A Dynamic Partial Equilibrium Model of Capital Gains Taxation

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Abstract

We analyze a multi-period partial equilibrium model with capital gains taxation. Relative to an economy without taxation, a capital gains tax tends to lower prices and raise expected returns, but it has little effect on volatility. Abstracting from tax redistribution policies, we find that a taxable investor’s welfare falls, a nontaxable investor’s welfare rises, and, depending on the tax rate, social welfare may either rise or fall under a capital gains tax. Furthermore, the taxable investor’s tax-timing option may either increase or decrease tax revenue. Implications for empirical asset pricing are identified.

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1 Introduction

The taxation of capital gains is ubiquitous, applying to financial securities, real estate, patents, and other types of capital assets. Capital gains taxation is also unique in the sense that tax consequences arise only upon the sale or disposition of a taxable asset, i.e., at the time a gain or loss is realized. This feature distinguishes capital gains taxation from the taxation of dividends, which typically generates a tax consequence at the time a dividend is received or accrued. The realization-based nature of capital gains taxation enables investors to retain control over the timing of tax consequences and, therefore, has the potential to affect prices, allocations, and other equilibrium attributes in ways that other types of taxation do not.

While capital gains taxation is undoubtedly important, modeling its effects is challenging. Compared to more conventional models with easily manageable state spaces that can be solved using standard backward induction techniques, any model incorporating capital gains taxation is inherently more complicated because an investor’s tax basis represents an additional endogenous state variable that both affects and reflects the distributions of future prices and allocations. Due to this complexity, the existing literature (discussed below) almost exclusively relies on exogenous price processes when analyzing the effects of capital gains taxation on investors’ portfolio decisions.

To evaluate the effects of capital gains taxation on equilibrium outcomes, we construct a multi-period model in which prices are determined endogenously and tax liability arises only upon the sale of a taxable asset. The model features two investors, one who is subject to taxation and one who is not. The investors trade two types of assets: a risky stock, which generates a tax consequence for the taxable investor whenever he sells, and a risk-free bond, which investors may trade without any tax consequences. Neither investor is endowed with any shares of stock, but each investor has the opportunity to purchase stock from the issuing firm during an initial offering. Both the timing and size of the taxable investor’s tax liability are endogenously determined in equilibrium and depend on his trading activity and tax basis.

Relative to a benchmark economy in which capital gains are not taxed, we find that tax-
ation generates a clientele effect whereby the taxable (nontaxable) investor tends to hold less (more) stock in equilibrium when capital gains are taxed. This result is consistent with prior studies that examine how capital gains taxation affects investors’ portfolio decisions in settings with exogenous price processes. An important distinction, however, is that the price is determined endogenously in our model, so the investors’ allocations correspond to equilibrium outcomes. For sufficiently high tax rates, the taxable investor abstains from holding equity, and the nontaxable investor holds all of the stock.

The distortions to the equilibrium allocations caused by capital gains taxation hinder efficient risk sharing and result in the nontaxable (taxable) investor bearing more (less) risk than is optimal relative to the benchmark economy without a capital gains tax. To compensate the nontaxable investor for holding more stock and bearing more risk, the equilibrium stock price falls on average when capital gains are taxed, resulting in higher pre-tax expected returns and a higher cost of capital for the issuing firm.

Our results pertaining to prices and allocations are consistent with both the capital gain lock-in effect, which stipulates that an investor who owns an asset with an embedded gain may refrain from selling the asset to avoid incurring a tax liability, and the capitalization effect, which implies that taxation gives rise to lower prices because investors capitalize future tax liabilities into current prices. Specifically, we find that the taxable investor realizes a smaller fraction of his embedded gains than his embedded losses (consistent with the lock-in effect) even though prices are lower on average (consistent with the capitalization effect).

The value of tax revenue generated by a capital gains tax in equilibrium follows a familiar Laffer curve. When the tax rate is low, increasing the rate generates more revenue because a larger percentage of realized gains accrue to the taxing authority. For higher tax rates, further increasing the rate generates less revenue because the taxable investor holds fewer shares of stock, resulting in a smaller tax liability. For sufficiently high tax rates, a capital gains tax generates no tax revenue because the nontaxable investors holds all of the stock.

Taxation’s effects on equilibrium prices, allocations, and tax revenue ultimately affect the
investors’ welfare. Abstracting from tax revenue redistribution policies, the taxable investor experiences a loss of welfare because the tax lowers his average payoff and he tends to hold a suboptimal portfolio relative to his portfolio without a capital gains tax. In contrast, the nontaxable investor’s welfare rises due to positive spillover effects, as she tends to hold more stock in equilibrium and realize higher returns.

Interestingly, capital gains taxation’s effect on social welfare (the aggregate welfare of the taxable and nontaxable investors plus the value of tax revenue and amount of capital raised by the firm during the initial offering), which is unaffected by state-independent redistribution policies, depends on the tax rate. Specifically, capital gains taxation increases social welfare for low tax rates but decreases welfare for high tax rates. The effect is due to risk sharing and the cost of capital. For low tax rates, taxation enhances risk sharing because the taxing authority bears a portion of the risk that the investors would otherwise bear without a tax on capital gains, as first shown by Domar and Musgrave (1944). Additionally, taxation only slightly raises the firm’s cost of capital because it causes only a minor distortion to the equilibrium price. For high tax rates, taxation diminishes risk sharing because the taxable investor abstains from the market, requiring the nontaxable investor to hold all of the stock and, therefore, bear all of the risk. Furthermore, taxation raises the firm’s cost of capital to a greater extent because there is greater price distortion.

Although different tax rates produce different amounts of revenue, as stated above, we find that a tax rate that changes over time can generate more revenue than a constant tax rate. In particular, a dynamic tax policy that begins with a low initial rate followed by an (unexpected) increase to a moderate rate generates greater revenue on average than a tax policy that simply implements a single constant rate over time. The low initial rate induces the taxable investor to buy more stock than he otherwise would if the tax rate was higher, creating a larger tax consequence following the rate increase. Perhaps surprisingly, a dynamic policy that comprises a rate decrease generates (weakly) less tax revenue on average than a policy that adopts the lower rate initially.
While the amount of revenue depends on the tax rate, the realization-based nature of the capital gains tax creates a tax-timing option for the taxable investor whereby he can defer the realization of gains and accelerate the realization of losses, à la Constantinides (1983) and Stiglitz (1983) (but cf. Constantinides 1984; Dammon, Dunn, and Spatt, 1989; Dammon and Spatt, 1996; Dai et al., 2015). To determine the value of the option, we compare the economy in our model where tax consequences arise only when the taxable investor sells the stock to an alternative economy in which tax consequences arise as gains and losses accrue regardless of the taxable investor’s trading activity. We find that the tax-timing option has a positive (negative) value for the taxable (nontaxable) investor. Furthermore, the tax-timing option tends to increase social welfare because it improves risk sharing by incentivizing the taxable investor to hold more stock in equilibrium. For a similar reason, the tax-timing option increases tax revenue on average if the tax rate is relatively high. However, the tax-timing option decreases revenue for low tax rates because the option enables the investor to reduce the present value of his tax liability.

Finally, we identify two potentially important implications of capital gains taxation for empirical asset pricing. First, we find that a capital gains tax increases the price of risk but decreases the quantity of risk, measured with ex post time-series realizations, because the ex post volatility of the stochastic discount factor (SDF) rises as the tax rate increases. Hence, capital gains taxation affects average ex post returns in a non-linear fashion. Second, the increase in the volatility of the SDF attenuates the correlation between the SDF and returns. This affects measures of model fit such as the Hansen and Jagannathan (1991) bound and suggests that non-risk distortions may contribute to the low empirical correlations between returns and asset pricing fundamentals (Cochrane and Hansen, 1992; Albuquerque et al., 2016).

To the best of our knowledge, we are the first to analyze the effects of capital gains taxation on equilibrium prices and allocations in a multi-period economy. Klein (1999) and Viard (2000) characterize the equilibrium effects of a capital gains tax, but their results are difficult to interpret because the expressions for the stock price and allocations in their models
are implicit functions of themselves.\(^1\) Both Klein (1999) and Viard (2000) demonstrate that, *ceteris paribus*, a capital gains tax may result in higher prices due to the lock-in effect. In contrast, we find that capital gains taxation tends to lower prices because the capitalization effect dominates the lock-in effect in equilibrium. Dammon and Spatt (1996) use no-arbitrage conditions to derive equilibrium prices and optimal trading strategies. However, they do not equate supply with demand, so the market in their model does not clear in the traditional sense. Dybvig and Ross (1986), Shackelford and Verrecchia (2002), Sahm (2008), and Sikes and Verrecchia (2012) consider single period models with capital gains taxes.

Many other studies examine portfolio implications of capital gains taxation in multi-period settings where prices are exogenous. Dammon, Spatt, and Zhang (2001) analyze a market with a single taxable asset and show that investors’ lifetime consumption and investment decisions are not separable when frictions such as short-sale constraints exist. Gallmeyer, Kaniel, and Tompaidis (2006) extend their analysis to a market with two taxable assets and find that incorporating additional assets can lead investors to implement tax-trading strategies that result in highly non-diversified portfolios when short sales are permitted. Dammon, Spatt, and Zhang (2004), Garlappi and Huang (2006), and Fischer and Gallmeyer (2017) examine how investors should allocate their investments between taxable and tax-deferred accounts. Marekwica (2012) and Ehling et al. (2017) investigate portfolio trading strategies when the use of losses to offset gains is limited.

Most of the empirical literature related to capital gains taxation focuses on the effect of taxation on prices and trading behavior. The empirical evidence regarding the effect of a capital gains tax on asset prices is mixed. Many studies find that taxation tends to raise prices due to the lock-in effect (Landsman and Shackelford, 1995; Klein, 2001; Ayers, Lefanowicz, and Robinson, 2003; George and Hwang, 2007). However, several other studies find that taxation tends to lower prices due to the capitalization effect (Guenther and Willenborg, 1999; Lang

\(^1\)Specifically, the price in those models is a function of the future price, which is a function of the future tax basis, which is a function of the current price. Hence, the price is an implicit function of itself. Because allocations depend on prices, the allocations are also implicit functions of themselves.
and Shackelford, 2000; Dai et al., 2008; Blouin, Hail, and Yetman, 2009). As stated above, we find that the capitalization effect tends to dominate the lock-in effect in equilibrium, leading to a decrease in prices on average.

Capital gains taxation also appears to affect the trading behavior of investors in practice. Consistent with our finding that the taxable investor realizes a much smaller fraction of his embedded gains than his embedded losses, most empirical studies find that investors tend to defer the realization of capital gains and/or accelerate the realization of capital losses (see, e.g., Feldstein and Yitzhaki, 1978; Feldstein, Slemrod, and Yitzhaki, 1980; Reese, 1998; Blouin, Raedy, and Shackelford, 2003; Jin, 2006; Shan, 2011).

The remainder of the article is organized as follows. We first describe our model in Section 2 and characterize the equilibrium in Section 3. In Section 4, we discuss the effects of capital gains taxation on equilibrium attributes such as prices, allocations, tax revenue, and welfare. We also discuss implications for empirical asset pricing. In Section 5, we consider extensions of the model to examine the value of the tax-timing option and the impact of tax rate changes on tax revenue. Finally, Section 6 concludes.

2 Model

We consider a simple partial equilibrium model of capital gains taxation. Many of our assumptions are designed to enhance tractability and computational efficiency. There are $T$ trading dates indexed by $t = 1, 2, \ldots, T$. Two types of assets comprise the financial economy. A stock, which is in unit supply and undergoes an initial offering at $t = 1$, pays a random amount, $\tilde{Y}$, at $T + 1$ equal to the sum of $T$ random components, $\tilde{X}_t$. Hence, $\tilde{Y}$ is given by

$$\tilde{Y} = \sum_{t=1}^{T} \tilde{X}_t. \quad (1)$$

The random components of the stock payoff are realized as time progresses; $X_t$ is realized at $t + 1$. Each random component of the stock payoff takes a value that is either high, $H$, or
low, \( L \). The probability that \( X_t = L \), which is denoted by \( \pi \), is exogenous and constant over time; \( X_t = H \) with complementary probability \( 1 - \pi \).

A (unmodeled) competitive market maker sets ask and bid prices for the stock. Investors pay the ask when they buy the stock and receive the bid when they sell. The time-\( t \) ask and bid, which are determined endogenously, are denoted by \( A_t \) and \( B_t \), respectively. As explained in more detail below, the ask and bid are selected so as to minimize the bid-ask spread and clear the market while avoiding arbitrage opportunities. Modeling asks and bids rather than a single price allows for different marginal rates of substitution and, therefore, avoids the potential nonexistence of equilibrium that can arise in the presence of taxation, as highlighted by Dammon and Green (1987).

The other type of asset is a series of one-period bonds. For simplicity, we assume that each one-period bond has an exogenous interest rate, \( r \in \mathbb{R}^{++} \), and that the supply of each bond is elastic. Thus, a bond purchased at \( t \) for one unit of account pays \( 1 + r \) at \( t + 1 \).

There are two investors: a taxable investor (he) and a nontaxable investor (she). In reality, some types of investors are subject to taxation (e.g., individuals) whereas others are not (e.g., pension funds). The investors obtain utility from consuming their individual wealth after liquidating their respective portfolios at \( T + 1 \). The taxable and nontaxable investors have preferences characterized by constant absolute risk aversion (CARA) with respective risk aversion coefficients \( \delta \) and \( \hat{\delta} \). As discussed below, the CARA assumption reduces the dimensionality — and, thereby, eases the computational burden — because these preferences do not exhibit wealth effects.\(^2\)

\(^2\)Allowing for wealth effects is not computationally feasible because incorporating wealth effects would necessitate two additional state variables, namely, each investors’ time-\( t \) wealth. Although avoiding wealth effects is not innocuous, the equilibrium allocations and trading behavior generated by our model are consistent with the lock-in effect: the fraction of the time that the insider sells with an embedded gain is smaller than the fraction of the time that he sells with an embedded loss. Other types of preferences (e.g., constant relative risk aversion) would produce qualitatively similar results given that the investors receive identical endowments. Moreover, although CARA preferences generally result in no-trade equilibria in the absence of frictions (once initial equilibrium allocations are reached), capital gains taxation generates trading volume in the time series because the realization-based nature of the tax and the dependence of the basis on transaction prices endogenously alter the taxable investor’s risk-return tradeoff over time. This time variation in the risk-return tradeoff creates an incentive for trade even in the absence of wealth effects.
Each investor holds a portfolio of financial assets, the compositions of which they may alter over time. Let $S_t$ and $W_t$ ($\hat{S}_t$ and $\hat{W}_t$) denote the respective quantities of the stock and bond held by the taxable (nontaxable) investor between $t$ and $t+1$. The taxable (nontaxable) investor also receives an exogenous endowment of the bond, denoted by $W_0$ ($\hat{W}_0$), before trading at $t = 1$. The investors are not endowed with any stock, so $S_0 = \hat{S}_0 = 0$. For tractability, we assume that investors cannot short the stock, either outright or against the box.\(^3\) Wash sales also are prohibited.\(^4\)

The taxable investor must pay a tax at rate $\theta \in (0, 1)$ on realized capital gains. This capital gains tax is paid at the time the gain is realized, i.e., when the stock is sold. Additionally, the taxable investor receives a tax rebate for realized capital losses, which reflects the current law that permits individuals to deduct capital losses.\(^5\) The nontaxable investor is not subject to a capital gains tax. To isolate the effects of capital gains taxation, we assume that neither investor pays tax on interest income. Modeling the capital gains tax as a realization-based tax is critical because, as Balcer and Judd (1987) show, an accrual-based tax does not accurately summarize the effects of a realization-based tax.

Following standard practice in the literature, we assume that the taxable investor’s time-$t$ tax basis per share, $Q_t$, is a weighted average of the prices at which the stock was previously acquired. Thus, the tax basis evolves according to

$$Q_t = \begin{cases} 
Q_{t-1} & \text{if } S_t \leq S_{t-1} \\
\frac{1}{S_t} [S_{t-1} Q_{t-1} + (S_t - S_{t-1}) A_t] & \text{otherwise,}
\end{cases}$$

\(^3\)Short selling against the box involves shorting a stock while simultaneously holding a long position in the stock. Current tax law requires an investor to recognize a gain from short selling against the box if the investor would be required to recognize a gain from selling the stock outright (I.R.C. §1259).

\(^4\)A wash sale occurs when an investor sells a stock at a loss and buys the stock within 30 days before or after the sale. Current tax law prohibits tax deductions from wash sales. (I.R.C. §1091).

\(^5\)Under I.R.C. §1211, individuals may deduct up to $3,000 of capital losses from ordinary income. For simplicity, we assume that the taxable investor in our model can claim a tax rebate (determined by the capital gains tax rate) for a realized capital loss regardless of the size of the loss. Ehling et al. (2017) find that limiting the use of losses to offset gains affects portfolio decisions in the short term but that limiting the use of losses becomes less relevant over longer horizons.
where $Q_0 = 0$ because the investor is not endowed with any stock. In a multiperiod setting with an exogenous price process, DeMiguel and Uppal (2005) find that investors make very similar portfolio decisions when they use an exact tax basis to calculate gains instead of a weighted-average basis. The taxable investor's tax liability at time $t \leq T$ is

$$L_t = \begin{cases} 
(S_{t-1} - S_t)(B_t - Q_{t-1})\theta & \text{if } S_t \leq S_{t-1} \\
0 & \text{otherwise}
\end{cases}$$

(3)

because he incurs a tax liability only when he realizes a gain (or loss). When portfolios are liquidated at $T + 1$, his tax liability is $L_{T+1} = S_T(\bar{Y} - Q_T)\theta$.

3 Equilibrium

The equilibrium concept is standard. At each trading date $t$, the taxable and nontaxable investors choose stock allocations to maximize their respective expected utilities, $U_{t+1}$ and $\hat{U}_{t+1}$, taking the ask and bid prices as given. Ask and bid prices are determined through market-clearing and no-arbitrage conditions. For the market to clear, the aggregate amount of stock demanded by the investors at $t$ must equal the outstanding supply of stock. Additionally, to prevent arbitrage opportunities, the time-$t$ ask must not be less than the time-$t$ bid. Because there may be more than a single pair of ask and bid prices that clear the market, we also impose a market-efficiency condition. This condition, which reflects a competitive market-making environment, stipulates that if more than one pair of ask and bid prices clears the market, then the pair that emerges in equilibrium is the one that minimizes the bid-ask spread.

The following definition formalizes the equilibrium concept.

Definition of Equilibrium. An equilibrium at time $t$ is defined by ask and bid prices, $A_t$ and $B_t$, and stock allocations, $S_t$ and $\hat{S}_t$, such that the following four conditions hold:

(i) Utility maximization: The respective stock allocations, $S_t$ and $\hat{S}_t$, maximize the taxable and nontaxable investors’ expected utilities, i.e.,
\[ S_t = \arg \max E_t[\tilde{U}_{t+1}] \]  
\[ \hat{S}_t = \arg \max E_t[\tilde{U}_{t+1}] . \]  

(ii) **Market clearing**: Aggregate stock demand equals supply, i.e., 
\[ S_t + \hat{S}_t = 1. \]  

(iii) **No arbitrage**: The bid does not exceed the ask, i.e., 
\[ B_t \leq A_t. \]  

(iv) **Market efficiency**: The bid-ask spread is minimized, i.e., for any alternative ask and bid prices, \( A'_t \) and \( B'_t \), that satisfy (6) and (7), 
\[ A_t - B_t \leq A'_t - B'_t. \]

We recursively solve for the equilibrium at each date using a non-recombining binomial tree. The nodes of the tree represent trading dates, and the branches represent the possible realizations of \( \tilde{X}_t \). The tree is non-recombining because the equilibrium is path dependent in the presence of capital gains taxation.

In the absence of taxation, the equilibrium stock prices and allocations at any given \( t \) depend only on investors’ expectations about the future payoff and prices. Hence, the time-\( t \) equilibrium in a benchmark setting without a capital gains tax (i.e., \( \theta = 0 \)) is independent of past prices and allocations. In contrast, the stock prices and allocations in the presence of a capital gains tax depend on both expectations about future returns as well as past prices and allocations because the taxable investor’s tax liability — and, thus, his portfolio allocation — depends on, *inter alia*, his tax basis, which is determined by past prices and allocations.

Although the equilibrium at \( t \) depends on the entire history of past prices and allocations, this history can be summarized by two relevant state variables: the taxable investor’s stock allocation and tax basis at \( t - 1 \). Using numerical methods, which is a common practice in the literature (see, e.g., Dammon and Spatt, 1996; Dammon, Spatt, and Zhang, 2001; DeMiguel and Uppal, 2005; Gallmeyer, Kaniel, and Tompaidis, 2006; Dai et al. 2015), we compute the
time-$t$ equilibrium ask and bid prices and stock allocations over a grid of these state variables. Beginning at $T$ and working backwards, at each node in the tree we compute the time-$t$ equilibrium prices and allocations over a grid of $S_{t-1}$ and $Q_{t-1}$. For each $\{S_{t-1}, Q_{t-1}\}$ pair, we employ a root-finding algorithm to determine the time-$t$ equilibrium ask and bid prices, and we solve for the time-$t$ equilibrium allocations using a grid. The time-$t$ allocations must satisfy equilibrium condition (i) and may take any value on the grid $S = \{0, \frac{1}{N_S}, \frac{2}{N_S}, \ldots, 1\}$, where $N_S$ denotes the fineness of the grid. The time-$t$ ask and bid prices must satisfy equilibrium conditions (ii)–(iv) and may take any value in $\mathbb{R}^{++}$. We then “roll back” the tree and compute the equilibrium at $t - 1$ over a grid of $S_{t-2}$ and $Q_{t-2}$ using bilinear interpolation. Ultimately, we compute an equilibrium at each node and, thereby, obtain a time series of equilibrium prices and allocations for every possible path in the tree. The solution method is described in greater detail in the following sections, and detailed algorithms are provided in the appendix.

### 3.1 Equilibrium at $T$

At date $T$, the taxable and nontaxable investors allocate their wealth among the stock and bond to maximize their respective expected utilities from consumption, $C$ and $\hat{C}$, subject to budget constraints. When selecting his portfolio at $T$, the taxable investor faces a tradeoff between bearing a preferred amount of risk and delaying the realization of any accrued capital gains until $T + 1$ (or, alternatively, accelerating the realization of any accrued capital losses to $T$). The taxable investor’s problem at $T$ is:

$$\max_{S_T} \mathbb{E}_T[\exp[-\delta \tilde{C}] \mid X_1, X_2, \ldots, X_{T-1}]$$  \hspace{1cm} (8)$$

s.t.  \hspace{1cm} \tilde{C} = W_T(1 + r) + S_T \tilde{Y} - \tilde{L}_{T+1}  \hspace{1cm} (9)$$

$$W_T = W_{T-1}(1 + r) + \max\{S_{T-1} - S_T, 0\} B_T - \max\{S_T - S_{T-1}, 0\} A_T - L_T. \hspace{1cm} (10)$$
The nontaxable investor faces a similar problem at \( T \). The only difference is that her utility is not directly affected by the tax. Thus, the nontaxable investor’s problem at \( T \) is:

\[
\max_{\hat{S}_T} \quad \mathbb{E}_T\left[-\exp[-\delta \hat{C}] \mid X_1, X_2, \ldots, X_{T-1}\right]
\]

s.t. \( \hat{C} = \hat{W}_T(1+r) + \hat{S}_T \hat{Y} \) (12)

\[
\hat{W}_T = \hat{W}_{T-1}(1+r) + \max\{\hat{S}_{T-1} - \hat{S}_T, 0\} B_T - \max\{\hat{S}_T - \hat{S}_{T-1}, 0\} A_T.
\] (13)

Although the tax does not directly affect the nontaxable investor’s utility, the tax indirectly affects her welfare because, as we demonstrate below in Section 4, capital gains taxation distorts the equilibrium prices and allocations.

Analytic expressions for the investors’ time-\( T \) demand functions can be derived by solving the first-order conditions of their utility-maximization problems. Furthermore, the demand functions can be aggregated to obtain analytic expressions for time-\( T \) equilibrium ask and bid prices. However, as we discuss below in Section 3.2, tractable analytic expressions for the investors’ demand functions at \( t < T \) are unobtainable. Therefore, we rely on numerical methods to compute the equilibrium prices and allocations at \( t < T \). For consistency, we numerically compute the time-\( T \) equilibrium, as well.

In addition to the realizations of the components of the stock payoff, \( X_t \), there are five state variables that may affect the investors’ utility-maximization problems. These five state variables are: the taxable investor’s previous stock and bond holdings, \( S_{T-1} \) and \( W_{T-1} \); the taxable investor’s basis, \( Q_{T-1} \); and the nontaxable investor’s previous stock and bond holdings, \( \hat{S}_{T-1} \) and \( \hat{W}_{T-1} \). Because aggregate demand must equal supply, however, \( \hat{S}_{T-1} = 1 - S_{T-1} \).

Furthermore, because there are no wealth effects with CARA preferences, we can ignore the bond holdings at \( T - 1 \) when determining the time-\( T \) equilibrium stock prices and allocations, though we account for the impact of these bond holdings when we compute the equilibrium at \( T - 1 \), as discussed below in Section 3.2. This reduces the dimensionality, leaving only two relevant state variables: \( S_{T-1} \) and \( Q_{T-1} \).
As illustrated in Figure 1, we compute time-$T$ ask and bid prices along with stock allocations over a grid of state variables, i.e., over an array of combinations of $S_{T-1}$ and $Q_{T-1}$, at each time-$T$ node in the tree. To compute the equilibrium for a given $\{S_{T-1}, Q_{T-1}\}$ pair, we employ an iterative algorithm that involves conjecturing ask and bid prices, calculating demands, and updating the prices until the four equilibrium conditions are satisfied. More specifically, we first conjecture ask and bid prices that satisfy the no-arbitrage condition, (iii). We then compute the investors’ expected utilities over a grid of possible allocations, taking into account the effects of the allocations and the (conjectured) ask and bid prices on the taxable investor’s tax basis and liability. Because there are no wealth effects, we temporarily set $W_{T-1}$ and $\hat{W}_{T-1}$ equal zero to reduce the dimensionality, though for the time being the bond holdings can take any arbitrary value. Next, we select the allocation for each investor that maximizes that investor’s expected utility, i.e., the allocations that satisfy equilibrium condition (i). We then update the conjectured ask and bid prices with a bisection method until the market clears and the bid-ask spread is minimized (equilibrium conditions (ii) and (iv)).\footnote{Our numerical methodology selects the highest bid and ask prices such that all four conditions of the equilibrium are satisfied. Our results, which are presented in Section 4 below, are qualitatively robust to an alternative methodology that selects the lowest bid and ask prices.} This procedure, which is described in greater detail in the appendix, generates equilibrium ask and bid prices and stock allocations for all possible states (defined by $S_{T-1}$ and $Q_{T-1}$) at a given node at $T$. We use these state-dependent time-$T$ prices and allocations to compute equilibrium prices and allocations at earlier dates, as discussed in the next section.

### 3.2 Equilibrium at $t < T$

After computing the time-$T$ equilibrium prices and allocations, we roll back the tree and compute the equilibrium prices and allocations at earlier dates using backward induction. Deriving the equilibrium prices and allocations at $t < T$ is more computationally intensive than at $T$ because tractable analytic expressions for the investors’ utility functions do not exist. Nonetheless, the equilibrium concept is the same.
At every date \( t < T \), the investors choose portfolios of the stock and bond to maximize their expectations of their respective expected utilities at \( t + 1 \), subject to budget constraints. The taxable investor’s problem at \( t < T \) is:

\[
\max_{S_t} \quad E_t[\tilde{U}_{t+1} | X_1, X_2, \ldots, X_{t-1}] \\
\text{s.t.} \quad W_t = W_{t-1}(1 + r) + \max\{S_{t-1} - S_t, 0\}B_t - \max\{S_t - S_{t-1}, 0\}A_t - L_t.
\]

The taxable investor faces a tradeoff at \( t < T \) similar to the one faced at \( T \); he may potentially obtain a more desirable risk exposure by rebalancing his portfolio, but rebalancing may trigger a tax liability. If he rebalances, then his time-\( t \) tax liability is captured by \( L_t \) in (15). All potential future tax liabilities are embedded within \( \tilde{U}_{t+1} \) in (14).

Like at \( T \), the nontaxable investor is affected by a capital gains tax only indirectly through the distortions to the equilibrium prices and allocations. Her problem at \( t < T \) is:

\[
\max_{\hat{S}_t} \quad E_t[\hat{U}_{t+1} | X_1, X_2, \ldots, X_{t-1}] \\
\text{s.t.} \quad \hat{W}_t = \hat{W}_{t-1}(1 + r) + \max\{\hat{S}_{t-1} - \hat{S}_t, 0\}B_t - \max\{\hat{S}_t - \hat{S}_{t-1}, 0\}A_t.
\]

Although the investors’ objectives at \( t < T \) are fairly straightforward, deriving the equilibrium prices and allocations is complicated by the fact that, in general, tractable analytic expressions for the investors’ expected utilities, \( U_{t+1} \) and \( \tilde{U}_{t+1} \), do not exist. Capital gains taxation gives rise to intractable utility functions because the investors’ expected utilities at \( t + 1 \) depend on the direction of the taxable investor’s trade at \( t \). Nevertheless, we can compute a numerical value for each investor’s expected utility using the numerical values for the distributions of prices and allocations at \( t + 1 \).

As discussed above, we ignore the investors’ time-\( t \) bond holdings when computing the

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\(^7\)Klein (1999), for example, characterizes equilibrium prices and allocations in the presence of a capital gains tax, but the expressions are not in closed form because the prices and allocations at any point in time depend on future prices, which in turn depend on the present prices and allocations.
equilibrium prices and allocations at $t + 1$ to reduce the dimensionality of the problem. This is feasible because CARA preferences do not exhibit wealth effects. Although the time-$t$ bond holdings do not affect the equilibrium prices and allocations at $t + 1$, they do affect the investors’ expected utilities at $t + 1$. Therefore, we must account for the impact of the time-$t$ bond holdings on the investors’ expected utilities at $t + 1$ when computing the time-$t$ equilibrium stock prices and allocations. We temporarily ignore the bond holdings at $t - 1$ because they have no effect on the time-$t$ prices and allocations.

We incorporate the impact of the time-$t$ bond holdings, which are affected by trades made at $t$, on the investors’ expected utilities at $t + 1$ using the following procedure. At any given time-$t + 1$ node in the tree, a state-dependent (i.e., dependent on $S_t$ and $Q_t$) “quasi-expected utility” can be computed after determining the state-dependent equilibrium prices and allocations at that node. The quasi-expected utility at $t + 1$, which we denote by $U_{t+1}'$ for the taxable investor and by $\hat{U}_{t+1}'$ for the nontaxable investor, is not an investor’s actual expected utility because the investors’ time-$t$ bond holdings are undetermined when we compute the equilibrium at $t + 1$.\footnote{At $T - 1$, the time-$T$ state-dependent quasi-expected utility for the taxable investor is found by substituting the state-defining values of $S_{T-1}$ and $Q_{T-1}$ along with (3), (9), (10), $W_{T-1} = 0$, and the numerically-derived values of $S_T$ and either $A_T$ or $B_T$, depending on whether he buys or sells, into (8). The nontaxable investor’s time-$T$ state-dependent quasi-expected utility is derived in an analogous fashion. For $t < T$, the investors’ quasi-expected utilities are computed as an intermediate step in the algorithm used to derive the equilibrium prices and allocations, as described below.}

To convert an investor’s state-dependent quasi-expected utility into an “actual expected utility,” we first invert the investor’s state-dependent quasi-expected utility function.\footnote{We use the term “actual expected utility” to describe an investor’s time-$t+1$ utility after incorporating the time-$t$ bond holdings. Although the actual expected utility does not account for the investor’s bond holdings at $t - 1$, the prior bond holdings do not affect the investor’s stock allocation at $t$. However, we account for the investors’ initial bond endowments when computing their time-1 expected utilities, which we use to evaluate the effects of capital gains taxation on welfare in Section 4.} This inversion, which is viable due to the absence of wealth effects, generates a state-dependent certainty equivalent level of consumption at $t + 1$ for a given $\{S_t, Q_t\}$ pair. The taxable investor’s state-dependent certainty equivalent is given by

$$F_{t+1} = -\frac{1}{\delta} \log[-U_{t+1}']. \tag{18}$$
An analogous expression describes the nontaxable investor’s certainty equivalent, \( \hat{F}_{t+1} \).

Next, for each possible time-\( t \) allocation along with (conjectured) ask and bid prices, we compute the time-\( t \) tax basis with (2) and use the corresponding \( \{S_t, Q_t\} \) pair to approximate the taxable investor’s state-dependent certainty equivalent by bilinearly interpolating \( F_{t+1} \) over \( S_t \) and \( Q_t \). Approximation is necessary because computational limitations restrict the granularity of the grid over which the time-\( t + 1 \) equilibrium is computed. Other researchers who rely on interpolation techniques include, for example, Dammon, Spatt, and Zhang (2001), Dammon, Spatt, and Zhang (2004), and Gallmeyer, Kaniel, and Tompaidis (2006). Then, we compute the taxable investor’s time-\( t \) bond holdings with (3) and (15) using the (conjectured) ask and bid prices along with the time-\( t \) stock allocation. We add the investor’s bond holdings to his certainty equivalent, which yields an expression for his actual state-dependent expected utility at \( t + 1 \),

\[
U_{t+1} = -\exp[-\delta(F'_{t+1} + W_t(1 + r)^{T-t-1})],
\]

where \( F'_{t+1} \) denotes the certainty equivalent approximated by bilinearly interpolating over \( S_t \) and \( Q_t \). The nontaxable investor’s actual time-\( t + 1 \) expected utility, \( \hat{U}_{t+1} \), is derived in a similar fashion.

Figure 2 illustrates the solution method at \( T - 1 \), but we use the same process to derive the equilibrium at all \( t < T \). We compute the time-\( t < T \) prices and allocations over a grid of \( S_{t-1} \) and \( Q_{t-1} \) using an iterative algorithm similar to the one used at \( T \). First, we conjecture ask and bid prices that satisfy equilibrium condition (iii). Next, we compute the investors’ actual expected utilities at \( t + 1 \) as described above over a grid of possible time-\( t \) allocations, taking into account the effects of the (conjectured) prices and allocations on the taxable investor’s tax liability and basis. The taxable investor’s time-\( t \) allocation and corresponding tax basis determine the elements of the \( F_{t+1} \) and \( \hat{F}_{t+1} \) grids over which to interpolate to approximate the investors’ expected utilities. Again, we temporarily set \( W_{t-1} \) and \( \hat{W}_{t-1} \) equal to zero to reduce the dimensionality. We then select the utility-maximizing allocation for each investor (equilibrium condition (i)) and update the conjectured ask and bid prices with a
bisection method until the spread is minimized and the market clears (equilibrium conditions (ii) and (iv)). Once we determine the time-\(t\) equilibrium prices and allocations, we compute the investors’ expected utilities at \(t\). These expected utilities serve as the investors’ time-\(t\) quasi-expected utilities when computing the equilibrium prices and allocations at \(t - 1\).

### 3.3 Numerical Analysis

Table 1 lists the parameter values for the numerical analysis. We consider four different parameterizations to confirm the robustness of our results and to evaluate how interest rates, market conditions, and risk sharing influence the effects of capital gains taxation on equilibrium outcomes. Within each parameterization, we consider tax rates ranging from 0% to 90%. The current U.S. tax rate on long-term capital gains varies between 0% and 23.8%, depending on a household’s income level. Historically, the tax rate has been as high as 40%.

Many parameters are constant across all four parameterizations. The time horizon, \(T\), is 10. This horizon provides an ample number of periods for us to study equilibrium dynamics while maintaining computational tractability. The \(t\)-th component of the stock payoff, \(X_t\), takes a value of zero in the low state (i.e., \(L = 0\)) and a value of \(1/T\) in the high state (i.e., \(H = 0.1\)). Without loss of generality, the investors’ bond endowments, \(W_0\) and \(\hat{W}_0\), are zero.

In our baseline parameterization #1, we set the interest rate, \(r\), at 5%. For simplicity, we set the probability of the high state occurring equal to the probability of the low state occurring at each node, so \(\pi = \frac{1}{2}\). The investors’ risk aversion coefficients, \(\delta\) and \(\hat{\delta}\), are set to 5. The remaining parameterizations (#2–#4) consider comparative statics of these parameters. In parameterization #2, the interest rate is 7.5%, which enables us to study the effect of interest rates on capital gains taxation. In parameterization #3, the probability of the low state occurring at each node is \(\frac{2}{3}\), which allows us to analyze the impact of capital gains taxation under different economic growth scenarios. Finally, in parameterization #4, the taxable investor’s risk aversion coefficient is increased to 10, which permits us to evaluate the impact of capital gains taxation in different risk sharing environments.
At each time-$t$ node in the tree, we solve for an equilibrium over a grid of $S_{t-1}$ and $Q_{t-1}$. Each dimension of the grid ranges from 0 to 1, with a coarseness of 0.01. Overall, we compute an equilibrium over a grid of 10,201 ($101^2$) states at 1,023 ($2^7 - 1$) nodes in the binomial payoff tree, for a total of 10,435,623 possible equilibria ($10,201 \times 1,023$). The coarseness of the time-$t$ allocation grid, $\frac{1}{N_S}$, is 0.0001.

4 Results

To evaluate the effects of capital gains taxation, we compare equilibrium outcomes with taxation to equilibrium outcomes without taxation. Results are presented for the four parameterizations summarized in Table 1.

4.1 Allocations

Capital gains taxation alters the taxable investor’s payoff from owning the stock in two distinct ways. First, the tax lowers his average payoff because the stock price tends to rise over time (due to both the positive interest rate and the investors’ risk aversion) and the taxing authority confiscates a fraction of his gains. Second, it lowers the variance of his payoff because, in addition to appropriating a portion of his gains, the taxing authority also rebates a fraction of his losses. Because the taxable investor is risk averse, the reduction in both the mean and volatility of the payoff may either raise or lower his demand for the stock, depending on which effect dominates. We find that capital gains taxation results in a smaller average equilibrium stock allocation for the taxable investor.

Figure 3(a), which depicts the difference between the taxable investor’s average stock allocation for a given tax rate and his average allocation without taxation (i.e., $\theta = 0$) across time for the baseline parameterization #1, shows that the taxable investor tends to hold less stock when he is subject to taxation. Moreover, the taxable investor’s average equilibrium allocation is monotonically decreasing in the tax rate, and he holds no stock when the tax
rate is sufficiently high (i.e., $\theta > 0.5$).

Taxation’s effect on the equilibrium allocations has important risk-sharing implications. Because the nontaxable investor holds a greater quantity of stock in equilibrium while the taxable investor holds less, taxation shifts (at least some of) the economic risk exposure from the taxable investor to the nontaxable investor. This outcome is not unique to capital gains taxation, as tax clientele effects arise in other contexts, e.g., capital structure (Miller, 1977; Zechner, 1990), portfolio choice (Elton and Gruber, 1978; Dybvig and Ross, 1986; Desai and Jin, 2011; Sialm and Starks, 2012), yield curves (Green, 1993), and payout distribution policies (Allen, Bernardo, and Welch, 2000; 2003, 2003; Green and Hollifield, 2003).

The effect of taxation on allocations is robust to the other parameterizations, as shown by Figure 3(b), which plots the mean (over time) of the difference between the taxable investor’s average stock allocation for a given tax rate and his average allocation without taxation. Comparing the different parameterizations indicates that taxation has a greater impact on allocations when the interest rate $r$ is higher (parameterization #1 vs. #2), there is a smaller probability $\pi$ of a low stock payoff (parameterization #1 vs. #3), or the taxable investor’s risk aversion coefficient $\delta$ is smaller (parameterization #1 vs. #4). A higher interest rate, ceteris paribus, gives rise to a lower initial stock price and larger price increases over time simply due to time discounting. Consequently, larger capital gains occur and, hence, greater tax liability arises when $r$ is higher, which compels the taxable investor to hold less stock. Similarly, a smaller probability of a low stock payoff means that the price is more likely to increase over time, resulting in greater tax liability. Therefore, the taxable investor holds less stock when $\pi$ is smaller. Finally, in the absence of taxation (i.e., $\theta = 0$), the taxable investor holds less stock in equilibrium when he is more risk averse than the nontaxable investor.$^{10}$ Because a larger allocation results in greater tax liability, taxation has a greater effect on allocations

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$^{10}$It is straightforward to show that the taxable and nontaxable investors hold equal quantities of stock $(S_t = \hat{S}_t = \frac{1}{2})$ in the absence of taxation if they are equally risk averse but that the taxable investor holds less stock $(S_t = \frac{1}{2} \text{ and } \hat{S}_t = \frac{2}{3})$ when $\delta = 10$ and $\delta = 5$) in the absence of taxation if he is more risk averse than the nontaxable investor. This explains why the impact of taxation on allocations in the figure plateaus at $-\frac{1}{2}$ for parameterizations #1–#3 but at $-\frac{3}{4}$ for parameterization #4.
when the taxable investor is less risk averse and would otherwise hold more stock.

Interestingly, the investors rebalance their portfolios relatively frequently when capital gains are taxed. In contrast, without a tax on capital gains, the investors hold a constant amount of stock in all periods, which is a standard outcome for investors with CARA preferences. Hence, taxation induces all trading in the model. Moreover, trading volume tends to be higher when the taxable investor has an embedded loss, consistent with empirical evidence documented by Dyl (1977). This result is shown in Table 2, which reports the effect of taxation on the difference between average volume when the taxable investor has an embedded loss and when he has an embedded gain. Trading volume tends to be higher when the taxable investor has an embedded loss because he receives a rebate when selling at a loss, which encourages him to trade, but must pay a tax when selling at a gain, which discourages him from trading.\textsuperscript{11} This result is also consistent with existing theory (e.g., Constantinides, 1983; Dammon, Spatt, and Zhang, 2001) and empirical evidence (e.g., Feldstein and Yitzhaki, 1978; Feldstein, Slemrod, and Yitzhaki, 1980; Reese, 1998; Blouin, Raedy, and Shackelford, 2003; Jin, 2006; Shan, 2011) that investors with embedded gains are reluctant to rebalance their portfolios and, thus, locked into their positions.

4.2 Prices

Existing theories predict that capital gains taxation may either increase or decrease asset prices. On the one hand, an investor subject to taxation should demand a lower price to purchase an asset because taxes paid on future realized gains will decrease his return. This is known as the capitalization effect. On the other hand, an investor should demand a higher price to sell an asset with an embedded gain because the tax liability from realizing the gain lowers the effective rate of return on the sale proceeds reinvested in another asset. This is known as the lock-in effect. Depending on which effect dominates, a capital gains tax could

\textsuperscript{11}This effect on trading volume should be robust to an alternative environment in which the taxable investor does not receive a rebate for realized capital losses because it would still be costlier for him to sell at a gain than at a loss.
conceivably either raise or lower prices. We find that imposing a tax on capital gains results in lower prices on average.

We define the time-\(t\) equilibrium price as the midpoint of the bid-ask spread,\(^{12}\)

\[
P_t \equiv \frac{1}{2}(B_t + A_t).
\]  

(20)

Capital gains taxation has a substantial effect on equilibrium prices, as shown in Figure 4(a), which plots the difference between the average stock price for a given tax rate and the average price without taxation for parameterization #1. The figure indicates that average prices are lower when capital gains are subject to taxation, meaning that the capitalization effect dominates the lock-in effect. Additionally, taxation tends to reduce the price to a greater extent when the tax rate is higher. Prices tend to fall when a tax is imposed because taxation lowers the taxable investor’s demand for the stock but does not directly affect the nontaxable investor’s demand. Because taxation lowers aggregate demand, \textit{ceteris paribus}, the average price must fall for the market to clear. Moreover, the average price is decreasing in the tax rate even when the taxable investor holds no stock (e.g., for \(\theta > 0.5\) in the baseline parameterization) simply because he is the marginal buyer in those cases and is willing to pay a lower price for the stock when the tax rate is higher.\(^ {13}\)

Taxation has a similar effect on the stock price in the other parameterizations. Figure 4(b) plots the mean (over time) of the difference between the average stock price for a given tax rate and the average price without taxation. Like its effect on equilibrium allocations, taxation has a stronger effect on the equilibrium stock price when the interest rate is higher, there is a greater probability of a high stock payoff, or the taxable investor is less risk averse for the same reasons as discussed in Section 4.1. Furthermore, our finding that capital gains

\(^{12}\)At \(t = 1\) when the stock is issued, the price equals the ask (i.e., \(P_1 = A_1\)) because neither investor sells the stock.

\(^{13}\)For higher tax rates where the taxable investor holds no stock, taxation has a lesser effect on the price at \(t = 1\) than at subsequent dates. The reason is that the nontaxable investor, who is willing to pay more for the stock than the nontaxable investor, is the marginal buyer at \(t = 1\) because she purchases shares at the initial offering.
taxation results in a lower price on average is consistent with much empirical evidence (see, e.g., Guenther and Willenborg, 1999; Lang and Shackelford, 2000; Dai et al., 2008; Blouin, Hail, and Yetman, 2009; but cf. Landsman and Shackelford, 1995; Klein, 2001; Ayers, Lefanowicz, and Robinson, 2003; George and Hwang, 2007).

The drop in prices has important implications for firms’ cost of capital. Because the firm receives fewer proceeds from issuing equity when capital gains are taxed, its cost of capital increases, which consistent is empirical evidence (e.g., Dhaliwal, Krull, and Moser, 2005; Dhaliwal, Krull, and Li, 2007).

Although taxation results in lower average prices, the average ex ante pretax expected return rises under a capital gains tax, as Figure 5(a) shows. The expected return rises for reasons analogous to why prices fall: the taxable investor commands a greater pretax return because taxation lowers his net return on average, and the nontaxable investor commands a greater risk premium because his risk exposure rises as his allocation increases. Thus, like the effect on prices, the magnitude of the effect of taxation on expected returns and Sharpe ratios is greater whenever the interest rate is higher, there is a smaller probability of a low stock payoff, or the taxable investor is less risk averse. This result is consistent with empirical evidence provided by Hail, Sikes, and Wang (2017), who document that returns rise with capital gains tax rates but that the effect is attenuated when interest rates are low or the risk premium is high. Furthermore, we find that taxation has only a minor effect on volatility, as depicted by Figure 5(b), which contrasts with empirical evidence that return volatility decreases with capital gains tax rates (Dai, Shackelford, and Zhang, 2013).

In addition to affecting average prices and returns, capital gains taxation tends to increase bid-ask spreads. We measure the time-\(t\) spread as a percentage of the price, \(\frac{A_t - B_t}{P_t}\). Table 2 reports the effect of a capital gains tax on the difference between the average spread when the investor has an embedded loss and the average spread when he has an embedded gain.

\[14\] We compute the ex ante expected return at each node in the binomial tree as the weighted mean of the two possible returns from \(t\) to \(t+1\) \((P_{t+1}/P_t - 1)\). We then take the weighted mean across all nodes to compute the average expected return. We compute the average ex ante stock return volatility in an analogous way.

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While taxation may raise spreads when the taxable investor has either an embedded gain or an embedded loss, the table indicates that taxation increases the average spread to a greater extent when he has an embedded loss. The reason is that the tax rebate received from realizing a loss alters the taxable investor’s marginal rate of substitution between the stock and bond and, therefore, induces him to sell the stock at a lower price than he otherwise would if he did not receive a rebate, which, *ceteris paribus*, results in a lower bid. Because the tax rebate does not directly affect the nontaxable investor’s marginal rate of substitution when buying the stock, the ask is not substantially affected by the presence of an embedded loss. Thus, the spread is wider on average when the taxable investor has an embedded loss.

### 4.3 Tax Revenue

The equilibrium price and allocation dynamics determine the amount of tax revenue generated by the capital gains tax. Figure 6, which plots the ex ante expected value (discounted at the risk-free rate) of tax revenue over a range of tax rates for the different parameterizations, indicates that the value of the tax revenue follows a Laffer curve. That is, the value of tax revenue is increasing in the tax rate for lower tax rates but is decreasing in the tax rate for higher tax rates. When the tax rate is low, raising the rate increases revenue because the taxing authority appropriates a larger percentage of realized gains. Conversely, when the tax rate is high, further raising the rate decreases revenue because the taxable investor partially withdraws from the market. If the tax rate is sufficiently high, then the capital gains tax generates no tax revenue because the taxable investor never trades the stock.

Although the value of tax revenue follows a Laffer curve in all of the parameterizations, the underlying parameters quantitatively affect the value of tax revenue generated by the capital gains tax. In general, more tax revenue is generated when the taxable investor holds a greater quantity of stock. This occurs when he is less risk averse, the interest rate is lower, or there is a higher probability of a low stock payoff. Interestingly, the revenue-maximizing tax rate also varies across the different parameterizations. The revenue-maximizing rate is higher when the
interest rate is lower or there is a higher probability of a low stock payoff because taxation reduces the taxable investor’s stock allocation to a lesser extent in these cases.

4.4 Welfare

The distortions to the equilibrium prices and allocations ultimately affect the investors’ welfare. We measure welfare by computing a certainty equivalent of capital gains taxation. The certainty equivalent is the amount of initial wealth that must be endowed to an investor in the absence of taxation for that investor to be indifferent between an economy with a capital gains tax and an economy without such a tax. A positive (negative) certainty equivalent indicates that an investor is better (worse) off when capital gains are taxed. Although we ignore the potential redistribution of tax revenue when computing the investors’ certainty equivalents, the qualitative effects on the individuals’ welfare should be robust to a large set of redistribution policies, as discussed below.

Figures 7(a) and 7(b) plot the taxable and nontaxable investors’ respective certainty equivalents. The figures show that capital gains taxation lowers the taxable investor’s welfare but raises the nontaxable investor’s welfare. The taxable investor experiences a loss of welfare because the tax lowers his average payoff and distorts his equilibrium allocation, resulting in a suboptimal portfolio (relative to his portfolio without taxation). Conversely, the nontaxable investor experience a rise in welfare even though she holds a suboptimal portfolio because she tends to buy more stock at a lower price, which increases her returns. Taxation has a greater effect on the investors’ welfare when the interest rate is high, the probability of a low stock payoff is high, or the taxable investor is less risk averse because the tax distorts prices and allocations to a greater extent in these cases.

These results highlight the fact that taxation’s distortions to the equilibrium prices and allocations are important channels through which a capital gains tax affects welfare. To wit, simply redistributing all of the tax revenue back to the taxable investor does not offset the effects of taxation. In particular, the highest tax rates cause the largest distortions and have
the biggest effects on welfare, but they produce no tax revenue. Hence, the decrease in the taxable investor’s welfare is not purely due to the direct effect of an incurred tax liability on his consumption. Furthermore, the nontaxable investor’s welfare rises even though she is not directly affected by the tax.

As stated above, we do not account for any potential redistribution of tax revenue when evaluating the effect of taxation on the individual investors’ welfare. Nevertheless, the qualitative results should be robust to any redistribution policy that involves a non-negligible amount of revenue not being redistributed to the taxable investor because, as Figures 6 and 7 show, the nontaxable investor’s certainty equivalent is positive and the magnitude of the taxable investor’s negative certainty equivalent is either greater than or similar to the amount of tax revenue. Although we cannot draw any definitive conclusions about the effect of capital gains taxation on the individual investors’ welfare without making additional assumptions regarding tax revenue redistribution policies, we can assess taxation’s effect on social welfare because social welfare does not depend on how the tax revenue is distributed, provided, of course, that the revenue redistribution policy does not depend on realized states of nature.

We measure the effect of capital gains taxation on social welfare in the following way. We first aggregate the taxable and nontaxable investors’ certainty equivalents with the expected present value of tax revenue. Including tax revenue in the social welfare calculation implicitly assumes that the (present value of) revenue is costlessly redistributed within the economy. We then add the difference between the initial stock price with taxation and the initial price without taxation. Incorporating the difference in the price accounts for the different amounts of capital that would accrue to the (unmodeled) firm under the different tax regimes (recall, the stock undergoes an initial offering at $t = 1$, and the proceeds from the stock issuance must

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15 The tax revenue redistribution policy is beyond the scope of this article because such an analysis would require many additional assumptions (e.g., revenue redistribution allocations and costs related to collection and administration). Furthermore, although the redistribution policy itself could potentially affect taxation’s impact on welfare (by altering the equilibrium prices and allocations), this would be unlikely to have a significant effect if there were many investors because each investor would reap only a tiny fraction of the redistributed tax revenue, so the potential redistribution of revenue would exert only a minor influence over the investors’ portfolio decisions.
Figure 7(c), which depicts the effect of taxation on social welfare, shows that, perhaps surprisingly, welfare rises for low tax rates but falls for high tax rates. This result stems from taxation’s effect on risk sharing and the cost of capital. For low tax rates, a capital gains tax improves risk sharing because the taxing authority bears some of the risk that would otherwise be borne by the investors in the absence of taxation, as shown by Domar and Musgrave (1944). Moreover, taxation distorts the initial stock price to a lesser extent when the tax rate is low, leading to only a small increase in the firm’s cost of capital. Conversely, for high tax rates, a capital gains tax reduces risk sharing because the nontaxable investor holds all of the stock and, therefore, neither the taxable investor nor the taxing authority bear any risk. The distortion to the initial stock price is also greater when the tax rate is higher, resulting in a larger increase in the firm’s cost of capital.

4.5 Asset Pricing Implications

In this section, we discuss potentially important implications for empirical asset pricing. As discussed above in Section 4.2, taxation tends to raise expected returns. However, the effects depicted in Figure 5 are computed from an ex ante perspective; that is, they are based on the actual — or forward-looking — return distributions. Although using forward-looking distributions to compute moments of returns accurately captures the risk-return trade-off faced by investors, empirical asset pricing studies must rely on historical realized returns because actual forward-looking return distributions are not observable in practice. Therefore, we examine the effects of taxation on the realized time-series of returns to evaluate the potential impact of capital gains taxation on empirical asset pricing.

Fundamental asset pricing theory implies

\[ 1 = E[mR] = E[m]E[R] + \rho_{m,R}\sigma_R\sigma_m, \]

(21)
where \( m \) denotes the SDF. When the moments of the one-period returns and SDF are computed from an ex ante perspective at each node in the payoff tree, the SDF and returns are perfectly negatively correlated, \( \rho_{m,R} = -1 \), at every node. This result is consistent with classical asset pricing theory. Interestingly, however, when the moments of returns and the SDF are computed using realized time-series values, as an econometrician would observe in practice, the correlation between the SDF and returns attenuates, as shown by Figure 8(a), which plots the average (across all time-series paths) correlation between the SDF and returns.

Two distinct effects drive the correlation attenuation, giving rise to two different regions in Figure 8(a). For relatively low tax rates (i.e., \( \theta < 0.5 \)), taxation directly affects the correlation by altering the marginal rate of substitution between assets and, hence, by altering the state-price density. For higher tax rates, taxation indirectly affects the correlation by completely eliminating risk sharing, as the nontaxable investors holds all of the stock in equilibrium. As the figure indicates, the magnitude of the latter effect is much larger than the magnitude of the former. Nonetheless, the direct effect of taxation appears to meaningfully attenuate the correlation between the SDF and returns. Focusing on the leftward side of Figure 8(a) where the tax rate is relatively low, the correlation in the baseline parameterization #1, for example, attenuates from \(-0.99\) to \(-0.82\) as the tax rate increases from \( \theta = 0 \) to \( \theta = 0.55 \).

The attenuation of the correlation as the tax rate increases is primarily due to the increase in the volatility of the SDF (depicted in Figure 8(b)) rather than to changes in either the standard deviation of returns (depicted in Figure 8(c)) or the covariance of the SDF with returns (depicted in Figure 8(d)), both of which are relatively constant across tax rates. We note two implications for asset pricing studies. First, capital gains taxation impacts both the perceived quantity of risk and the price of risk. Figure 8(e) plots the quantity of risk, \( \beta_{m,R} \), and Figure 8(f) plots the price of risk, \( \lambda_m \), from the following rearrangement of (21) into a beta pricing model, using the fact that \( \text{E}[m] = \frac{1}{r} \):

\[
\text{E}[R] = r - \frac{\sigma_{m,R}}{\sigma_m^2} \frac{\text{E}[m]}{\text{E}[m]} = r + \beta_{m,R} \lambda_m.  
\] (22)
Because the attenuation of the correlation between returns and the SDF is driven by an increase in the volatility of the SDF, the quantity (price) of risk monotonically decreases (increases) as the tax rate increases. Consequently, average returns are non-linearly related to capital gains tax rates, and the magnitude of the impact of a tax rate change on average returns depends on the relation between taxation’s effect on the quantity of risk and its effect on the price of risk.

The second implication is related to the correlation puzzle noted by Cochrane and Hansen (1992), empirically explored by Otrok, Ravikumar, and Whiteman (2002), and explicitly modeled by Albuquerque et al. (2016). In representative agent models, an asset’s expected return is determined by the covariance of the agent’s SDF and the asset payoff. The correlation puzzle arises because measurable fundamentals such as consumption growth and returns have low correlation. Even though the endogenous impact of capital gains taxation on returns does not affect the covariance of the SDF and payoffs, the increase in the volatility of the SDF distorts the correlation between the SDF and returns, and this distortion directly affects measures of model fit such as the well-known Hansen and Jagannathan (1991) bound,

$$\sigma_m \geq E[m] \left| \frac{E[R] - r}{\sigma_R} \right|. \quad (23)$$

Figure 9, which plots the bounds when $\theta = 0$ and $\theta = 0.55$ for the baseline parameterization #1, shows that a higher capital gains tax rate results in a tighter bound for low risk-free rates. Perhaps surprisingly, the bounds, which are derived from ex post returns, increase at a slower rate than the variance of the average equilibrium SDF $m^*$, denoted in the figure by $\ast$ for $\theta = 0$ and by $\triangle$ for $\theta = 0.55$. The deviation between $m^*$ and the bound at higher tax rates is due to the attenuation in the correlation between the SDF and returns, which in turn arises primarily from the increase in the volatility of the SDF rather than through a change in the covariance, as discussed above. This suggests that non-risk distortions in conjunction with models that account for demand shocks (e.g., Albuquerque et al., 2016) may help explain the low empirical correlations between returns and asset pricing fundamentals.
5 Extensions

We consider two extensions of the model. In Section 5.1, we evaluate how an unexpected change in the tax rate may affect tax revenue. In Section 5.2, we analyze the taxable investor’s tax-timing option.

5.1 Tax Rate Change

The analysis in Section 4 is based on tax rates remaining constant over time. In reality, however, tax rates tend to fluctuate. The U.S. capital gains tax rate, for example, has been altered at least 20 times since it was first instituted in 1913, with the most recent change occurring in 2013. In this section, we examine how (unexpected) changes in the tax rate affect revenue. Our analysis here differs from the comparative static analysis reported in Section 4.3, which demonstrates how tax revenue varies across different constant tax rates.

We consider a single tax rate change (from an “initial” rate to an “amended” rate) that occurs halfway through the life of the stock ($t = 5$). Due to computational constraints, we assume that the change is unanticipated.\footnote{The unexpected nature of the tax rate change means that investors do not anticipate its occurrence. Permitting investors to anticipate tax rate shocks by assigning a nonzero probability to a tax rate change would increase the dimensionality of the numerical analysis and is, therefore, not computationally feasible. Nonetheless, the results presented in this section generally should be qualitatively robust to an alternative environment in which investors anticipate the possibility of a tax rate change, provided that the probability of a rate change is sufficiently small. Although investors undoubtedly factor potential near-term tax rate changes into their portfolio decisions in practice, forecasting tax rate changes that might occur in the distant future is likely to exert only a small influence over investors’ portfolio decisions.} Figure 10 depicts the average present value of tax revenue for various combinations of initial and amended tax rates for parameterization #1 (the other parameterizations generate similar results, but we do not report them for brevity). The darker-shaded diagonal represents an economy with a constant tax rate, and the revenue along this diagonal is identical to that reported in Figure 6. Points to the right of the diagonal correspond to a rate increase, whereas points to the left correspond to a rate decrease.

Notably, a dynamic tax policy can generate greater tax revenue than a static one. For parameterization #1, the constant tax rate that maximizes revenue is 25%. In contrast, the...
dynamic tax policy (limited to a single rate change) that maximizes revenue consists of an initial tax rate of 0% followed by a rate increase to 35%. This dynamic policy increases tax revenue by roughly 35% relative to the maximum expected revenue generated under a static policy.\textsuperscript{17} The reason is as follows. A low initial tax rate incentivizes the taxable investor to hold more stock than he otherwise would if the tax rate were higher (see Figure 3) because the tax reduces his expected payoff. The larger allocation then results in greater tax liability on average after the tax rate is raised. Raising the tax rate too high, however, can reduce tax revenue. If the tax rate is sufficiently high, then the taxable investors holds no stock in equilibrium. Consequently, the stock price drops (see Figure 4) to compensate the nontaxable investor for bearing all of the risk. The lower stock price tends to generate a capital loss for the taxable investor, which results in a tax rebate and, hence, lower tax revenue.

### 5.2 Tax-Timing Option

With a realization-based capital gains tax (like in the economy described in Section 2), the taxable investor incurs a tax consequence only when he sells the stock. This creates a tax-timing option for the investor whereby he may delay a sale to avoid paying a tax when he has an embedded gain but accelerate a sale to receive a rebate when he has an embedded loss. Here, we analyze the value of the investor’s tax-timing option and its effect on tax revenue.

To determine the value of the tax-timing option, we consider an alternative economy in which tax consequences are recognized as gains and losses are accrued, not just when they are realized. Under an accrual-based capital gains tax, gains are taxed and losses are rebated regardless of whether the taxable investor realizes his gain or loss by selling the stock. We make two key modifications to the realization-based tax model described in Section 2. First, unlike with a realization-based tax, a single price clears the market with an accrual-based tax\textsuperscript{17}Interestingly, a dynamic policy that involves a rate decrease results in (weakly) less tax revenue than simply adopting the lower rate initially. The dynamic policy that generates an amount of revenue most similar to the maximum amount generated by a static policy is a rate increase from 10% to 20%. Because the model is not calibrated, the quantitative results should not be taken literally. Nevertheless, the results appear to be qualitatively significant and, as stated above, are qualitatively similar across the various parameterizations.
because transactions do not trigger a tax consequence with an accrual-based tax; the ask and
bid endogenously converge to a single time-\(t\) price, \(P^a_t\), where a superscript “\(a\)” denotes the
accrual tax regime and distinguishes relevant variables from the setting with a realization-
based tax. Second, the time-\(t\) tax liability described in (3) reduces to

\[
L^a_t = S^a_{t-1}(P^a_t - P^a_{t-1})\theta. (24)
\]

The remaining assumptions are unaltered. The solution method with an accrual-based tax is
analogous to the one described in Section 2, except that the time-\(t\) equilibrium price and allo-
cations are not computed over a grid of \(S^a_{t-1}\) and \(Q^a_{t-1}\) because the taxable investor’s marginal
rate of substitution between the stock and bond does not depend on either the direction of
his trade (due to the accrual-based nature of the tax) or his basis (which, effectively, equals
the prior-period price) like it does with a realization-based tax.

We measure the value of the tax-timing option to the investors as their certainty equivalents
of a realization-based tax relative to an accrual-based tax, holding the tax rate fixed.\(^{18}\) Figure
11(a) shows that the tax-timing option value is positive for the taxable investor, whereas Fig-
ure 11(b) shows that the option value is generally negative for the nontaxable investor. For
both investors, the tax-timing option value is non-monotonic in the tax rate. The value to the
taxable investor increases with the tax rate for low rates because there is a bigger benefit from
deferring gains when they are taxed at a higher rate. Once the tax rate is sufficiently high,
however, the investor holds less stock, which lowers the tax-timing option value. Similarly,
the value to the nontaxable investor decreases with the tax rate for low rates because the
greater benefit to the taxable investor induces him to hold more stock in equilibrium, thereby
increasing the price and lowering the nontaxable investor’s return. For higher tax rates, the
magnitude of the tax-timing option to the nontaxable investor is smaller because the option

\(^{18}\)Similar to the effects of capital gains taxation on the individual investors’ welfare discussed in Section 4.4,
we do not account for any potential redistribution of tax revenue when determining each investor’s value of
the tax-timing option. Moreover, the social value of the tax-timing option, discussed below, is independent of
the redistribution policy.
has less of an effect on the taxable investor’s demand. The magnitude of the tax-timing option value is greater when either the interest rate is higher, there is a greater probability of a high stock payoff, or the taxable investor is less risk averse because, ceteris paribus, greater tax liability is more likely to arise in these cases, as discussed above.

Figure 11(c) depicts the effect of the tax-timing option on tax revenue. The figure, which plots the difference between the average present value of tax revenue under a realization-based tax and the average present value under an accrual-based tax, shows that, perhaps surprisingly, the tax-timing option may either increase or decrease tax revenue. For low tax rates, the timing option decreases tax revenue because the taxable investor’s equilibrium stock allocation is comparable under the two tax regimes, but the timing option allows him to reduce his tax liability. For higher tax rates, the timing option increases tax revenue because the ability to defer gains induces the taxable investor to hold more stock, which creates more revenue when he eventually unwinds his position. Similar to the option’s value to the taxable investor, the tax-timing option tends to have a greater effect on tax revenue when \( r \) is higher, \( \pi \) is smaller, or \( \delta \) is lower.

Figure 11(d) portrays the tax-timing option’s effect on social welfare, which we measure analogously as described in Section 4.4. The figure indicates that the tax-timing option tends to result in greater social welfare than requiring the taxable investor to recognize gains and losses as they accrue. Relative to an accrual-based tax, a realization-based tax increases the taxable investor’s demand for the stock, which gives rise to better risk sharing and, hence, results in greater social welfare.

6 Conclusion

We study a multi-period partial equilibrium model with capital gains taxation. We find that a capital gains tax tends to lower prices but raise expected returns. This results largely from a clientele effect whereby the taxable (nontaxable) investor tends to hold less (more)
stock in equilibrium under a capital gains tax, which diminishes risk sharing among the two traders. However, because the taxing authority shares some of the risk when capital gains are subject to taxation, social welfare may rise. Furthermore, we find that the taxable investor’s tax-timing option may increase the present value of tax revenue for high tax rates even though the timing option allows him to defer his tax liability.

Our analysis presents several opportunities for future research. Our finding that capital gains taxation attenuates the time-series correlation between the SDF and returns has potentially important but unexplored implications for empirical asset pricing. Additionally, to accurately reflect current tax regulations, our model prohibits wash sales and short selling against the box. An extension of our model that allowed investors to engage in such activities would isolate the effects of these regulations on equilibrium outcomes.

References


Table 1: Parameter values.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Symbol</th>
<th>#1</th>
<th>#2</th>
<th>#3</th>
<th>#4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time horizon</td>
<td>$T$</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>Payoff in low state</td>
<td>$L$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Payoff in high state</td>
<td>$H$</td>
<td>0.1</td>
<td>0.1</td>
<td>0.1</td>
<td>0.1</td>
</tr>
<tr>
<td>Interest rate</td>
<td>$r$</td>
<td>0.05</td>
<td>0.075</td>
<td>0.05</td>
<td>0.05</td>
</tr>
<tr>
<td>Probability of low state</td>
<td>$\pi$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{2}{3}$</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>Taxable investor's risk aversion</td>
<td>$\delta$</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>10</td>
</tr>
<tr>
<td>Nontaxable investor's risk aversion</td>
<td>$\hat{\delta}$</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>Taxable investor's bond endowment</td>
<td>$W_0$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Nontaxable investor's bond endowment</td>
<td>$\hat{W}_0$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
Table 2: Volume and spread. The difference in the difference between the average trading volume or bid-ask spread when the taxable investor has an embedded loss and when he has an embedded gain for nonzero tax rates and the difference between the average trading volume or bid-ask spread when the taxable investor has an embedded loss and when he has an embedded gain without taxation is reported.

<table>
<thead>
<tr>
<th>Parameterization</th>
<th>#1 (baseline)</th>
<th>#2 (r lower)</th>
<th>#3 (π greater)</th>
<th>#4 (δ higher)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Volume</td>
<td>0.040</td>
<td>0.050</td>
<td>0.043</td>
<td>0.059</td>
</tr>
<tr>
<td>Spread</td>
<td>0.016</td>
<td>0.010</td>
<td>0.018</td>
<td>0.008</td>
</tr>
</tbody>
</table>
Figure 1: Equilibrium computation at $T$. At each time-$T$ node in the payoff tree, equilibrium prices and allocations are computed over a grid of potential values for the taxable investor’s time-$T-1$ allocation and basis.
Figure 2: Equilibrium computation at \( t < T \). At each time-\( t < T \) node in the payoff tree, equilibrium prices and allocations are computed over a grid of potential values for the taxable investor’s time-\( t-1 \) allocation and basis, taking into account the effect of the potential time-\( t-1 \) allocation and basis and the time-\( t \) equilibrium prices and allocations on the investors’ expected utility at \( t+1 \).
Figure 3: Taxable investor’s stock allocation. For various tax rates, $\theta$, (a) plots the difference between the taxable investor’s average stock allocation for a given tax rate and his average allocation without taxation at each date $t$ for the baseline parameterization #1 and (b) plots the difference between the taxable investor’s average stock allocation for a given tax rate and his average allocation without taxation over time for multiple parameterizations.
Figure 4: Stock price. For various tax rates, $\theta$, (a) plots the difference between the average stock price for a given tax rate and the average price without taxation at each date $t$ for the baseline parameterization #1 and (b) plots the difference between the average stock price for a given tax rate and the average price without taxation over time for multiple parameterizations.
Figure 5: Expected return and volatility. For various tax rates, $\theta$, and multiple parameterizations, (a) plots the difference between the average ex ante expected stock return for a given tax rate and the average ex ante expected stock return without taxation and (b) plots the difference between the average ex ante stock return volatility for a given tax rate and the average ex ante stock return volatility without taxation.
**Figure 6: Tax revenue.** For various tax rates, $\theta$, and multiple parameterizations, the average present value of tax revenue is plotted.
Figure 7: Welfare. For various tax rates, $\theta$, and multiple parameterizations, (a) and (b), respectively, plot the taxable and nontaxable investors’ certainty equivalents of capital gains taxation, whereas (c) plots the effect of capital gains taxation on social welfare.
Figure 8: Ex post statistics. For various tax rates, $\theta$, and multiple parameterizations, (a) plots the correlation between stock returns and the SDF, (b) plots the volatility of the SDF, (c) plots the volatility of returns, (d) plots the covariance between stock returns and the SDF, (e) plots the quantity of risk, $\beta_{m,R}$, and (f) plots the price of risk, $\lambda_m$. 
Figure 9: Hansen-Jagannathan bounds. The Hansen-Jagannathan bounds and the equilibrium SDF ($m^*$) are plotted for $\theta = 0$ and $\theta = 0.55$ in parameterization #1.
Figure 10: Tax rate change. The average present value of tax revenue is plotted for various combinations of initial and amended tax rates for the baseline parameterization #1. The darker shaded diagonal corresponds to a constant tax rate.
Figure 11: Tax-timing option value. For various tax rates, $\theta$, and multiple parameterizations, (a) plots the taxable investor’s certainty equivalent of the tax-timing option, (b) plots the nontaxable investor’s certainty equivalent of the tax-timing option, (c) plots the difference between the average present value of tax revenue with a realization-based tax versus an accrual-based tax, and (d) plots the effect of the tax-timing option on social welfare.
Appendix: Capital Gains Taxation Algorithms

Table A.1 provides a description of the variables used in the numerical analysis. A superscript “∗” on certain variables indicates a conjectured equilibrium value. Algorithms 1 through 8 describe the solution method. In general, a set of time-\(t\) state-dependent equilibrium prices, allocations, and expected utilities are derived over a grid of the taxable investor’s time-\(t−1\) allocation and tax basis at each time-\(t\) node in the payoff tree, as described in Algorithm 1. The time-\(t\) states are defined by the node in the payoff tree along with the time-\(t−1\) allocation and basis. Once all of the possible state-dependent prices and allocations are determined, realized equilibrium prices and allocations are computed for every possible path in the payoff tree, as described in Algorithm 8 below.

Algorithm 2 describes the nested bisection method that is used to determine the time-\(t\) ask and bid prices for a given state. First, lower and upper bounds on the time-\(t\) ask are assigned, and a value for the ask is conjectured as the midpoint of the two bounds. Next, lower and upper bounds for the time-\(t\) bid are assigned, with the upper bound equaling the conjectured ask. A value for the bid is then conjectured as the midpoint of the two bounds, and aggregate demand is computed with the conjectured ask and bid using Algorithm 3. If there is excess supply, then the conjectured bid replaces the upper bound on the bid; otherwise, the conjectured bid replaces the lower bound on the bid. The conjectured bid is updated (holding the conjectured ask fixed) until the bid bounds converge to one another. After the conjectured bid is finished updating, the conjectured ask is updated in an analogous fashion. If there is excess supply, the conjectured ask replaces the upper bound on the ask; otherwise, the conjectured ask replaces the lower bound on the ask. The conjectured ask is updated until the ask bounds converge to one another and the spread between the conjectured ask and bid equals zero (or the maximum number of iterations is reached), with the conjectured bid being updated within each iteration.

Algorithm 3 describes how the allocations are computed for a given state using the conjectured ask and bid as inputs. First, the investors’ time-\(t\) expected utilities are computed over
a grid of possible time-\( t \) stock allocations with Algorithms 4 through 7. Then, the investors respective stock demands are selected as the allocations that generate the highest expected utilities.

The particular algorithms used to compute the investors’ expected utilities depend on \( t \). Algorithms 4 and 5 are used to calculate the taxable and nontaxable investors’ respective expected utilities at \( T \). The taxable investor’s time-\( T \) expected utility is determined by computing his consumption at \( T + 1 \) with (1), (2), (3), (9), and (10), taking the time-\( T \) prices and stock allocation and the time-\( T - 1 \) basis and allocation as given. Similarly, the nontaxable investor’s time-\( T \) expected utility is determined by computing her consumption at \( T + 1 \) with (1), (12), and (13).

Algorithms 6 and 7 describe the computation of the taxable and nontaxable investors’ respective expected utilities at \( t < T \). As described in the text, the time-\( t \) tax basis and liability are updated based on the price and allocation at \( t \) and the basis and allocation at \( t - 1 \), which are all taken as given. Then, the time-\( t \) bond holdings for each investor are computed with (15) and (17), temporarily assuming that the time-\( t - 1 \) bond holdings are zero. Next, the investors’ time-\( t + 1 \) quasi-expected utilities are determined from the \( S_t - Q_t \) grid over which the time-\( t + 1 \) expected utilities were previously computed, based on the time-\( t \) allocation and basis. The investors’ time-\( t \) bond holdings are then added to their respective certainty equivalents that are bilinearly interpolated from their time-\( t + 1 \) expected utilities.

Finally, Algorithm 8 describes how the realized equilibrium prices and allocations are computed for a given path in the payoff tree. The stock endowments and initial tax basis are set equal to zero. Then, at each \( t \), the position on the \( S_{t-1} - Q_{t-1} \) grid is determined from the time-\( t - 1 \) stock allocation and basis. The time-\( t \) prices and allocations are selected and the tax basis is updated accordingly.
Table A.1: List of variables.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Description</th>
<th>Dimension</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_t$</td>
<td>equilibrium ask price at $t$</td>
<td>$1 \times 1$</td>
</tr>
<tr>
<td>$B_t$</td>
<td>equilibrium bid price at $t$</td>
<td>$1 \times 1$</td>
</tr>
<tr>
<td>$S_t$</td>
<td>equilibrium allocation at $t$</td>
<td>$1 \times 1$</td>
</tr>
<tr>
<td>$W_t$</td>
<td>bond holdings at $t$</td>
<td>$1 \times 1$</td>
</tr>
<tr>
<td>$Q_t$</td>
<td>tax basis at $t$</td>
<td>$1 \times 1$</td>
</tr>
<tr>
<td>$L_t$</td>
<td>tax liability at $t$</td>
<td>$1 \times 1$</td>
</tr>
<tr>
<td>$C$</td>
<td>consumption at $T+1$</td>
<td>$1 \times 1$</td>
</tr>
<tr>
<td>$A'_t$</td>
<td>conjectured ask at $t$</td>
<td>$1 \times 1$</td>
</tr>
<tr>
<td>$B'_t$</td>
<td>conjectured bid at $t$</td>
<td>$1 \times 1$</td>
</tr>
<tr>
<td>$S_t$</td>
<td>$S_{t-1}$ grid</td>
<td>$101 \times 1$</td>
</tr>
<tr>
<td>$S'_{t-1}$</td>
<td>conjectured allocation at $t-1$</td>
<td>$1 \times 1$</td>
</tr>
<tr>
<td>$Q_t$</td>
<td>$Q_{t-1}$ grid</td>
<td>$101 \times 1$</td>
</tr>
<tr>
<td>$Q'_{t-1}$</td>
<td>conjectured tax basis at $t-1$</td>
<td>$1 \times 1$</td>
</tr>
<tr>
<td>$S_t$</td>
<td>$S_t$ grid</td>
<td>$10001 \times 1$</td>
</tr>
<tr>
<td>$S'_t$</td>
<td>conjectured allocation at $t$</td>
<td>$1 \times 1$</td>
</tr>
<tr>
<td>$U_{t+1}$</td>
<td>conjectured quasi-expected utility at $t+1$</td>
<td>$1 \times 1$</td>
</tr>
<tr>
<td>$F_{t+1}$</td>
<td>conjectured certainty equivalent at $t+1$</td>
<td>$1 \times 1$</td>
</tr>
<tr>
<td>$F'_{t+1}$</td>
<td>bilinearly interpolated conjectured certainty equivalent at $t+1$</td>
<td>$1 \times 1$</td>
</tr>
<tr>
<td>$U_{t+1}$</td>
<td>conjectured actual utility at $t+1$</td>
<td>$1 \times 1$</td>
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<tr>
<td>$U'_t$</td>
<td>conjectured utility at $t$</td>
<td>$1 \times 1$</td>
</tr>
<tr>
<td>$U_t$</td>
<td>grid of $U'_t$</td>
<td>$101 \times 1$</td>
</tr>
<tr>
<td>$A_t$</td>
<td>state-dependent ask prices at $t$</td>
<td>$101 \times 101$</td>
</tr>
<tr>
<td>$B_t$</td>
<td>state-dependent bid prices at $t$</td>
<td>$101 \times 101$</td>
</tr>
<tr>
<td>$S_t$</td>
<td>state-dependent allocation at $t$</td>
<td>$101 \times 101$</td>
</tr>
<tr>
<td>$U_t$</td>
<td>state-dependent utility at $t$</td>
<td>$101 \times 101$</td>
</tr>
<tr>
<td>$a_t$</td>
<td>lower bound on $A_t$</td>
<td>$1 \times 1$</td>
</tr>
<tr>
<td>$\bar{a}_t$</td>
<td>upper bound on $A_t$</td>
<td>$1 \times 1$</td>
</tr>
<tr>
<td>$b_t$</td>
<td>lower bound on $B_t$</td>
<td>$1 \times 1$</td>
</tr>
<tr>
<td>$a_{t'}$</td>
<td>iteratively updated lower bound on $A_t$</td>
<td>$1 \times 1$</td>
</tr>
<tr>
<td>$a_{t'}$</td>
<td>iteratively updated upper bound on $A_t$</td>
<td>$1 \times 1$</td>
</tr>
<tr>
<td>$b_{t'}$</td>
<td>iteratively updated lower bound on $B_t$</td>
<td>$1 \times 1$</td>
</tr>
<tr>
<td>$b_{t'}$</td>
<td>iteratively updated upper bound on $B_t$</td>
<td>$1 \times 1$</td>
</tr>
</tbody>
</table>
Algorithm 1 for each node at $t$

input $X_1, X_2, \ldots, X_{t-1}$

for $i \leftarrow 1$ to $\|S\|$ do

$S^*_t \leftarrow S(i)$

for $j \leftarrow 1$ to $\|Q\|$ do

$Q^*_t \leftarrow Q(j)$

compute $A_t, B_t, S_t, \hat{S}_t, U_t,$ and $\hat{U}_t$

$A_t(i, j) \leftarrow A_t$

$B_t(i, j) \leftarrow B_t$

$S_t(i, j) \leftarrow S_t$

$\hat{S}_t(i, j) \leftarrow \hat{S}_t$

$U_t(i, j) \leftarrow U_t$

$\hat{U}_t(i, j) \leftarrow \hat{U}_t$

end for

end for

return $A_t, B_t, S_t, \hat{S}_t, U_t, \hat{U}_t$
Algorithm 2 equilibrium prices and allocations at $t$

```
input $X_1, X_2, \ldots, X_{t-1}$, $i$, $S_{t-1}^*$, $Q_{t-1}^*$

$a_\ell \leftarrow a$
$a_h \leftarrow a$

while $a_h - a_\ell > 0$ or $A_t^* - B_t^* > 0$ do

\[ A_t^* \leftarrow \frac{1}{2}(a_\ell + a_h) \]  \hspace{2cm} ▷ conjecture $A_t$

\[ b_\ell \leftarrow b \]  \hspace{2cm} ▷ $B_t$ lower bound

\[ b_h \leftarrow A_t^* \]  \hspace{2cm} ▷ $B_t$ upper bound

while $b_h - b_\ell > 0$ or $A_t^* - B_t^* > 0$ do

\[ B_t^* \leftarrow \frac{1}{2}(b_\ell + b_h) \]  \hspace{2cm} ▷ conjecture $B_t$

compute $S_t$ and $\hat{S}_t$

if $S_t + \hat{S}_t \geq 1$ then

\[ b_\ell = B_t^* \]  \hspace{2cm} ▷ Alg. 3

else if $S_t + \hat{S}_t < 1$ then

\[ b_h = B_t^* \]  \hspace{2cm} ▷ Alg. 3

end if

end while

compute $S_t$ and $\hat{S}_t$

if $S_t + \hat{S}_t \geq 1$ then

\[ a_\ell = A_t^* \]  \hspace{2cm} ▷ Alg. 3

else if $S_t + \hat{S}_t < 1$ then

\[ a_h = A_t^* \]  \hspace{2cm} ▷ Alg. 3

end if

end while

$A_t \leftarrow A_t^*$

$B_t \leftarrow B_t^*$

compute $S_t, \hat{S}_t, U_t$, and $\hat{U}_t$

return $A_t, B_t, S_t, \hat{S}_t, U_t, \hat{U}_t$
```

Algorithm 3 demand at $t$

```plaintext
input $X_1, X_2, \ldots, X_{t-1}, i, S_{t-1}^*, Q_{t-1}^*, A_t^*, B_t^*$

$n \leftarrow (i - 1) \frac{\| S \|}{\| S \|} + 1$

for $k \leftarrow 1$ to $n$ do
  $S_t^* \leftarrow S(k)$  \hspace{1em} $\triangleright$ taxable investor sells
  $\triangleright$ conjecture $S_t$ is $k$-th element of $S$
  compute taxable investor’s expected utility, $U_i^*(B_t^*, S_t^*)$  \hspace{1em} $\triangleright$ Alg. 4 or 6
  $\hat{U}_t(k) \leftarrow U_i^*$  \hspace{1em} $\triangleright$ store conjectured utility
  $\hat{S}_t^* \leftarrow 1 - S_t^*$  \hspace{1em} $\triangleright$ conjecture $\hat{S}_t$ is $\| S \| + 1 - k$-th element of $S$
  compute nontaxable investor’s expected utility, $\hat{U}_i^*(A_t^*, \hat{S}_t^*)$  \hspace{1em} $\triangleright$ Alg. 5 or 7
  $\hat{\hat{U}}_t(k) \leftarrow \hat{U}_i^*$  \hspace{1em} $\triangleright$ store conjectured utility

end for

for $k \leftarrow n + 1$ to $\| S \|$ do
  $S_t^* \leftarrow S(k)$  \hspace{1em} $\triangleright$ taxable investor buys
  $\triangleright$ conjecture $S_t$ is $k$-th element of $S$
  compute taxable investor’s expected utility, $U_i^*(A_t^*, S_t^*)$  \hspace{1em} $\triangleright$ Alg. 4 or 6
  $\hat{U}_t(k) \leftarrow U_i^*$  \hspace{1em} $\triangleright$ store conjectured utility
  $\hat{S}_t^* \leftarrow 1 - S_t^*$  \hspace{1em} $\triangleright$ conjecture $\hat{S}_t$ is $\| S \| + 1 - k$-th element of $S$
  compute nontaxable investor’s expected utility, $\hat{U}_i^*(B_t^*, \hat{S}_t^*)$  \hspace{1em} $\triangleright$ Alg. 5 or 7
  $\hat{\hat{U}}_t(k) \leftarrow \hat{U}_i^*$  \hspace{1em} $\triangleright$ store conjectured utility

end for

$S_t \leftarrow \text{arg max } \hat{U}_t$  \hspace{1em} $\triangleright$ taxable investor’s demand

$\hat{S}_t \leftarrow \text{arg max } \hat{\hat{U}}_t$  \hspace{1em} $\triangleright$ nontaxable investor’s demand

compute $U_t(A_t^*, B_t^*, S_t)$  \hspace{1em} $\triangleright$ Alg. 4 or 6

compute $\hat{U}_t(A_t^*, B_t^*, S_t)$  \hspace{1em} $\triangleright$ Alg. 5 or 7

return $S_t, \hat{S}_t, U_t, \hat{\hat{U}}_t$
```

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Algorithm 4 taxable investor’s expected utility at $T$

input $X_1, X_2, \ldots, X_{T-1}, S_{T-1}^*, Q_{T-1}^*, A_T^*, B_T^*, S_T^*$

$W_{T-1} \leftarrow 0$

if $S_T^* \leq S_{T-1}^*$ then

$Q_T \leftarrow Q_{T-1}^*$  \hspace{1cm} \triangleright \text{tax basis}

$L_T \leftarrow (S_{T-1}^* - S_T^*)(B_T^* - Q_{T-1}^*)\theta$  \hspace{1cm} \triangleright \text{tax liability}

$W_T \leftarrow W_{T-1}(1 + r) + (S_{T-1}^* - S_T^*)B_T^* - L_T$  \hspace{1cm} \triangleright \text{budget constraint}

else

$Q_T \leftarrow \frac{S_{T-1}^*Q_{T-1}^* + (S_T^* - S_{T-1}^*)A_T^*}{S_T^*}$  \hspace{1cm} \triangleright \text{tax basis}

$L_T \leftarrow 0$  \hspace{1cm} \triangleright \text{tax liability}

$W_T \leftarrow W_{T-1}(1 + r) - (S_T^* - S_{T-1}^*)A_T^* - L_T$  \hspace{1cm} \triangleright \text{budget constraint}

end if

$L_{T+1}(L) \leftarrow S_T^*(\sum_{t=1}^{T-1} X_t + L - Q_T)\theta$  \hspace{1cm} \triangleright \text{tax liability in low state}

$L_{T+1}(H) \leftarrow S_T^*(\sum_{t=1}^{T-1} X_t + H - Q_T)\theta$  \hspace{1cm} \triangleright \text{tax liability in high state}

$C(L) \leftarrow W_T(1 + r) + S_T^*(\sum_{t=1}^{T-1} X_t + L) - L_{T+1}(L)$  \hspace{1cm} \triangleright \text{consumption in low state}

$C(H) \leftarrow W_T(1 + r) + S_T^*(\sum_{t=1}^{T-1} X_t + H) - L_{T+1}(H)$  \hspace{1cm} \triangleright \text{consumption in high state}

$U_T^* \leftarrow -\pi \exp[-\delta C(L)] - (1 - \pi) \exp[-\delta C(H)]$  \hspace{1cm} \triangleright \text{expected utility}

return $U_T^*$
Algorithm 5 nontaxable investor’s expected utility at \( T \)

```
input \( X_1, X_2, \ldots, X_{T-1}, \hat{S}_{T-1}^*, A_T^*, B_T^*, \hat{S}_T^* \)
\( \hat{W}_{T-1} \leftarrow 0 \)
if \( \hat{S}_T^* \leq \hat{S}_{T-1}^* \) then
   \( \hat{W}_T \leftarrow \hat{W}_{T-1}(1 + r) + (\hat{S}_{T-1}^* - \hat{S}_T^*)B_T^* \) \hspace{1cm} ▷ budget constraint
else
   \( \hat{W}_T \leftarrow \hat{W}_{T-1}(1 + r) - (\hat{S}_T^* - \hat{S}_{T-1}^*)A_T^* \) \hspace{1cm} ▷ budget constraint
end if
\( \hat{C}(L) \leftarrow \hat{W}_T(1 + r) + \hat{S}_T^*(\sum_{t=1}^{T-1} X_t + L) \) \hspace{1cm} ▷ consumption in low state
\( \hat{C}(H) \leftarrow \hat{W}_T(1 + r) + \hat{S}_T^*(\sum_{t=1}^{T-1} X_t + H) \) \hspace{1cm} ▷ consumption in high state
\( \hat{U}_T^* \leftarrow -\pi \exp[-\delta \hat{C}(L)] - (1 - \pi) \exp[-\delta \hat{C}(H)] \) \hspace{1cm} ▷ expected utility
return \( \hat{U}_T^* \)
```
Algorithm 6 taxable investor’s expected utility at $t < T$

**input** $X_1, X_2, \ldots, X_{T-1}, S_{t-1}^*, Q_{t-1}^*, A_t^*, B_t^*, S_t^*, U_{t+1}$

$W_{t-1} \leftarrow 0$

if $S_t^* \leq S_{t-1}^*$ then

$Q_t \leftarrow Q_{t-1}$ \> tax basis

$L_t \leftarrow (S_{t-1}^* - S_t^*)(B_t^* - Q_{t-1}^*)\theta$ \> tax liability

$W_t \leftarrow W_{t-1}(1 + r) + (S_{t-1}^* - S_t^*)B_t^* - L_t$ \> budget constraint

else

$Q_t \leftarrow \left[ S_{t-1}^*Q_{t-1}^* + (S_t^* - S_{t-1}^*)A_t^* \right]/S_t^*$ \> tax basis

$L_t \leftarrow 0$ \> tax liability

$W_t \leftarrow W_{t-1}(1 + r) - (S_t^* - S_{t-1}^*)A_t^* - L_t$ \> budget constraint

end if

$U_{t+1}^*(S_t^*, Q_t, L) \leftarrow U_{t+1}(S_t^*, Q_t, L)$ \> quasi-expected utility in low state

$U_{t+1}^*(S_t^*, Q_t, H) \leftarrow U_{t+1}(S_t^*, Q_t, H)$ \> quasi-expected utility in high state

$F_{t+1}(S_t^*, Q_t, L) \leftarrow -\frac{1}{\delta} \log[-U_{t+1}^*(S_t^*, Q_t, L)]$ \> certainty equivalent in low state

$F_{t+1}(S_t^*, Q_t, H) \leftarrow -\frac{1}{\delta} \log[-U_{t+1}^*(S_t^*, Q_t, H)]$ \> certainty equivalent in high state

$F_{t+1}^*(S_t^*, Q_t, L) \leftarrow \text{bilinear interpolation of } F_{t+1}(S_t^*, Q_t, L) \text{ over } S_t \text{ and } Q_t$

$F_{t+1}^*(S_t^*, Q_t, H) \leftarrow \text{bilinear interpolation of } F_{t+1}(S_t^*, Q_t, H) \text{ over } S_t \text{ and } Q_t$

$U_{t+1}(S_t^*, Q_t, L) \leftarrow -\exp\left[-\delta(F_{t+1}^*(S_t^*, Q_t, L) + W_t(1 + r)^{T-t+1}) \right]$ \> utility in low state

$U_{t+1}(S_t^*, Q_t, H) \leftarrow -\exp\left[-\delta(F_{t+1}^*(S_t^*, Q_t, H) + W_t(1 + r)^{T-t+1}) \right]$ \> utility in high state

$U_t^* \leftarrow \pi U_{t+1}(S_t^*, Q_t, L) + (1 - \pi)U_{t+1}(S_t^*, Q_t, H)$ \> expected utility

**return** $U_t^*$
Algorithm 7 nontaxable investor’s expected utility at $t < T$

\[ \text{input } X_1, X_2, \ldots, X_{T-1}, S_{t-1}^*, Q_{t-1}^*, \hat{S}_t^*, \hat{A}_t^*, B_t^*, \hat{S}_t^*, \hat{U}_{t+1} \]

\[ \hat{W}_{t-1} \leftarrow 0 \]
\[ S_t^* \leftarrow 1 - \hat{S}_t^* \]

\[ \text{if } \hat{S}_t^* \leq \hat{S}_{t-1}^* \text{ then} \]
\[ \hat{W}_t \leftarrow \hat{W}_{t-1}(1 + r) + (\hat{S}_{t-1}^* - \hat{S}_t^*)B_t^* \]
\[ Q_t \leftarrow [S_{t-1}Q_{t-1}^* + (S_t^* - S_{t-1}^*)A_t^*]/S_t^* \]
\[ \text{else} \]
\[ \hat{W}_t \leftarrow \hat{W}_{t-1}(1 + r) - (\hat{S}_t^* - S_{t-1}^*)A_t^* \]
\[ Q_t \leftarrow Q_{t-1}^* \]

\[ \text{end if} \]
\[ \hat{U}_{t+1}'(\hat{S}_t^*, Q_t, L) \leftarrow \mathcal{U}_{t+1}(\hat{S}_t^*, Q_t, L) \]
\[ \hat{U}_{t+1}'(\hat{S}_t^*, Q_t, H) \leftarrow \mathcal{U}_{t+1}(\hat{S}_t^*, Q_t, H) \]
\[ \hat{F}_{t+1}(\hat{S}_t^*, Q_t, L) \leftarrow -\frac{1}{\delta} \log[-\hat{U}_{t+1}'(\hat{S}_t^*, Q_t, L)] \]
\[ \hat{F}_{t+1}(\hat{S}_t^*, Q_t, H) \leftarrow -\frac{1}{\delta} \log[-\hat{U}_{t+1}'(\hat{S}_t^*, Q_t, H)] \]
\[ \hat{F}_{t+1}'(\hat{S}_t^*, Q_t, L) \leftarrow \text{bilinear interpolation of } \hat{F}_{t+1}(\hat{S}_t^*, Q_t, L) \text{ over } \hat{S}_t \text{ and } Q_t \]
\[ \hat{F}_{t+1}'(\hat{S}_t^*, Q_t, H) \leftarrow \text{bilinear interpolation of } \hat{F}_{t+1}(\hat{S}_t^*, Q_t, H) \text{ over } \hat{S}_t \text{ and } Q_t \]
\[ \hat{U}_{t+1}(\hat{S}_t^*, Q_t, L) \leftarrow -\exp[-\delta(\hat{F}_{t+1}'(\hat{S}_t^*, Q_t, L) + W_t(1 + r)^{T-t+1})] \]
\[ \hat{U}_{t+1}(\hat{S}_t^*, Q_t, H) \leftarrow -\exp[-\delta(\hat{F}_{t+1}'(\hat{S}_t^*, Q_t, H) + W_t(1 + r)^{T-t+1})] \]
\[ \hat{U}_t^* \leftarrow \pi \hat{U}_{t+1}(\hat{S}_t^*, Q_t, L) + (1 - \pi)\hat{U}_{t+1}(\hat{S}_t^*, Q_t, H) \]

\[ \text{return } \hat{U}_t^* \]
Algorithm 8 realized equilibrium prices and allocations

input $\mathcal{A}_t, \mathcal{B}_t, S_t, \hat{S}_t$

$S_0 \leftarrow 0 \quad \triangleright \text{taxable investor’s stock endowment}$
$\hat{S}_0 \leftarrow 0 \quad \triangleright \text{nontaxable investor’s stock endowment}$
$Q_0 \leftarrow 0 \quad \triangleright \text{initial tax basis}$

for $t \leftarrow 1$ to $T$ do

$i \leftarrow S_{t-1}(\|S\| - 1) + 1 \quad \triangleright \text{allocation position on grid}$
$j \leftarrow Q_{t-1}(\|Q\| - 1) + 1 \quad \triangleright \text{basis position on grid}$
$A_t \leftarrow \mathcal{A}_t(i, j) \quad \triangleright \text{realized equilibrium ask}$
$B_t \leftarrow \mathcal{B}_t(i, j) \quad \triangleright \text{realized equilibrium bid}$
$S_t \leftarrow S_t(i, j) \quad \triangleright \text{taxable investor’s realized equilibrium allocation}$
$\hat{S}_t \leftarrow \hat{S}_t(i, j) \quad \triangleright \text{nontaxable investor’s realized equilibrium allocation}$

if $S_t > S_{t-1}$ then

$Q_t \leftarrow [S_{t-1}Q_{t-1} + (S_t - S_{t-1})A_t]/S_t \quad \triangleright \text{update basis}$

else

$Q_t \leftarrow Q_{t-1} \quad \triangleright \text{maintain basis}$

end if

end for

return $A_t, B_t, S_t, \hat{S}_t, Q_t$