function, and $\varepsilon > 0$ is a small parameter. Equation (1) is a special discretization (discrete version) of the differential delay equation

$$\varepsilon\dot{x}(t) + x(t) = f(x(t-1)). \tag{2}$$

Equation (2) appears in a great number of applications. It serves as a mathematical model of real life phenomena in a variety of natural and applied sciences, such as mathematical biology, epidemiology and physiology, physics of circuits and laser optics, economics and life sciences [3, 4, 6, 8]. In some of those models the parameter $\varepsilon > 0$ in not necessarily small, and the equation comes in a slightly different but equivalent form, such as, for example, $\dot{x}(t) = -\mu x(t) + f(x(t-1))$. It also results as an exact reduction of nonlinear boundary value problems for hyperbolic partial differential equations where the singular term $\varepsilon\dot{x}(t)$ represents a small viscosity or capacity [10]. In those reductions one typically obtains a more general neutral type equation $\varepsilon[\dot{x}(t)+a\,\dot{x}(t-1)]+x(t) = f(x(t-1))$, see e.g. [9, 12]. Virtually no results on global dynamics of the latter neutral equation are known. However, its special case $a = 0$, which is equation (2), has received a lot of attention from researchers in the past 20 years or so, and is much better studied and understood.

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One fruitful approach to study properties of equation (2) is to treat it as a singular perturbation of the continuous time difference equation

$$x(t) = f(x(t-1)), \qquad t \in \mathbb{R}^+. \tag{3}$$

The dynamical properties of equation (3) are determined by the one-dimensional map f and are well understood [10]. Many results are obtained in this direction, showing similarities and differences in the dynamics of the above continuous time difference equation and its singular perturbations. However, many interesting and difficult questions still remain unanswered here. For more details see e.g. the review paper [4].

Another approach is to look at special cases of equation (2) when function $f(x)$ is a piece-wise constant one. In this case the solutions are easily found by integration for all $t \geq 0$ as continuously matched pieces of exponential functions. The dynamics of solutions of equation (2) on certain subsets of the phase space $\mathcal{C} := C[-1, 0]$ can be reduced to finite dimensional maps, in particular, - to one dimensional interval maps. This approach has produced numerous interesting results on global dynamics in equation (2). Loosely speaking, the dynamics of the differential delay equation (2) can be as complicated as those of general interval maps. For more details and additional references see for example [4, 7].

An interesting blend of the two approaches is the study of equation (2) with a piece-wise constant nonlinearity f which has a globally attracting cycle of period two [4, 7]. The map f is defined in this case as

$$\begin{array}{ll} f(x) = -1 \quad \text{for} \quad x \geq \theta_1, 0 < \theta_1 < 1, & f(x) = -A < -1 \quad \text{for} \quad x \in (0, \theta_1), \\ f(x) = 1 \quad \text{for} \quad x \leq \theta_2, 0 > \theta_2 > -1, & f(x) = B > 1 \quad \text{for} \quad x \in (\theta_2, 0), \\ f(0) = 0. & \end{array}$$

It is easily seen that this map f only has a globally attracting cycle of period two $\{-1, 1\}$: any initial point $x_0 \neq 0$ results in a trajectory that is attracted by the cycle. The dynamics of the corresponding difference equation (3) in this case is also simple: it typically has only solutions attracted by discontinuous piece-wise constant functions (square-wave solutions), whose shape is determined by the initial function and its range with respect to the cycle $\{-1, 1\}$ (see [10] for more details). On the contrary, the dynamics of the differential delay equation (2) can be quite complicated for all $0 < \varepsilon \ll 1$. The dynamics of its solutions on specially chosen subsets of initial functions are exactly reduced to those of continuous piece-wise linear-fractional maps; see [4, 7] for more details of the reduction and the exact form of the maps. Generally, the dynamics of such maps can be quite complicated, including the existence of multiple stable/unstable cycles, period doubling bifurcations, and chaos. However, all the complexity in equation (2) in this case happens within a small Hausdorff neighborhood (for small $\varepsilon > 0$) of the generalized periodic solution P_{AB} of the difference equation (3) given by: $P_{AB}(t) = 1, t \in (0, 1)$, $P_{AB}(t) = -1, t \in (1, 2)$, $P_{AB}(0) = P_{AB}(1) = [-A, B]$. Therefore, from the practical point of view it can be indistinguishable from the periodic motion $P_{AB}(t)$ for sufficiently small $\varepsilon > 0$.

In spite of a large number of publications and significant progress made in studying equation (2), many important and interesting problems about its dynamics remain unsolved. One of them is the existence of a stable periodic solution close for small $\varepsilon > 0$ to a respective attracting cycle of the map f; in particular, - for the case of a parameter dependent family f_λ when it goes through a sequence of period doubling bifurcations as λ varies (e.g., the well-known family $f_\lambda = \lambda x(1-x)$). It is

known that equation (2) does possess a square-wave periodic solution close to the globally attracting cycle of the map f , which includes the case of globally attracting 2-cycle [7]. However, even in this simplest, from the point of view of the interval map f, case the stability of the periodic solution is not established yet (except the case of local Hopf bifurcation). Therefore, as equation (2) and its neutral version are proposed as models of numerous real life phenomena, with the expectation that many of its properties are inherited from the one-dimensional map f, a large number of those results still remain to be proved. The situation in the correspondence of the dynamics is much better for the differential-difference equation (1), as it will be seen from new results presented in this paper.

Equation (1) also appears as a special dicretization of equation (2) when only a finite number of values of the initial function is used [1]. Let $K \in \mathbb{N}$ be a fixed positive integer, and $h := 1/K$ be the disretization step. Assume the values of an initial function $\phi \in \mathcal{C}$ are given at the discrete times $t_i = -i \cdot h, 1 \leq i \leq K : \ \phi(t_i) = \phi_i$. In order to solve equation (2) for $t \in [0, 1]$ one naturally extends the initial data to the intervals $(-i \cdot h, (-i + 1) \cdot h)$ from the given values $\phi(t_i) = \phi_i$ by setting $\phi(s) \equiv h_i, 1 \leq i \leq K$. Equation (2) then becomes equivalent on the interval $[0, 1]$ to

$$\varepsilon \dot{x}(t) + x(t) = f(x\left(h\left[\frac{t-1}{h}\right]\right)) = f(x(h\left[(t-1)K\right])). \tag{4}$$

The simplest partial case of equation (4) when $K = 1$ is the subject of our study in this paper. We consider the general case of arbitrary K in a separate paper [5]. For other discretizations of equation (2), such as Euler backward discretization, and their dynamical properties in relevance to the original differential delay equations see, for example, [1, 2].

In this paper we only consider a partial case of equation (4), the differential-difference equation (1). We derive sufficient conditions for the existence, stability, and asymptotic shape as $\varepsilon \to 0$ of periodic solutions. The analysis is based on the existence and attractivity properties of respective cycles of the map f. The periodic solutions are shown to be close (in the sense specified in section 2) to the respective attracting cycles of the map f.

Those results show close connections in dynamical properties between the interval map f (difference equation (3)) and its singular perturbation in the form of equation (1). Such connections, though expected, are not proved yet for the differential delay equation (2).

2. **Preliminaries.**

2.1. **Assumptions and existence of solutions.**

The basic assumption on the nonlinearity f throughout this paper is that it is a continuous function for all real x. Sometimes we will need stronger assumptions imposed on f, such as the differentiability along a particular cycle. In all those cases the additional hypotheses will be explicitly stated.

In order to solve a differential equation with delay one needs an initial function. For both equations (2) and (1) the set of initial functions is $\mathcal{C} := C([-1, 0])$. Given $\phi \in \mathcal{C}$ the corresponding solution of equation (2) is constructed for all $t \geq 0$ by step-by-step integration of the corresponding non-homogeneous linear equation on successive intervals $[i - 1, i], i \in \mathbb{N}$. Likewise, the solution of equation (1) is found by the same step-by-step integration procedure, except that one has to solve an ordinary linear differential equation with a constant non-homogeneous term on each

of the intervals $[i-1,i], i \in \mathbb{N}$. As the result, the corresponding solution of equation (1) exists for all $t \geq -1$ and is a piece-wise exponential function for $t \geq 0$ with the exponential pieces continuously matched at $t = i, i \in \mathbb{N} \cup \{0\}$. Given $\phi \in \mathcal{C}$ the solution $x = x_\phi^\varepsilon(t)$ of either equation (1) or equation (2) can be treated at any $t \geq 0$ as a point in $\mathbb{R}$ or as an element x_t of the state space $\mathcal{C}, x_t := x_\phi^\varepsilon(t+s), s \in [-1,0]$.

2.2. **One-dimensional maps.** In this subsection we briefly recall some basic notions and elementary facts from the interval map theory necessary for the relevant exposition in this paper.

A fixed point x_* of the map $f, f(x_*) = x_*$, is called *attracting* if there is an open neighborhood $\mathcal{U} \ni x_*$ such that for every $x \in \mathcal{U}$ one has $f^n(x) \to x_*$ an $n \to \infty$. Here $f^n(x) = f(f(\dots f(x)\dots))$ stands for the n-th iteration of the map f. The maximal connected interval $\mathcal{U}$ satisfying this property is called the *domain of immediate attraction*. A set of pairwise distinct points $\mathcal{B} = \{b_1, b_2, \dots, b_m\}$ is called a *cycle* of period m of the map f if $f(b_i) = b_{i+1}, i = 1, 2, \dots, m-1, f(b_m) = b_1$. Each point b_i of the cycle $\mathcal{B}$ is periodic with period m for the map f. Clearly every b_i is also a fixed point of the map f^m: $f^m(x_i) = x_i$, $\forall i \in \{1, 2, \dots, m\}$. If $\mathcal{U}_i$ is its domain of immediate attraction then the set $\mathcal{U}_1 \cup \mathcal{U}_2 \cup \dots \mathcal{U}_m$ is called the domain of immediate attraction of the cycle $\mathcal{B}$. It is well known that $f(\mathcal{U}_i) = \mathcal{U}_{i+1}, i = 1, 2, \dots, m-1$ and $f(\mathcal{U}_m) = \mathcal{U}_1$.

For these and other basic concepts and definitions from the interval map theory see, for example, [11].

2.3. **Continuous time difference equation.** In this subsection we consider the continuous time difference equation

$$x(t) = f(x(t-\tau)), \quad t \in \mathbb{R}^+. \tag{5}$$

with the constant delay $\tau > 0$. Note that this equation is the limiting case $\varepsilon = 0$ of equation (2) with $\tau = 1$. The formal limiting equation (1) when $\varepsilon = 0$ is the difference equation $x(t) = f(x([t-1]))$ whose variable delay $t - [t-1]$ ranges in the interval $[1, 2)$. For the analysis of equation (1) we will use difference equation (5) with $\tau = 2$ as its limiting case.

Since $x(0) = f(x(-\tau))$ in order to solve equation (5) one needs an initial function ψ defined on the initial interval $[-\tau, 0)$. Let $\mathcal{C}_-^\tau$ be the set of all initial functions from $\mathcal{C}^\tau := C[-\tau, 0]$ whose domain is narrowed to the interval $[-\tau, 0)$. For every initial function $\psi \in \mathcal{C}_-^\tau$ the corresponding solution $x = x(t, \psi)$ of equation (5) is found by successive iterations. It is generally discontinuous at the values $t = i \cdot \tau, i \in \mathbb{N} \cup \{0\}$ of the independent variable. However, if the consistency condition $\lim_{t \to 0-} \psi(t) = f(\psi(-\tau))$ is met the corresponding solution $x(t, \psi)$ is continuous for all $t \geq -\tau$.

Let $\mathcal{B} = \{b_1, \dots, b_m\}$ be a cycle of period m of the interval map f. With $\tau = 2$ consider the following periodic solution $x = b(t), t \geq -2$, of difference equation (5)

$$\mathbf{b}_0 = b(t) = b_{i(\mathrm{mod}\, m)} \quad \text{for} \quad t \in [2i-2, 2i),\ i \in \mathbb{N} \cup \{0\}.$$

The periodic solution $\mathbf{b}_0$ will be referred to as the *square wave* solution corresponding to the cycle $\mathcal{B} = \{b_1, \dots, b_m\}$. Note that choosing a different first point in the cycle $\mathcal{B}$, say $\{b_2, b_3, \dots, b_m, b_1\}$, one obtains a formally different periodic solution of equation (5). It is, however, just a time shirt of the original periodic solution $\mathbf{b}_0 = b(t)$. In view of this we will only consider the periodic solution $\mathbf{b}_0$ associated with the ordering $\{b_1, \dots, b_m\}$ in the cycle $\mathcal{B}$. Remark 1 in [5] discusses other possible periodic solutions (geometrically different) associated with the cycle $\mathcal{B}$.

As it will be seen in the next section, the discontinuous square wave periodic solution $\mathbf{b}_0 = b(t)$ is the limiting profile as $\varepsilon \to 0$ of continuous solutions of equation (1). We need to define the appropriate convergence in this case.

Definition 2.1. Let $g(t, \lambda)$ be a parameter dependent family of continuous in t on the interval $[0, 2m]$ functions. We say that $g(t, \lambda)$ converges to $b(t)$ as $\lambda \to \lambda_0$ if for arbitrary $\sigma > 0$ and $\mu > 0$ there exists $\delta = \delta(\sigma, \mu) > 0$ such that $|g(t, \lambda) - b(t)| < \sigma$ for all $t \in \cup_{i=1}^{m} [2(i-1) + \mu, 2i - \mu]$ provided $|\lambda - \lambda_0| < \delta$.

The convergence of $g(t, \lambda)$ to $b(t)$ is uniform on the interval $[0, 2m]$ except a small μ-neighborhood of points $t = 2i, i = 1, 2, \ldots, m$ (which can be made arbitrarily small).

3. **Main results.**

The following two theorems are the principal new results of this paper.

Theorem 3.1. *Suppose that the map f has an attracting cycle $\mathcal{B} = \{b_1, \ldots, b_m\}$ of period m. There exists an $\varepsilon_0 > 0$ such that for every $0 < \varepsilon \le \varepsilon_0$ equation (1) has a periodic solution $x = p_\varepsilon(t)$ of period $T = 2m$. $p_\varepsilon(t)$ converges to the square wave solution $b(t)$ of equation (5) as $\varepsilon \to 0+$.*

Note that the uniqueness of the periodic solution of equation (1) is not claimed by Theorem 3.1. We think that the non-uniqueness can happen, but we are not able at this time to provide a relevant example. As the next theorem suggests, this can only be an exceptional situation which can happen when map f has a non-hyperbolic cycle.

Theorem 3.2. *Suppose that the continuously differentiable map f has an attracting cycle $\mathcal{B} = \{b_1, \ldots, b_m\}$ of period m with the multiplier $\mu(\mathcal{B}) = f'(b_1) \cdot f'(b_2) \cdot \ldots \cdot f'(b_m)$ satisfying $|\mu| < 1$. Then the periodic solution $x = p_\varepsilon(t)$ from Theorem 3.1 is asymptotically stable with the asymptotic phase.*

In order to prove Theorems 3.1 and 3.2 we need several auxiliary statements and lemmas.

Proposition 1. *For every initial function $\phi \in \mathcal{C}$ there exists a unique solution $x = x_\phi^\varepsilon(t)$ of equation (1) defined for all $t \ge -1$. The solution is uniquely determined by $\phi(-1)$ and $\phi(0)$: any two initial functions ϕ and ψ with $\phi(-1) = \psi(-1)$ and $\phi(0) = \psi(0)$ result in the same solution.*

Proof. Given $\phi(s) \in \mathcal{C}$ the solution $x = x_\phi^\varepsilon(t)$ is found from the initial value problem:

$$\varepsilon \dot{x}(t) + x(t) = f(\phi(-1)),\ x(0) = \phi(0),\ t \in [0, 1]. \tag{6}$$

Therefore, it is given by $x_\phi^\varepsilon(t) = f(\phi(-1)) + [\phi(0) - f(\phi(-1))]\, e^{-t/\varepsilon}, t \in [0, 1]$, and is uniquely determined by $\phi(-1)$ and $\phi(0)$ for all $t \ge 0$. □

Proposition 2. *Let $T(t)$ be the shift operator along solutions of equation (1) in the phase space $\mathcal{C}$: $T(t) := x_t \in \mathcal{C}$, where $x(t) = x_\phi^\varepsilon(t)$ is the solution with the initial function $\phi \in \mathcal{C}$. The operator $T(1)$ is equivalent to the two-dimensional map:*

$$F:\ (u, v) \to (v, f(u) + [v - f(u)]\, e^{-1/\varepsilon}). \tag{7}$$

Proof. Given $\phi(s) \in \mathcal{C}$ the corresponding solution $x = x_\phi^\varepsilon(t)$ is found from the initial value problem (6). With $\phi(-1) := u$ and $\phi(0) := v$ the value of $x_\phi^\varepsilon(1)$ is given by

$x^{\varepsilon}_{\phi}(1) := f(u) + [v - f(u)]\, e^{-1/\varepsilon}$. Therefore, the operator $T(1) : \mathcal{C} \to \mathcal{C}$ is uniquely determined by the values of $u = \phi(-1)$ and $v = \phi(0)$, and is equivalent to the map

$$u_1 = v, \quad v_1 = f(u) + [v - f(u)]\, e^{-1/\varepsilon},$$

which is the map (7). □

Proposition 2 shows that the dynamics of solutions of equation (1) are completely determined by the two-dimensional map F given by (7).

The following statement shows a continuous dependence on parameter ε between solutions of equations (1) and (5). It is an analogy of a similar statement for solutions of equations (2) and (1) established in [4].

Proposition 3. *For every $\phi \in \mathcal{C}$ and arbitrary $\sigma > 0$ and $\mu > 0$ there exists $\varepsilon_0 = \varepsilon_0(\phi, \sigma, \mu)$ such that $|x^{\varepsilon}_{\phi}(t) - f(\phi(-1)| \le \sigma, \forall t \in [\mu, 1]$ and all $0 < \varepsilon \le \varepsilon_0$.*

Proof. With $\phi \in \mathcal{C}$ fixed the corresponding solution x of equation (1) is given by the expression $x^{\varepsilon}_{\phi}(t) = f(\phi(-1)) + [\phi(0) - f(\phi(-1))]\, e^{-t/\varepsilon}, t \in [0, 1]$. Since for every fixed $t > 0$ the function $f(\phi(-1)) + [\phi(0) - f(\phi(-1))]\, e^{-t/\varepsilon}$ converges to $f(\phi(-1))$ as $\varepsilon \to 0+$, given $\phi(0), \phi(-1), \sigma > 0$ and $\mu > 0$ an ε_0 can be found such that $|[\phi(0) - f(\phi(-1))]\, e^{-\mu/\varepsilon_0}| = \sigma$. It is easily seen then that $|[\phi(0) - f(\phi(-1))]\, e^{-t/\varepsilon_0}| \le \sigma$ for all $t \in [\mu, 1]$ and all $0 < \varepsilon \le \varepsilon_0$. □

Proof of Theorem 3.1. (a) Existence. To prove the existence part we shall use the operator $T(1)$ and its equivalence to the two-dimensional map F (Proposition 2).

Let two closed intervals $[a, b] := J_1$ and $[c, d] := J_2$ be given, and set $f([a, b]) := [a_1, b_1] = f(J_1)$, $f([c, d]) := [c_1, d_1] = f(J_2)$. Suppose $u \in [a, b], v \in [c, d]$ are arbitrary initial data. The first iteration of the map F is given by $u_1 = v, v_1 = f(u) + [v - f(u)]\, e^{-1/\varepsilon}$. Therefore, $u_1 \in [c, d]$. Since $|v - f(u)| \le M$ is uniformly bounded, for arbitrary $\mu > 0$ there exists $\varepsilon_0 = \varepsilon_0(\mu)$ such that $v_1 \in [a_1 - \mu, b_1 + \mu]$ for all $0 < \varepsilon \le \varepsilon_0$. Likewise, the second iteration F^2 is given by $u_2 = v_1, v_2 = f(u_1) + [v_1 - f(u_1)]\, e^{-1/\varepsilon}$ and it has the property

$$u_2 \in [a_1 - \mu, b_1 + \mu],\ v_2 \in [c_1 - \mu, d_1 - \mu],\ \forall\, 0 < \varepsilon \le \varepsilon_0.$$

The latter inclusions mean that the components u_2 and v_2 of the second iteration $(u_2, v_2) = F^2(u, v)$ of the initial values $u \in J_1, v \in J_2$ lie within the μ-proximity of the respective intervals $f(J_1)$ and $f(J_2)$ for all sufficiently small $\varepsilon > 0$.

Let $\mathcal{B} = \{b_1, b_2, \ldots, b_m\}$ be an attracting cycle of the map f with the domain $\mathcal{U} = \mathcal{U}_1 \cup \mathcal{U}_2 \cup \cdots \cup \mathcal{U}_m$ of immediate attraction. Since b_1 is an attracting fixed point of the map f^m there exists a closed interval $[\alpha_1, \beta_1] \ni b_1, [\alpha_1, \beta_1] \subset \mathcal{U}_1$, such that $f^m([\alpha_1, \beta_1]) \subset (\alpha_1, \beta_1)$ and $\cap_{n \ge 1} f^{mn}([\alpha_1, \beta_1]) = b_1$. Let $f([\alpha_1, \beta_1]) := [\alpha_2, \beta_2], f([\alpha_2, \beta_2]) := [\alpha_3, \beta_3], \ldots, f([\alpha_m, \beta_m]) := [\alpha_*, \beta_*]$. Then $f^m([\alpha_1, \beta_1]) = [\alpha_*, \beta_*] \subset (\alpha_1, \beta_1)$. For every fixed $\mu_i > 0$ there exists $\mu_{i-1} = \mu_{i-1}(\mu_i)$ such that one has $f([\alpha_{i-1} - \mu, \beta_{i-1} + \mu]) \subseteq [\alpha_i - \mu_i, \beta_i + \mu_i], i = 2, \ldots, m$ for all $\mu \in [0, \mu_{i-1}]$. In view of the inclusion $f([\alpha_m, \beta_m]) := [\alpha_*, \beta_*]$ a constant $\mu_m > 0$ can be chosen in such a way that $f([\alpha_m - \mu, \beta_m + \mu]) \subseteq [\alpha_1, \beta_1]$ for all $\mu \in [0, \mu_m]$. Given $\mu_m > 0$ select the remaining $\mu_i, i = m - 1, m - 2, \ldots, 2, 1$ in accordance with the above described algorithm.

Consider the following initial set for the map F: $(u, v) \in [\alpha_1, \beta_1] \times [\alpha_1, \beta_1] := \mathcal{D}_0$. Given $\mu_2 > 0$ there exists $\varepsilon_2 > 0$ such that the second iteration $(u_2, v_2) = F^2(u, v)$ has the property

$$u_2 \in [\alpha_2 - \mu_2, \beta_2 + \mu_2],\ v_2 \in [\alpha_2 - \mu_2, \beta_2 + \mu_2],\ \forall\, 0 < \varepsilon \le \varepsilon_2$$

for all $(u,v) \in \mathcal{D}_0$. Likewise, given $\mu_3 > 0$ the above constant μ_2 can be chosen in such a way that for every initial pair (u,v) from the set $\mathcal{D}_1 := [\alpha_2 - \mu_2, \beta_2 + \mu_2] \times [\alpha_2 - \mu_2, \beta_2 + \mu_2]$ its second iteration $(u_2, v_2) = F^2(u,v)$ has the property

$$u_2 \in [\alpha_3 - \mu_3, \beta_3 + \mu_3],\ v_2 \in [\alpha_3 - \mu_3, \beta_3 + \mu_3],\ \forall\, 0 < \varepsilon \le \varepsilon_2.$$

Continuing by induction, and in view of $f^m([\alpha_1, \beta_1]) = [\alpha_*, \beta_*] \subset (\alpha_1, \beta_1)$, the second iteration F^2 has the property

$$u_2 \in [\alpha_1, \beta_1],\ v_2 \in [\alpha_1, \beta_1],\ \forall\, 0 < \varepsilon \le \varepsilon_m$$

for all initial values (u,v) from the set $\mathcal{D}_{m-1} := [\alpha_m - \mu_m, \beta_m + \mu_m] \times [\alpha_m - \mu_m, \beta_m + \mu_m]$. Therefore, for all $0 < \varepsilon \le \varepsilon_0 := \min\{\varepsilon_i, i = 1, 2, \ldots, m\}$ the $2m^{\text{th}}$ iteration F^{2m} of the map F maps the set $\mathcal{D}_0$ into itself, and has a fixed point (u_1^*, v_1^*) there. As it is easily seen the fixed point generates a periodic solution of equation (1) with period $2m$.

(b) Shape of the periodic solution and convergence as $\varepsilon \to 0+$. For the fixed point (u_1^*, v_1^*) of the map F^{2m} from part (a) set

$$(u_i^*, v_i^*) := f^2(u_{i-1}^*, v_{i-1}^*) = f^{2(i-1)}(u_1^*, v_1^*),\ i = 2, 3, \ldots, m.$$

By the construction, the periodic solution $p = p_\varepsilon(t)$ of equation (1) from part (a) consists of pieces of the exponential function $A + B\, e^{-t/\varepsilon}$ continuously connecting the points $u_i^*, v_i^*, i = 1, 2, \ldots, m$.

For given $u, v \in \mathbb{R}$ and $t_0 \ge -1$ define the (u, v, t_0)-exponent, $\gamma_{u,v,t_0}(t)$, for t in the interval $[t_0, t_0 + 1]$ by

$$\gamma_{u,v,t_0}(t) := A + [u - A]\, e^{-(t-t_0)/\varepsilon},\ t \in [t_0, t_0+1],\ \text{where } A = (v - u\, e^{-1/\varepsilon})/(1 - e^{-1/\varepsilon}).$$

It is easily verified that $\gamma_{u,v,t_0}(t_0) = u,\ \gamma_{u,v,t_0}(t_0 + 1) = v$.

The periodic solution $p = p_\varepsilon(t), t \in [-1, 2m - 1]$, whose existence is proved in part (a), is given then by

$$\begin{aligned}
p(t) &= \gamma(u_1^*, v_1^*, -1) \text{ for } t \in [-1, 0],\\
p(t) &= \gamma(v_1^*, u_2^*, 0) \text{ for } t \in [0, 1],\\
p(t) &= \gamma(u_2^*, v_2^*, 1) \text{ for } t \in [1, 2],\\
p(t) &= \gamma(v_2^*, u_3^*, 2) \text{ for } t \in [2, 3],\\
&\ldots\\
p(t) &= \gamma(v_{m-1}^*, u_m^*, 2m - 3) \text{ for } t \in [2m - 3, 2m - 2],\\
p(t) &= \gamma(u_m^*, v_m^*, 2m - 2) \text{ for } t \in [2m - 2, 2m - 1].
\end{aligned}$$

We shall show the convergence of $p_\varepsilon(t)$ to the square wave function $\mathbf{b}_0 = b(t+1), t \in [-1, 2m - 1]$ as $\varepsilon \to 0+$ in the sense of Definition 2.1.

Note that each of the constants $\mu_i, i = 1, 2, \ldots, m$ satisfies $\mu_i \to 0$ as $\varepsilon \to 0+$, since the interval $[\alpha_1, \beta_1]$ can be chosen sufficiently small around point b_1. This implies that

$$u_1^* \to b_1, v_1^* \to b_1; u_2^* \to b_2, v_2^* \to b_2; \ldots u_m^* \to b_m, v_m^* \to b_m \quad \text{as} \quad \varepsilon \to 0+.$$

Therefore, the above periodic solution $p_\varepsilon(t)$ has the property that $p_\varepsilon(t) \to b_{i+1}$ as $\varepsilon \to 0+$ uniformly for all $t \in [2i - 1, 2i], i = 0, 1, \ldots, m - 1$.

Since $p(t) = \gamma_{v_1^*, u_2^*, 0}(t) = A + (v_1^* - A)\, e^{-t/\varepsilon}$ for $t \in [0, 1]$, where $A = (u_2^* - v_1^*\, e^{-1/\varepsilon})/(1 - e^{-1/\varepsilon})$, for every $\sigma_1 > 0$ there exists $\mu_1 = \mu_1(\sigma_1) > 0$ such that $|p_\varepsilon(t) - b_2| \le \sigma_1$ for all $t \in [\mu_1, 1]$. On the interval $[0, \mu]$ the solution $p_\varepsilon(t)$ is the exponent, $\gamma_{v_1^*, u_2^*, 0}(t)$, ranging from v_1^* to $u_2^* - \sigma_1$. This means the convergence to b_2 on $[\mu_1, 1]$ as $\varepsilon \to 0+$. Likewise, the convergence to b_3 can be shown on an interval

$[2+\mu_2, 3]$ for some $\mu_2 > 0$, etc. We leave the remaining details to the reader. This completes the proof of Theorem 3.1.

Proof of Theorem 3.2. The existence of the periodic solution, $p_\varepsilon(t)$, follows from Theorem 3.1. Since $T(1)$ is given by the two-dimensional map (7) we shall show the local asymptotic stability of $p_\varepsilon(t)$ by proving the local attractivity of the fixed point (u_1^*, v_1^*) under the map $F^{2m} = (F^2)^m$. Its stability is determined by the norm of the linearisation of the map F along the cycle generated by the point (u_1^*, v_1^*).

An easy calculation shows that the second iteration F^2 of the map F is given by

$$F^2: \quad (u,v) \to (f(u)+[v-f(u)]\,e^{-1/\varepsilon}, f(v)+[f(u)-f(v)]\,e^{-1/\varepsilon}+[(v-f(u))]\,e^{-2/\varepsilon}).$$

Its linearization A at any point (u,v) is given by

$$A = \begin{pmatrix} f'(u)(1-e^{-1/\varepsilon}) & e^{-1/\varepsilon} \\ f'(u)(e^{-1/\varepsilon}-e^{-2/\varepsilon}) & f'(v)(1-e^{-1/\varepsilon})+e^{-2/\varepsilon} \end{pmatrix}.$$

The linearization of F^{2m} along the fixed point (u_1^*, v_1^*) is given then by $A_\varepsilon := A_m \cdot A_{m-1} \cdot \ldots \cdot A_2 \cdot A_1$, where

$$A_i = \begin{pmatrix} f'(u_i^*)(1-e^{-1/\varepsilon}) & e^{-1/\varepsilon} \\ f'(u_i^*)(e^{-1/\varepsilon}-e^{-2/\varepsilon}) & f'(v_i^*)(1-e^{-1/\varepsilon})+e^{-2/\varepsilon} \end{pmatrix}.$$

For small $\varepsilon > 0$ matrix A_ε is a small perturbation of the limiting case $\varepsilon = 0$ given by

$$A_0 = \begin{pmatrix} f'(u_m^*) & 0 \\ 0 & f'(v_m^*) \end{pmatrix} \cdot \ldots \cdot \begin{pmatrix} f'(u_2^*) & 0 \\ 0 & f'(v_2^*) \end{pmatrix} \cdot \begin{pmatrix} f'(u_1^*) & 0 \\ 0 & f'(v_1^*) \end{pmatrix}.$$

The norm $||A_0||$ of matrix A_0 is given by

$$||A_0|| = |[f'(u_1^*)\ldots f'(u_m^*)] \cdot [f'(v_1^*)\ldots f'(v_m^*)]|.$$

Since $f'(x)$ is continuous, the interval $[\alpha_1, \beta_1]$ in the proof of Theorem 3.1 can be chosen small enough so that $\max|f'(x_1)\cdot f'(x_2)\cdot\ldots\cdot f'(x_m)| < 1$, where the maximum is taken over the set: $x_1 \in [\alpha_1, \beta_1], x_2 \in [\alpha_2, \beta_2], \ldots, x_m \in [\alpha_m, \beta_m]$. This shows that $||A_0|| < 1$ implying the local attractivity of (u_1^*, v_1^*) under F^{2m}. The proof of Theorem 3.2 is complete.

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