1) The Markowitz Portfolio selection model

a) In our simple world with only two risky assets we have seen that the solution to the investor's problem is a two-step process.

(1) Using the attributes of the individual securities, the investor finds the *optimal risky portfolio* by maximizing the Sharpe ratio.

(2) Next the investor finds the utility maximizing level of investment in the risky asset, which in turn defines the weights to assign to all the assets in her portfolio.

b) The implication of the two-asset world is that the rational utility maximizing investor will only invest in portfolios along the CAL.

(1) Turning that statement around, the investor faces an *investment opportunity set* defined by portfolios that lie along the CAL.

c) The results thus far beg the question: What happens when the investor has many different risky assets from which to build an optimally risky portfolio?

i) In the real world where an investor faces a multitude of risky assets, differing borrowing and lending rates, and possible investment restrictions the problem becomes only slightly more complicated.

ii) If data for expected returns, variances, and covariance's exist, the investor needs to find the risk-return opportunities available.

iii) In effect, the investor finds the weights for all the assets yielding the different portfolios that solve one of the following equivalent constrained optimization problems.

(1) Minimize the variance of a portfolio of all the assets for different given levels of expected return.

(2) This problem may be written in matrix notation as:

$$\min_{x} \sigma_{rp}^2 = w'\Omega w$$
$$s.t.\ \ \mathrm{E}\left(r_p\right) = \bar{R}'w$$
$$q'w = 1$$

Here we have

w = a vector of portfolio weights

Ω = the variance covariance matrix

R = a vector of asset returns

$E(r_P) = \bar{R}'w$ the portfolio return

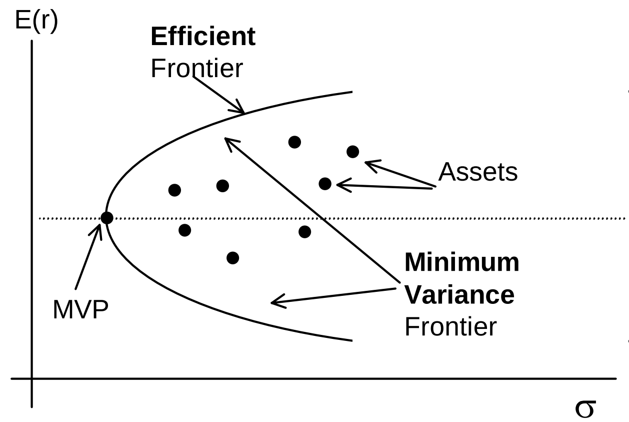
$\sigma^2_P = w'\Omega w$ the portfolio variance

q = vector of ones.

(3) Or maximize the expected return of all the assets for different levels of variance.

$$\max_{w_i} E(r_p) = \sum_{i=1}^{n} w_i E(r_i)$$
$$s.t.\ \sigma_p^2 \in \{\sigma_1^2, \sigma_2^2, \sigma_3^2, \sigma_4^2, \ldots\}$$

iv) A plot of the portfolios that solve the optimization problems for different constraints yields the *minimum variance frontier*.




(1) The portfolio lying along the Minimum Variance Frontier with the least amount of variance is called the *minimum variance portfolio* (MVP).
(2) The portion of the Minimum Variance Frontier lying above the MVP is known as the *Efficient Frontier* because the best risk-return opportunities are along this part of the frontier.
(3) A line from the risk-free rate to the tangency point on the Efficient Frontier (the optimal risky portfolio) is called the Capital Market Line or CML rather than a CAL, since we are looking at all the assets in the capital market.

v) There are two rather amazing implications that come out of this analysis.
(1) Without differing borrowing and lending rates, and without other restrictions on investing, a portfolio manager would offer only one optimal risky portfolio to any client.
(a) This would be the portfolio on the frontier that maximized the Sharpe ratio.
(b) This also implies that the investment opportunity set for all investors lies along the CML.
(2) **Separation principle**: The property that portfolio choice can be separated into two independent tasks:
(a) Determination of the optimal risky portfolio, which is purely a technical problem, and
(b) The personal choice of the best mix of the risky portfolio and the risk-free asset.

d) A Recipe for a portfolio manager

i) Determine the expected returns and variance/covariance structure for the assets included in the analysis.
   (1) Given the nature of the problem, this is where different investment firms really compete. As we have seen, there are several different ways to derive an expected value for an asset in the future. These included the following:
      (a) Use historical data
      (b) Use fundamental analysis to estimate next period prices and hence returns.
      (c) Use fundamental analysis coupled with sensitivity analysis.
      (d) Use stochastic models of stock prices such as Geometric Brownian Motion.

ii) Once the expected returns of all the assets have been accumulated and the variance/covariance matrix has been estimated from historical data, the optimal risky portfolio can be constructed by

$$\underset{W_i}{Max}\, S = \frac{E(r_p) - rf}{\sigma_P}$$

$$s.t. \sum_i w_i = 1$$

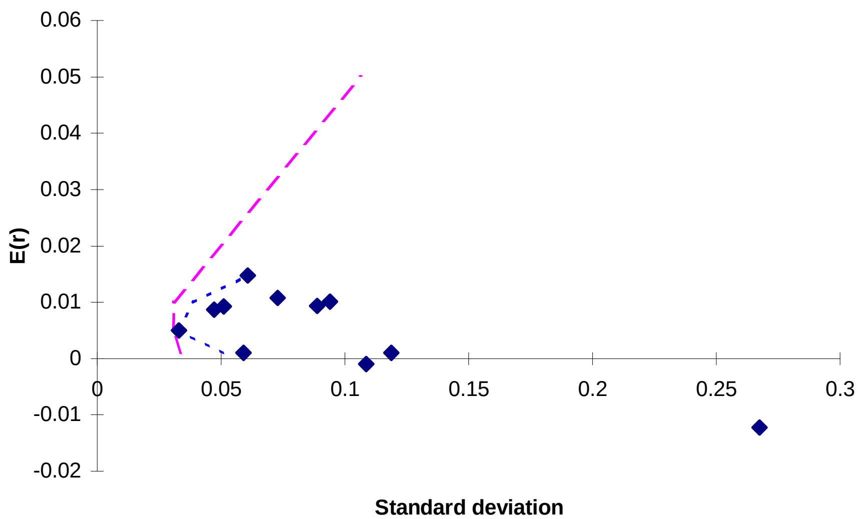
**As long as there are no restrictions on borrowing, etc.**

iii) The client's portfolio can then be created by

$$y^* = [E(r_p) - r_f] / [.01A\sigma_p^2]$$

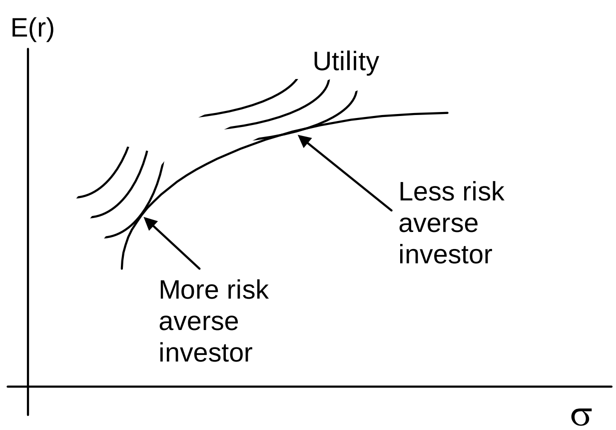
e) Incorporating different restrictions into the portfolio selection model
   i) Asset and/or position restrictions
      (1) *These types of restrictions reduce the Minimum Variance Frontier and hence the investment opportunity of the investor*.
      (2) These included:
         (a) No short-selling
         (b) Expected dividend-yield restrictions
         (c) Tax based restrictions
         (d) Asset class restrictions
            (i) Only investing in technology assets
            (ii) Green investing

   **ii) Example: Restrictions on short-selling**
      (1) Chose eleven assets randomly from the three exchanges and plot their historical expected values and standard deviations.
      (2) Using solver, calculate the minimum variance portfolio for a set of different expected values both with and without constraining the possibility of short sales.
      (3) Plot these frontiers.



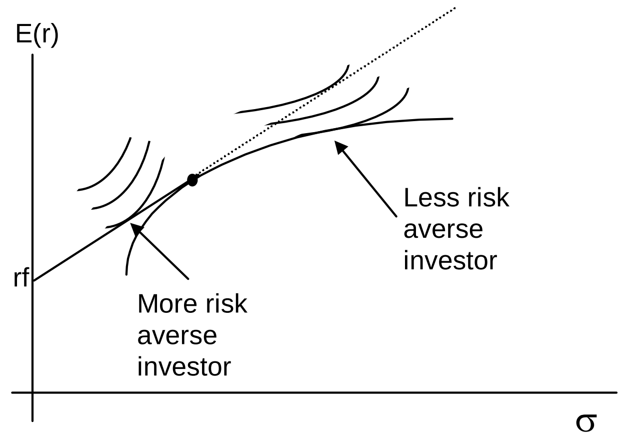

(4) The heavy dotted line is the MVF allowing for short sales = letting the weights be positive or negative.
(5) The light dotted line is the MVF without short sales = the weights must all be between zero and one.
(6) It is obvious that the restriction on short sales greatly reduces the investment opportunity set given this universe of assets.
(7) Any of the restrictions mentioned above will have a similar affect on the investment opportunity set.

iii) What happens when there are restrictions on the risk-free rate?
   (1) No risk-free asset
      (a) In this case the investor will choose a portfolio along the Efficient Frontier that maximizes utility. There is no optimal risky portfolio for all investors in this case.



(2) Risk-free lending only.

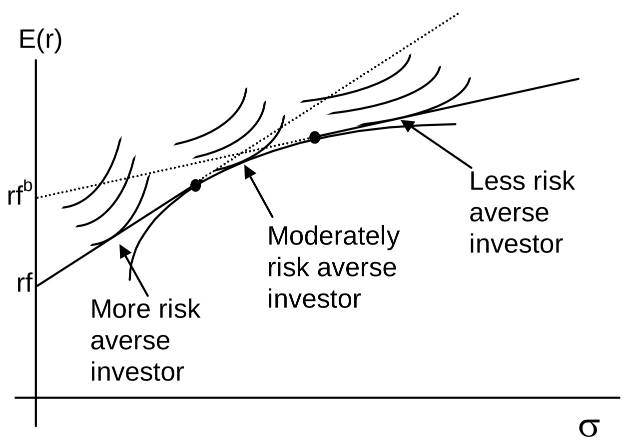
(a) In this case an optimal risky portfolio only exists for investors who are lending. For those who borrow the investor will again need to choose a portfolio along the Efficient Frontier that maximizes utility.




(3) Different borrowing and lending rates.

(a) We saw in the last set of notes how differential lending and borrowing rates put a kink in the CAL. This intuition carries over to a kink the CML.

(b) Here a portfolio manager will have to estimate two optimal portfolios; one for lending clients and one for borrowing clients.




2) Index models defined

a) Consider defining the returns on any asset by a ***single index model***

$$r_i = E(r_i) + \beta iF + e_i$$

i) In words we have that future returns are made up of two parts

(1) The expected part of returns, $E(r_i)$

(2) and two unexpected parts $\beta$iF and $e_i$

(3) F is known as a factor and represents a measure of things such as inflation, money supply changes, business cycles, technology advancements, etc. These are macro economic forces that move the whole market.
(4) $\beta i$ is the sensitivity of $r_i$ to this market factor and
(5) $e_i$ is the firm specific part of returns.

ii) Note: We could let ***F*** be anything we want and we could have more than one factor.

iii) For example purposes, let's define F as the S&P 500 (call it $r_m$ we can assume that this factor represents how many individual factors move the market).

iv) If we use this factor then we can rewrite the model for returns as

$$r_i - r_f = \alpha_i + \beta i(r_m - r_f) + e_i$$

v) We subtract the risk free rate since the level of stock returns represents the state of the macro economy only to the extent that it exceeds (or falls short of) the rate on T-bills.

vi) If we match terms in the previous two equations we can deduce that

$\alpha_i = E(R_i)$, it is the expected return when the market is neutral ($r_m = r_f$)
$\beta i(r_m - r_f)$ = the component of returns due to movement in the overall market where $\beta i$ is the assets sensitivity to that movement.
$e_i$ = Component of returns due to unexpected firm specific events. We expect this to be zero on average. Just as many good things can happen unexpectedly to a firm as bad things.

vii) To simplify the notation we can write
$R_i = r_i - r_f$ and
$R_m = r_m - r_f$ to get
$R_i = \alpha_i + \beta i R_m + e_i$

b) Now let's take a look at the variance of $R_i$
$var(R_i) = var(\alpha_i + \beta i R_m + e_i)$

i) But, a rule from statistics is that
$var(c + X) = var(X)$

when c is a constant and so we really have
$var(R_i) = var(\beta i R_m + e_i)$

ii) But we also know that
$var(aX + bY) = a^2 var(X) + b^2 var(Y) + 2*a*b*cov(X;Y)$
*and*
$cov(\beta i R_m; e_i) = 0$
since firm specific and macro shocks should be unrelated.

iii) These imply that

$var(R_i) = \beta i^2 var(R_m) + var(e_i)$ or
$\sigma^2_{ri} = \beta i^2 \sigma^2_{Rm} + \sigma^2_{ei}$

iv) This shows that the variance of returns in this model is comprised of the variance from common factors and the variance from firm specific risk.

c) We can also derive the covariance between two assets in this model. I don't want to type this out and so I leave as an exercise for the reader. To do this you will need the following tools and assumptions.
   i) Tools:
      $var(x) = E(x^2) - [E(x)]^2$
      $cov(x;y) = E(x*y) - E(x)E(y)$
   ii) Assumptions:
      (1) The expected value of $e_i = 0$
      (2) The covariance between two different firms specific shocks is zero. That is $E(e_ie_j) = 0$
      (3) The covariance between macro shocks and firm specific shocks is zero. $E(R_ie_i) = 0$.
   iii) Using these tools and assumptions it is straightforward to show that
      $cov(R_i; R_j) = \beta i \beta_j \sigma^2_M$

d) This result is very useful in real world portfolio selection problems.
   i) To see this consider an analysis of 50 stocks. To find the optimal portfolio is this case we need.
      n = 50 estimates of $E(r_i)$
      n = 50 estimates of $var(r_i)$
      $(n^2 - n)/2$ = 1,225 estimates of covariance's
      That is we need a total of 1,325 estimates to find the optimal risky portfolio. If we were to analyze all the assets on the NYSE, we would need around 1.5 million estimates.
   ii) Using the single index model we can obtain the covariance estimates via $cov(R_i; R_j) = \beta i \beta_j \sigma^2_M$. This reduces the problem to
      n = 50 estimates of $E(r_i)$
      n = 50 estimates of $\beta i$
      n = 50 estimates of $var(e_i)$
      1 estimate of $var(r_M)$
      That is we need a total of 151 estimates to find the optimal portfolio. This is a considerable difference.

e) Example:
   i) I found that by calculating historical covariance's that cov(asset 1; asset 2) = .003464. But, I also have $\beta_1 = 1.116513$ and $\beta_2 = 1.76187$ and $\sigma^2_M = .002191$. Hence, the cov(asset 1; asset 2) = $\beta_1\beta_2\sigma^2_M$ = (1.116513)( 1.76187)(.002191) = .00431.
   ii) Why don't the two covariance calculations match?

iii) One of the necessary assumptions of the single index model is that the $e_i$ are uncorrelated. This fails to hold in when dealing with real world data. This is one of the "costs" of using the model. We must make assumptions that may be unrealistic.

iv) *The question then is which way is best for calculating covariance?*

3) Estimation and interpretation of estimation results.

a) Suppose we use the single index model to find the β on GM stock. Here are the steps to doing this

i) Calculate the excess returns of the market (S&P 500) and GM over the 3-month T-Bill rate.

ii) Run the regression

$R_{GMt} = \alpha + \beta R_{Mt} + e_t$

iii) Now suppose we get the following output: standard errors are in parenthesis.

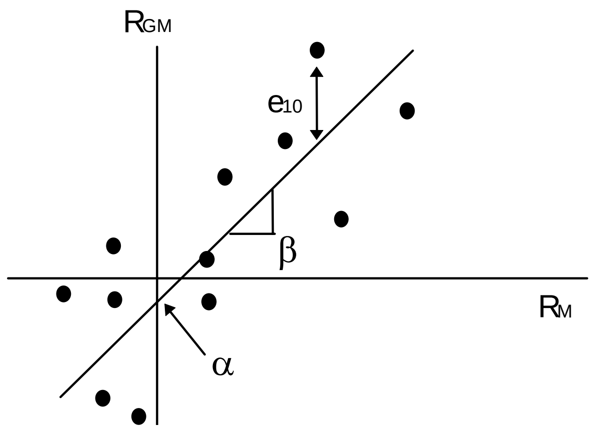
$\alpha = -2.590$ $\beta = 1.1357$ (0.309)

variance of residuals: = 12.0601

$R^2 = .575$

iv) What does this all mean?

b) Let's start with the coefficients. If we scatter plot the data we might have a picture that looks like the following




i) The α is the intercept and β is the slope of a line that fits the data the best in the sense that we have minimized the sum of squared residuals. The β is the estimated CAPM beta given that we have used a market index for our factor.

ii) The error terms from our regression represent the $e_i$ from our model. For instance, if we look at the $10^{th}$ month we can calculate what the firm specific part of returns was for that month by looking at the distance of the $10^{th}$ data point from our fitted line.

iii) We can use the standard errors from the output to calculate the t-ratios of the coefficient estimates. Just divide the coefficient value by the standard error for that coefficient.

   $t(\alpha) = -2.59/1.547 = 1.67$

   $t(\beta) = 1.1357/0.309 = 3.68$

   (1) These statistics are used to test whether the coefficients are statistically different from zero. A rule of thumb to remember is that if the t-statistic is greater than 2, we can reject the null hypothesis that the coefficient is statistically different from 0. In this case the $\alpha$ is NOT different from 0. Is this good or bad?

iv) The $e_i$ are the estimates of the monthly unexpected firm-specific component of $R_{GM}$. That means we can estimate the part of the variance in GM returns due to firm specific events by looking at the variance of the residuals. In this case the part of $var(R_{GM})$ due to firm specific shocks is 12.601.

v) What about the $R^2$ term?

   (1) Statistically $R^2$ = (Explained variance)/(total variance) and we can use this to determine how well the line fits the data. More precisely, we interpret $R^2$ as the amount of the variance in $R_{GM}$ that is "explained" by the independent variable $R_M$.

   (2) From an economic standpoint $R^2 = \beta^2\sigma^2_M/\sigma^2_r$ and is the ratio of systematic variance to total variance. This tells us what fraction of the firm's volatility is attributable to the market movements.

4) Diversification and the single index model

   a) Consider an equally weighted portfolio of assets: that is $w_i = 1/n$.

   b) Since

      $R_i = \alpha_i + \beta i R_M + e_i$

      i) this also holds for the portfolio of assets so that

      $R_P = \alpha_P + \beta_P R_M + e_P$.

      ii) and we know that

         $R_P = \Sigma w_i R_i$

         $= (1/n)\Sigma R_i$

         $= (1/n)\Sigma(\alpha_i + \beta i R_M + e_i)$

         $= (1/n)\Sigma(\alpha_i) + (1/n)\Sigma(\beta i)R_M + (1/n)\Sigma(e_i)$

   c) Now, match terms between this last expression and the second equation to see that

      $\alpha_P = (1/n)\Sigma\alpha_i$

      $\beta_P = (1/n)\Sigma(\beta i)$

      $e_P = (1/n)\Sigma e_i$

      i) and we also have

         $var(e_P) = (1/n)^2 var(\Sigma e_i) = (1/n)\overline{\sigma}_e^2$

ii) Now, notice that as the number of assets in the portfolio gets large the variance term due to firm specific shocks goes to zero. That is as $n \rightarrow$ infinity, $\sigma^2_e \rightarrow 0$

iii) In words, as we increase the number of assets in the portfolio, the variance of the portfolio returns attributable to firm specific events goes to zero. Diversification eliminates idiosyncratic risk!

5) Commercially available β's

a) Some sources of β calculate the single index model using total realized returns, which begs the question of how using total versus excess returns affects the comparison of the single index model with the CAPM.

i) If we start with the single index model we have

$r_i - r_f = \alpha_i + \beta i(r_M - r_f) + e_i$

$r_i = \alpha_i + r_f + \beta i r_M - \beta i r_f + e_i$

$r_i = \alpha_i + r_f(1 - \beta i) + \beta i r_M + e_i$

ii) This rearrangement implies that β doesn't change whether we use total or realized returns but the intercept α will change.

iii) In fact, α will become $\alpha_i + r_f(1 - \beta i)$

(1) The difference in the intercept will be small though, especially if the frequency of the data is high such as using daily or weekly returns. In this case $r_f$ will be very small and $(1 - \beta i)$ will also be small since most β's are close to 1 in magnitude.

b) Another issue with commercial β's is that over time, they tend towards 1.

c) Why might that happen?

i) This may happen because as firms age, they become more diversified in their activities and hence, are more like the market.

d) Since β's are estimated from data up to today and are used as forecasts of tomorrow's β's, we may want to adjust β in the direction that we know they move over time. That is we may want to adjust them towards 1.

i) The most common adjustment found in the industry is to calculate an adjusted β from

$\beta i' = (2/3)\beta i + 1/3$

ii) That is, a beta is calculated from historical data and then stuck in the formula.

iii) This adjustment is an attempt to account for sampling error in the data and measurement error in the estimation.

e) Another adjustment that is often made is done by

i) Calculating β this month, time t.

ii) Regressing β's up to time t on β's up to time t - 1 to get the coefficients a and b in the following formula.

$\beta_t = a + b(\beta_{t-1})$

iii) Then using the estimated coefficients a and b to form the adjusted β as $\beta_t' = a + b\beta_t$

iv) This adjustment is also an attempt to account for sampling error in the data and measurement error in the estimation.

6) Diversification is cool
   a) Reduces risk
   b) We don't have to care about the risk of any individual asset, only about how it affects the risk of our portfolio
   c) Reduces or eliminates idiosyncratic risk.
      i) Total risk = Market risk + Idiosyncratic risk
         (1) Market or systematic risk, which all stocks experience and to a greater or lesser extent, cannot be diversified away.
         (2) Idiosyncratic risk is unique risk or diversifiable or unsystematic risk and represents risks that are unique to a particular firm, industry, location, country, or asset group.

   d) Modern Portfolio Theory: Henry Markowitz "father of modern portfolio theory" 1952.
      i) Markowitz started drawing risk and return relationships for individual stocks and portfolios and noted two things:
         (1) Minimum variance frontier: set is all the feasible combo's of stocks - all feasible portfolios having the least amount of risk for any given amount of expected return.
         (2) Efficient frontier: the part of the MVF investors would be interested in, the portfolios with the highest return with for any given level of risk.
         (3) Optimal portfolio selection
            (a) infinite number of portfolios in the efficient set: just change the weights
            (b) none is better than the others - depends on the amount of risk aversion
         (4) What about the risk free asset?
            (a) Inclusion of this assets gives us the capital market line
               (i) dominates all other lines
               (ii) no matter what other line you choose you can always get a higher rate of return for the same level of risk investing along this line
               (iii) between points Rf and M you are lending (investing in a riskless asset) but points past M and on, you are borrowing. This is where you would be if you leveraged M.
            (b) If there is riskless lending and borrowing then the CML is now the efficient set and M is the only efficient portfolio of *risky* assets.