I. The Capital Asset Pricing Model
   A. Assumptions and implications
      1. Security markets are perfectly competitive.
         a) Many small investors
         b) Investors are price takers
      2. Markets are frictionless
         a) There are no taxes or transaction costs.
      3. Investors are myopic
         a) All investors have only one and the same holding period
      4. Investments are limited to publicly traded assets with unlimited borrowing and lending at the risk-free rate.
         a) Assets such as human capital are not part of the investment opportunity set.
      5. All investors are rational mean-variance optimizers
         a) Everyone uses the Markowitz portfolio selection method.
      6. Perfect Information
         a) All investors have access to the same information.
         b) All investors analyze information in the same manner.
      7. Everyone either has quadratic utility or has homogenous beliefs concerning the distribution of security returns.
         a) Everyone uses the same estimates of expected return and the same variance/covariance matrix.

   B. Equilibrium
      1. Given that all investors use identical analysis of the same universe of assets for the same holding period and all believe asset returns are normally distributed, it must be the case that the optimal risky portfolio for all investors is the same, call it *M*.
         a) Because every investor holds the same optimal risky portfolio with the same weights for each stock in the portfolio, it must be the case that each investor holds a share of the Market Portfolio.
         b) In fact, it must also be the case that each individual's portfolio weights any given stock with the same proportion as is found in the market index. That is, if IBM comprises 1% of the total value of the market portfolio, then each individual optimal risky portfolio will also be 1% IBM.
      2. Because the market portfolio is the only optimal risky portfolio, it is also a portfolio along the Efficient Frontier.
      3. The market portfolio is also the tangency portfolio for CML with the maximum Sharpe ratio.
      4. In equilibrium, the risk premium on the market portfolio will be proportional to its variance and the average risk aversion of investors.
      5. In the overall economy, any borrowing position is offset by a lending position. This means that net borrowing across all investors is zero and hence the average position in the optimal risky portfolio *M* is 100% or $y_{ave} = 1$.

         But if $y_{ave} = 1$ and $y_{ave} = [E(r_m) - rf]/.01A_{ave}\sigma_m^2$, then it must be that

         $E(r_m) - rf = .01\ A_{ave}\sigma_m^2$.

II. Derivation of the Capital Asset Pricing Model: A perturbation argument
   A. Consider an average investor in this economy. This investor holds 100% of her portfolio as the optimal risky portfolio, *M*.

For this investor
$E(r_c) = E(r_m)$ and
$\sigma_c^2 = \sigma_m^2$

B. If this investor changes her position so that she increases her holding of *M* by some small amount δ and finances the increase by borrowing at the risk-free rate she will have an expected return on her new portfolio of:
$E(r_c)' = E(r_m + \delta r_m - \delta rf)$
$= E(r_m) + \delta[E(r_m) - rf]$

C. And the change in her expected return will be
$E(r_c)' - E(r_c) = \Delta E(r_c) = \delta[E(r_m) - rf]$

D. The variance of the new complete portfolio will be
$\sigma_c^{2\prime} = (1+\delta)^2\sigma_m^2 + (-\delta)^2\sigma_{rf}^2 - 2(1+\delta)(-\delta)Cov(r_m,rf)$
$= (1+\delta)^2\sigma_m^2$
$= (1 + 2\delta + \delta^2)\sigma_m^2$
$\approx(1 + 2\delta)\sigma_m^2$

E. Since $\sigma_{rf}^2 = 0$ and δ^2 is very small. And the change in the variance of her new portfolio will be:
$\sigma_c^{2\prime} - \sigma_c^2 = \Delta\sigma_c^2 = 2\delta\sigma_m^2$

F. Therefore we can form the following ratio.
$\Delta E(r_c)/\Delta\sigma_c^2 = [E(r_m) - rf]/2\sigma_m^2$

This is the trade-off between the *incremental change in the risk premium* and the *incremental change in risk.*

1. Formally, the above ratio is known as the ***marginal price of risk*** and can be interpreted as *the additional risk-premium the average investor receives for a small increase in risk.*

G. By the same technique, we can derive the marginal price of risk for any stock. Simply investing the borrowed amount δ, in a particular stock (say IBM) and doing the algebra yields the marginal price of risk for IBM as
$\Delta E(r_c)/\Delta\sigma_c^2 = [E(r_{IBM}) - rf]/2Cov(r_{IBM}, r_m)$
1. **Note** that in equilibrium, the marginal price-of-risk for all assets must be equal or the investor could increase her portfolio average price-of-risk by investing in the higher price-of-risk securities.
2. That is securities that offer more of an additional risk-premium for small increases in risk would be more desirable and the forces of supply and demand would adjust until the market price-of-risk was equal across all assets.

H. But this implies that

$[E(r_{IBM}) - rf]/2Cov(r_{IBM}, r_m) = [E(r_m) - rf]/2\sigma_m^2$

$E(r_{IBM}) - rf = [Cov(r_{IBM}, r_m)/ \sigma_m^2] * [E(r_m) - rf]$

$E(r_{IBM}) = rf - \beta[E(r_m) - rf]$

Where $\beta = Cov(r_{IBM}, r_m)/ \sigma_m^2$

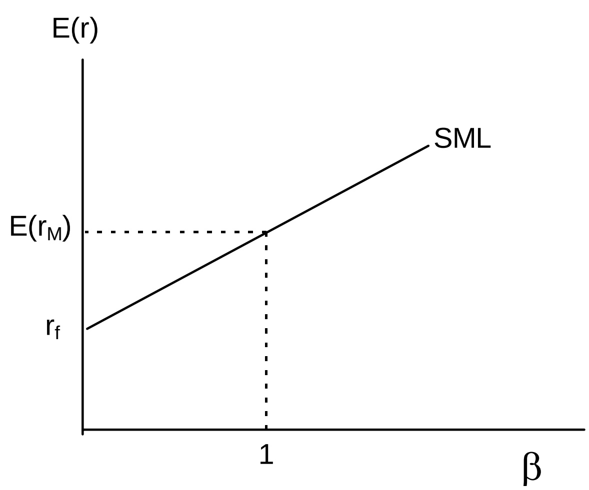
III. SML: The security market line

A. In the CAPM world

1. The market portfolio M is THE optimal risky portfolio for all investors,
2. Risk averse investors must be rewarded for variance by a risk premium that depends on how the risk of an individual asset contributes to the portfolio.
3. In particular, β is a measure of that risk.
   a) Represents the stocks contribution to the variance of the market portfolio as a fraction of the total portfolio variance.
   $\beta = Cov(r_i, r_m)/ \sigma_m^2$
4. But, this implies that the risk premium should depend on β.
5. The risk premium for all assets in the CAPM is
   $E(r_i) - r_f = \beta[E(r_m) - r_f]$

   a) In words that means that the risk premium of any asset is directly proportional to β and to the market risk premium.

IV. The risk-return relationship predicted by the CAPM can be shown graphically as follows.



A. The linear relation between risk and return is the primary result of the CAPM.

B. If the model is correct, then all assets should have expected returns that can be found by calculating β and finding the corresponding E(r) on the security market line.

C. The SML graphs the risk premium of individual assets as a function of asset risk.

D. The relevant measure of risk for *well-diversified* investors is not the standard deviation of an asset's returns, but rather how it contributes to the risk of the portfolio as measured by β.

E. Note that the SML is different from the CML.

1. The CML graphs risk premiums of efficient portfolios as a function of portfolio standard deviation.

2. This is useful since we need to find portfolios that are candidates for the "optimal" risky portfolio.
3. The CML helps us find portfolios while the SML predicts the expected return on individual assets from their β's.

V. Uses of the CAPM

A. You have probably already seen that we can use the CAPM to derive the required rate of return on equity.

B. Another use of the SML is as a benchmark for assets or fund manager performance.

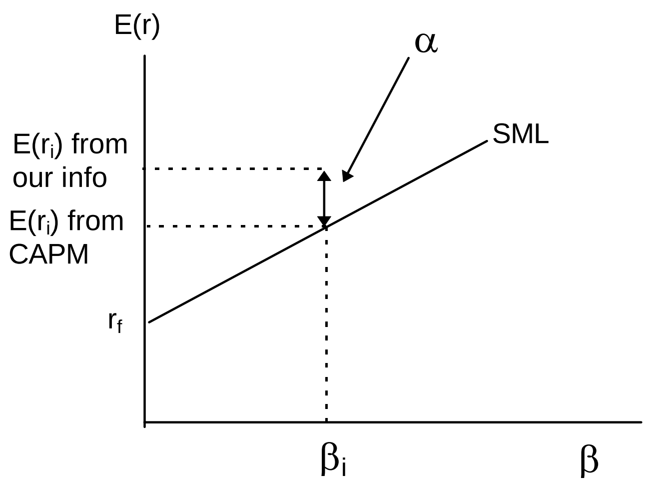
1. Suppose we have inside information on a particular stock that causes us to expect a return that is higher than that predicted by the CAPM. Or suppose that we calculate the β for a mutual fund manager and then observe the return the fund has generated. In particular the CAPM predicts that

$$E(r_i) = r_f + \beta[E(r_m) - r_f]$$

2. But the fund performance has been slightly better than the CAPM says it should be so that we have

$$E(r_i) = r_f + \beta[E(r_m) - r_f] + \alpha$$

3. Graphically we have the following result.



4. The difference between what the CAPM predicts and what the fund actually produced is known as **Jensen's alpha**.
5. If we have inside information and we predict a higher return than the CAPM we may consider the assets as being a good buy (under priced) since the "market" prediction of returns is below what we believe will happen.

VI. One obvious drawback of the CAPM is that it doesn't incorporate restrictions on the risk free rate that are present in the real world.

A. Fisher Black attacked this problem by showing that the CAPM holds even if there is NO risk-free rate.

B. Here is an outline of his argument.
   1. It can be shown that any portfolio constructed from combining efficient portfolios is also efficient.
   2. It can also be shown that for every portfolio on the efficient frontier there is a portfolio along the inefficient part of the frontier that has a zero covariance with the efficient portfolio.
   3. What Black showed is that the expected return on any asset can be shown to have an exact linear relationship with the expected returns of any two portfolios on the efficient frontier.
      a) In particular we can pick the efficient portfolio to be the market portfolio M and hence there is an exact linear relation between the expected return of any asset and M and it's corresponding zero covariance portfolio.
      b) This implies that the risk-free asset is the zero covariance portfolio and hence the CAPM holds.

C. The same type of logic can be applied to show that the CAPM holds in a world with differential borrowing and lending rates.

VII. Another draw back of the CAPM is that it is a static model one period model.

A. Fama showed that the CAPM could be extended to multiple periods if we are willing to make a few more assumptions.
   1. Namely we must assume that consumer preferences do not change over time,
   2. The risk-free rate must be constant or predictable, and
   3. The distributions of asset returns must be constant over time.
   4. Under these assumptions it can be shown that the CAPM holds intertemporally.

VIII. Review of the CAPM

A. Assumptions
   1. Security markets are perfectly competitive.
      a) Many small investors
      b) Investors are price takers
   2. Markets are frictionless
      a) There are no taxes or transaction costs.
   3. Investors are myopic
      a) All investors have only one and the same holding period
   4. Investments are limited to publicly traded assets with unlimited borrowing and lending at the risk-free rate.
   5. All investors use the Markowitz portfolio selection method.
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   7. Everyone either has quadratic utility or has homogenous beliefs concerning the distribution of security returns.
      a) Everyone uses the same estimates of expected return and the same variance/covariance matrix.

   8. Equilibrium

a) Because every investor holds the same optimal risky portfolio with the same weights for each stock in the portfolio, it must be the case that each investor holds a share of the Market Portfolio with each individuals portfolio weights any given stock being the same proportion as is found in the market index.
b) Because the market portfolio is the only optimal risky portfolio, it is also a portfolio along the Efficient Frontier.
c) In equilibrium, the risk premium on the market portfolio will be proportional to its variance and the average risk aversion of investors.
d) The risk premium on any asset is proportional to β and the market risk premium.
e) There is an exact linear relationship between risk and return with β being the relevant measure of risk.