Take-Home Quiz #2. Part B

Fall 2013

Due by 11:59 PM on November 3, 2013
To be submitted as a .pdf file via the drop-box at ANGEL

Section:

☐ 012 (11:15AM – 12:05PM) ☒ 042 (12:20 – 1:10PM) ☐ 051 (1:25 – 2:15PM)

Name: (First) Amedo (Last) Interiano

INSTRUCTIONS

- Write legibly and neatly the solutions to all the problems.
- Scan pages 1 through 14 with the solutions. Create a single pdf file with all 14 pages included. Use your name as the name of the file, for example, John_Smith.pdf
- Make sure all pages in the pdf file are legible and have the correct orientation.
- Submit the file via the Take-Home Quiz #2 Part B drop box at ANGEL.
- NOTE: To create a multi-page pdf file you don’t need to own Adobe Acrobat. In the University libraries, Adobe Acrobat is installed on the computers that are connected to the scanners. If you don’t know how to scan your pages and create a multi-page pdf file, ask a librarian to help. Please select GRAY IMAGE when scanning to reduce the size of the file. A color image will result in an extremely large file which may not upload and/or open properly.

1. Consider the function \( f(x) = \frac{x^2 + 2}{x + 1} \). Determine if it satisfies the hypotheses of Rolle’s Theorem on the interval \([0, 2]\). If yes, find the numbers \( c \) that satisfy the conclusion of the theorem. If no, explain why.

\[
\begin{align*}
&f(a) = 2 \quad f(b) = 2 \\
&f'(c) = \frac{(x^2 + 1)(2x) - (x)(x^2 + 2)}{(x+1)^2} = 0 \\
&f'(c) = \frac{2x^2 + 2x - 2}{(x-1)^2} = \frac{2c^2 + 2c - 2}{(c-1)^2} = 0 \\
&f'(c) = \frac{c^2 + 2c - 2}{(c-1)^2} = 0 \\
&c = \frac{\sqrt{8} - 1}{2} \\
&c = \frac{\sqrt{3} - 1}{2} \\
&c = \frac{3^{1/2} - 1}{2}
\end{align*}
\]
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2. Consider the function $f(x) = \sqrt{3x-2}$. Determine if it satisfies the hypotheses of the Mean Value Theorem on the interval $[1, 6]$. If yes, find the numbers $c$ that satisfy the conclusion of the theorem. If no, explain why.

\[ \frac{f(6) - f(1)}{6 - 1} = \frac{4 - 1}{5} = \frac{3}{5} \]

\[ f'(x) = \frac{3}{2\sqrt{3x-2}} \]

\[ f'(c) = \frac{3}{2\sqrt{3c-2}} = \frac{3}{5} \]

\[ c = \frac{11}{4} \]

3-6. Determine the domain and the equations of all asymptotes of the following functions.

3. $f(x) = \frac{x^3 + 8x - 2}{x^2 - 9}$

$\text{Domain} = (-\infty, 3) \cup (3, \infty)$

$\text{VA: } x^2 - 9 = 0$

$\pm 3$

$\text{HA: } \text{Domain excl. } x = \pm 3$

$\text{SA: } \text{Long Division}$

$\text{Asymptote: } y = x$
4. \( g(x) = \frac{5x^2 + 5x - 30}{2x^2 - 8} \)
   - \( D = -2, 2 \)
   - \( H_a = \lim_{x \to \infty} g'(x) = \frac{5x^2 + 5x - 30}{2x^2 - 8} = \frac{5}{2} \)
   - \( V_a = -2 \)
   - No slant

5. \( h(x) = \frac{x}{\sqrt{x^2 - 16}} \)
   - \( D = -4, 4 \)
   - \( H_a = \lim_{x \to \infty} h'(x) = \frac{1}{x \sqrt{x^2 - 16}} = 1 \)
   - \( V_a = -4 \)

6. \( j(x) = \frac{3x^2 + 3x - 6}{x^3 - 4x} \)
   - \( D = -2, 0, 2 \)
   - \( H_a = \lim_{x \to \infty} j'(x) = \frac{3x^2 + 3x - 6}{x^3 - 4x} = 0 \)
   - \( V_a = -2, 0, 2 \)
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7. The graph below shows the function $f(x)$.

(a) List the $x$-coordinates of the local maximums of $f(x)$.

\[ x = 1 \]

(b) List the $x$-coordinates of the local minimums of $f(x)$.

\[ x = 3 \]

(c) List the intervals, on which $f(x)$ is increasing.

\[ (-\infty, -1) \cup (-1, 1) \cup (3, \infty) \]

(d) List the intervals, on which $f(x)$ is decreasing.

\[ (1, 3) \]

(e) List the intervals, on which $f(x)$ is concave downward.

\[ (-1, 2) \]

(f) List the intervals, on which $f(x)$ is concave upward.

\[ (-\infty, -1) \cup (2, \infty) \]

(g) List the $x$-coordinates of the inflection points of $f(x)$.

\[ x = 2, x = 4 \]
8. The graph below shows the first derivative, \( g'(x) \), of a function \( g(x) \).

(a) List the \( x \)-coordinates of the local maximums of \( g(x) \).

(b) List the \( x \)-coordinates of the local minimums of \( g(x) \).

(c) List the intervals, on which \( g(x) \) is increasing.

(d) List the intervals, on which \( g(x) \) is decreasing.

(e) List the intervals, on which \( g(x) \) is concave downward.

(f) List the intervals, on which \( g(x) \) is concave upward.

(g) List the \( x \)-coordinates of the inflection points of \( g(x) \).
9. A function and its first two derivatives are

\[ f(x) = \frac{x}{(x^2-4)^2}, \quad f'(x) = \frac{-3x^2 - 4}{(x^2-4)^3} \quad \text{and} \quad f''(x) = \frac{12x(x^2 + 4)}{(x^2-4)^4}. \]

(a) Determine the domain of \( f(x) \).

\[ \{ x \in \mathbb{R} : x \neq -2, 2 \} \cup \begin{cases} \mathbb{R} & \text{for } x = -2, 2 \end{cases} \]

(b) Determine all x- and y-intercepts.

\[ x = 0, \quad y = 0 \quad \text{at } (0, 0) \]

(c) Is this function odd, even, or neither?

Odd

(d) State and classify all asymptotes of \( f(x) \). Compute the left and right limits at each of the vertical asymptotes.

Vertical Asymptotes:

\[ \lim_{{x \to -2^+}} \frac{x}{(x^2-4)^2} = \infty \]
\[ \lim_{{x \to -2^-}} \frac{x}{(x^2-4)^2} = -\infty \]
\[ \lim_{{x \to 2^+}} \frac{x}{(x^2-4)^2} = \infty \]
\[ \lim_{{x \to 2^-}} \frac{x}{(x^2-4)^2} = -\infty \]

(e) List all intervals on which the function is increasing.

Increasing on \( (-2, 2) \)

(f) State and classify all critical numbers of \( f(x) \).

\[ x = 0, \quad x = 2 \]

Critical Numbers:

\[ (0, 2) \cup (2, \infty) \]
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(g) List all intervals on which the graph of \( f(x) \) is concave up.
\[ x(x^2-4)=0 \]
\[ x=0 \text{ or } x=2 \]

\[ \text{Concave up } = (0, 2) \cup (2, \infty) \]

(h) State all inflection points of \( f(x) \). Give both coordinates of the points.
\[ \text{Inflection points } = (0, 0) \]

(i) Sketch the graph of \( f(x) \).

\[ \text{Graph of } f(x) \text{ showing concavity and inflection points.} \]
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10. A function and its first two derivatives are
\[ f(x) = \frac{4x^2 - 1}{x^2 + 1}, \quad f'(x) = \frac{10x}{(x^2 + 1)^2} \quad \text{and} \quad f''(x) = \frac{10(1 - 3x^2)}{(x^2 + 1)^3} \]

(a) Determine the domain of \( f(x) \).
\[ \mathbb{D} = \mathbb{R} \setminus \{-1, 1\} \]

(b) Determine all x- and y-intercepts.
\[ 0 = \frac{4x^2 - 1}{x^2 + 1} \quad \Rightarrow \quad x = \pm \frac{1}{\sqrt{2}} \]
\[ y = f(x) \quad \Rightarrow \quad y = \frac{4x^2 - 1}{x^2 + 1} \quad \Rightarrow \quad y = 0 \quad \text{when} \quad x = \pm 1 \]
\[ y = f(x) \quad \Rightarrow \quad y = \frac{4x^2 - 1}{x^2 + 1} \quad \Rightarrow \quad y = 1 \quad \text{when} \quad x = 0 \]

(c) Is this function odd, even, or neither?
**Even**

(d) State and classify all asymptotes of \( f(x) \). Compute the left and right limits at each of the vertical asymptotes.
\[ \lim_{x \to \pm \infty} f(x) = \lim_{x \to \pm \infty} \frac{4x^2 - 1}{x^2 + 1} = \frac{4}{1} = 4 \quad y = 4 \]
\[ \lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{4x^2 - 1}{x^2 + 1} = \frac{4}{1} = 4 \quad y = 4 \]

(e) List all intervals on which the function is increasing.
\[ f'(x) = \frac{10x}{(x^2 + 1)^2} \]
\[ f'(0) = 0 \quad \Rightarrow \quad x = 0 \]
\[ f'(-1) = -10 \quad \Rightarrow \quad x = -1 \]
\[ f'(1) = 10 \quad \Rightarrow \quad x = 1 \]
\[ \text{Increasing:} \quad (0, \infty) \]
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(f) State and classify all critical numbers of $f(x)$.

\[
\begin{align*}
&x = 1 \quad \text{and} \quad x = -1 \\
&\text{So the only critical # is } x = 0
\end{align*}
\]

(g) List all intervals on which the graph of $f(x)$ is concave up.

\[
\begin{align*}
&f''(x) = \frac{10(1-3x^2)}{(x^2+1)^3} \\
&0 = 10(1-3x^2) \\
&0 = (x^2 + 1)^3 \\
&x = \frac{\sqrt{3}}{3} \quad x = -\frac{\sqrt{3}}{3}
\end{align*}
\]

(h) State all inflection points of $f(x)$.

\[
\begin{align*}
&\left(-\frac{\sqrt{3}}{3}, \frac{1}{12}\right) \left(\frac{\sqrt{3}}{3}, \frac{1}{12}\right)
\end{align*}
\]

(i) Sketch the graph of $f(x)$. 

\[
\begin{align*}
&\text{Sketch graph of } f(x) \\
&\text{Plot points and sketch curve.}
\end{align*}
\]
11. Sketch the graph of a function, for which the following is known:

- $f(x)$ is defined and continuous on $(-3, -1) \cup (-1, \infty)$
- $f(-2) = 2$
- $f(0) = 0$
- $f(2) = 3$
- $\lim_{{x \to -1^-}} f(x) = \infty$
- $\lim_{{x \to -1^+}} f(x) = \infty$
- $\lim_{{x \to 3^-}} f(x) = -\infty$
- $\lim_{{x \to 3^+}} f(x) = 2$
- $f'(x) > 0$ on $(-3, -1)$ and $(0, 2)$
- $f'(x) < 0$ on $(-1, 0)$ and $(2, \infty)$
- $f''(x) > 0$ on $(-2, -1)$, $(-1, 1)$ and $(3, \infty)$
- $f''(x) < 0$ on $(-3, -2)$ and $(1, 3)$
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12. A box with square base and open top is to be constructed from a square sheet of cardboard by cutting out a square from each of the four corners and bending up the sides. The cardboard piece has an area of 81 cm$^2$. What is the largest possible volume of the box?

Let $x$ be the side length of the square base and $y$ be the height of the box. Then the area of the cardboard is $x^2 + 4xy = 81$.

The volume of the box is $V = x^2 y$.

From the area equation, we have $x^2 = 81 - 4xy$. Substituting this into the volume equation gives $V = x(81 - 4xy) = 81x - 4x^2 y$.

Taking the derivative with respect to $x$, we get $V' = 81 - 8xy$. Setting $V' = 0$ gives $81 = 8xy$, so $x = 9/y$. Substituting this back into the area equation gives $y = 3$.

Therefore, the largest possible volume is $V = 81(3) = 243$ cm$^3$.

13. The material used for the production of the base and the top of a cylindrical container costs $9 per m$^2$. The material used for the side of the container costs $6 per m^2$. The container must have a volume of $24\pi$ m$^3$. Determine the size of the container (radius and height), which will minimize the cost of the materials used.

The volume of the container is $V = \pi r^2 h = 24\pi$.

The cost of the materials is $C = 9(\pi r^2) + 6(2\pi rh) = 9\pi r^2 + 12\pi rh$.

Taking the derivative with respect to $r$, we get $C' = 18\pi r + 12\pi rh = 0$, so $r + 2h = 0$. Then $r = 2h$.

Substituting this into the volume equation gives $\pi (2h)^2 h = 24\pi$, so $4h^3 = 24$, or $h = \sqrt[3]{6}$.

Therefore, the size of the container is $r = \sqrt[3]{6}$ and $h = \frac{\sqrt[3]{6}}{2}$.
14. Determine the x-coordinate of the point on the line $y = 5x + 3$ that is closest to the point $(-1, 2)$.

$$d = \sqrt{(x-x_1)^2 + (y-y_1)^2}$$

$(x, y) = (-1, 2)$

$$y = 5x + 3$$

$$x^2 + 2x + 1 + y^2 + 4y + 4 = 0$$

$$x^2 + 2x + \frac{1}{4} + y^2 + 4y + 4 = 0$$

$$x + y = 0$$

$$x = -\frac{3}{13}$$

$(-\frac{2}{13}, \frac{4}{13})$

15. What is the largest rectangle that can be inscribed in the ellipse with equation $\frac{x^2}{4} + \frac{y^2}{9} = 1$?
16. A poster is to have an area of 180 in$^2$ with 1-inch margins at the bottom and sides and a 2-inch margin at the top. What dimensions will give the largest printed area?

Let $x$ and $y$ be the dimensions of the poster excluding the margins.

The area of the poster is $A = (x-2)(y-3)$. We need to maximize $A$ subject to $A = 180$.

Differentiating $A$ with respect to $x$, we get:

$$A' = (x-2)\frac{d}{dx}(y-3) + (y-3)\frac{d}{dx}(x-2) = x^2 - 12x + 30$$

Setting $A' = 0$, we find:

$$x = 6$$

Substituting $x = 6$ into $A = (x-2)(y-3)$, we get:

$$y = 5$$

Thus, the dimensions that give the largest printed area are $x = 6$ and $y = 5$.

17. A boy is in a boat 1 mi from a straight coastline. His home is on the beach, 3 mi from the point on the shore closest to the boat. The boy rows at 3 mph and runs at 5 mph. A storm is approaching and the boy must get home as soon as possible. What is the fastest he can get home?

Let $x$ be the distance the boy rows on the water and $y$ be the distance he runs on the land.

The total time $T$ is given by:

$$T = \frac{x}{3} + \frac{3-x}{5}$$

Differentiating $T$ with respect to $x$, we get:

$$T' = \frac{1}{3} - \frac{1}{5} = \frac{2}{15}$$

Setting $T' = 0$, we find:

$$x = 2$$

Substituting $x = 2$ into $y = \sqrt{\frac{x^2}{4}-1} + \frac{3-x}{5}$, we get:

$$y = \frac{5}{3}$$

The fastest he can get home is when he rows 2 miles and runs $\frac{5}{3}$ miles.
18. A particle moves following a trajectory that is a straight line. The position function of the particle is \( s(t) = t^4 - 8t^3 + 18t^2 - 4t + 12 \) for \( 0 \leq t \leq 4 \). The position, \( s \), is measured in meters and the time, \( t \), in seconds. At what time instant(s) in the time interval \((0, 4)\) is the particle moving forward at the highest velocity?

Velocity is maximum when acceleration is 0

\[
\begin{align*}
\frac{ds}{dt} &= v(t) = 4t^3 - 24t^2 + 36t - 4 \\
\frac{d^2s}{dt^2} &= a(t) = 12t^2 - 48t + 36
\end{align*}
\]

\( 12t^2 - 48t + 36 = 0 \)

\( 12(t^2 - 4t + 3) \)

\( 12(1 - t)(t-3) \)

\( t = 3 \quad \text{and} \quad t = 1 \)

19. A farmer wants to fence off a rectangular field of area 120 ft² adjacent to the house. He needs no fence along the house. The fencing will cost $20 per foot for the side that is parallel to the house and $12 per foot for the sides that are perpendicular to the house. What is the lowest cost of such fencing?

\[
\begin{align*}
\text{Cost} &= 12 \left( \frac{20}{x} \right) + 20 \left( \frac{20}{x} \right) \\
\text{Cost} &= 12 \left( \frac{20}{x} \right) + 20 \left( \frac{20}{x} \right) \\
\end{align*}
\]

Condition 1: \( \cos \theta = 0 \)

\( x = 10 \)

\( \cos \theta = 0 \)

\( \theta = 90° \)

\( \text{Cost} = 12 \left( \frac{20}{10} \right) + 20 \left( \frac{20}{10} \right) \)

\( \text{Cost} = \$480 \)